The Routh-Hurwitz Stability Criterion, Revisited

Robert N. Clark

In the mid-nineteenth century James C. Maxwell, and others, became interested in the stability of motion of dynamic systems. Maxwell's interest in stability stemmed in part from his work with an automatic control system — a speed governor he and his colleagues were using in laboratory measurements to establish the definition of the ohm. Maxwell was the first to publish a dynamic analysis of this feedback system using differential equations [1],[2]. In this analysis he defined the types of responses one could expect from the solutions of linearized equations of motion having constant coefficients. He identified the conditions which must prevail on the roots of the characteristic polynomial corresponding to the linear differential equation in order that the solution of the homogeneous equation be stable. (They must lie inside the left-half plane.) In this paper he also urged mathematicians to address the question of how the coefficients of the polynomial are related to its roots which, for polynomials of degree higher than three or four, was a difficult question in those days, but a vital one in the study of stability, as it is today.

In the mid-1870s Maxwell was on the judging committee for the Adams Prize, a biennial competition for the best essay on a scientific subject selected by the committee. The topic for the 1877 Adams prize was The Stability of Motion E.J. Routh won the competition that year for his essay which showed how the number of roots of the characteristic polynomial lying in the right half plane could be determined from the coefficients of the polynomial [3]. Some twenty years following Routh, the Swiss mathematician A. Hurwitz, unaware of Routh's work, but also inspired by a stability problem in a control system advanced by his engineer colleague Dr. A. Stodola (the speed regulation of a high pressure water turbine), presented the conditions on the coefficients of the polynomial which must prevail in order that its roots all have negative real parts [4]. Hurwitz's conditions are identical to those given by Routh for no right half plane roots, and are known today as the Routh-Hurwitz Stability Criterion.

Both Routh and Hurwitz recognized that their test functions would not account for roots on the $j\omega$ axis if the test functions were simply computed from the formulae of the test functions. If the $j\omega$ axis roots are simple the corresponding differential equation solution is neither stable nor unstable, it is usually called marginally stable, since it has an undamped sinusoidal mode. But if the $j\omega$ axis roots are repeated the solution will be unstable, with a mode of the form: $t[\sin(\omega t + \phi)]$, if the roots are double. The Routh-Hurwitz criteria, applied only by formula, will not reveal this form of instability. An example of such a case is the unit impulse response of a system having a transfer function $W(s)$:

$$W(s) = 16/(s^5 + s^4 + 8s^3 + 8s^2 + 16s + 16).$$

This impulse response, $w(t)$, is plotted in Fig. 1. It clearly shows an unstable response due to the secular term with the sinusoidal factor. The roots of the characteristic polynomial for this system are: -1, $j2$, $j2$, $-j2$, and $-j2$.

Routh's Array for this case, after invoking the *auxiliary polynomial* procedure on rows s^3 and s^1, and dividing all the derived rows by a positive constant, is shown in Table I. There are no sign changes in the first column of the array, indicating that there are no roots of the characteristic polynomial in the right half s-plane. The $j\omega$ axis roots appear as factors of the auxiliary polynomial, of course, but these could be overlooked by one who simply follows the procedures given in most text books. A recently published text,

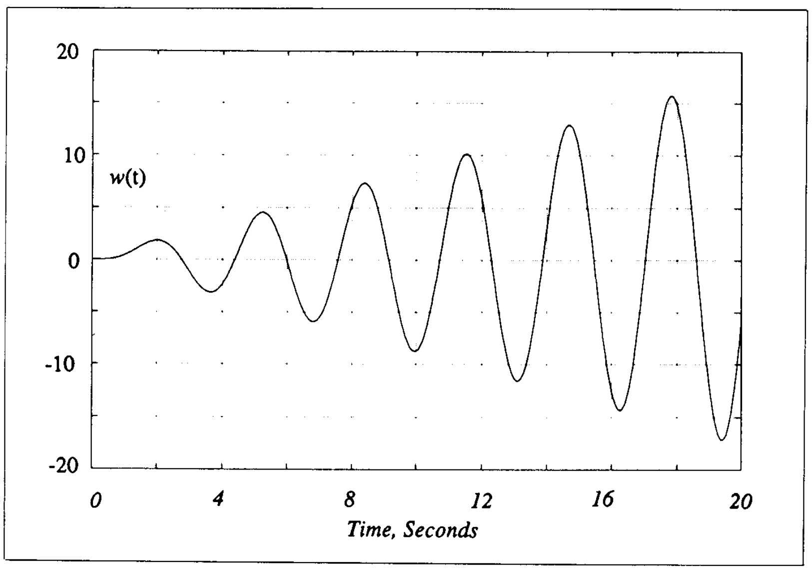

Fig. 1. Unit impulse response for the example.

Robert N. Clark is Professor of Electrical Engineering and of Aeronautics and Astronautics at the University of Washington, Seattle, WA 98195.

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Table I
Routh's Array for the Example

s^5:	1	8	16
s^4:	1	8	16
s^3:	1	4	
s^2:	1	4	
s^1:	1		
s^0:	1		

however, provides an appropriate emphasis on this point [5].

It is interesting to note that Routh himself was initially concerned about repeated roots, even those lying on the real axis in the left half plane relatively near the origin. He thought the modes of the time response corresponding to such roots, even though they would eventually die out, might cause the linear solution to be large enough, over a long enough period, to exceed the "linear range" of a basically nonlinear process. Later, Routh seemed to become less concerned about this possibility.

Systems whose characteristic polynomials have $j\omega$ axis roots, even repeated roots, are of considerable practical importance in the control of mechanical devices employing flexible appendages. Therefore the point of this paper can be pertinent to the analysis of practical systems. This is true even with present day computer analysis programs which can factor high order polynomials and display the roots in a fraction of a second, *provided the numerical algorithm used does not fail in the case at hand.* The Routh-Hurwitz criterion can provide an analytical check in such situations.

References

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[4] A. Hurwitz, "On the conditions under which an equation has only roots with negative real parts," *Mathematische Annelen*, vol. 46, pp. 273-284, 1895.

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Comments on "Automated Calibration of a Fuzzy Logic Controller Using a Cell State Space Algorithm"

Daniel P. Petersen

A recent paper by Smith and Comer [1] presents a method for "optimizing" the nonlinear input/output map generated by a set of fuzzy control rules. The procedure in [1] is quite involved and is not *per se* the subject of this comment. Rather, this comment concentrates on the specific result obtained when applied to a particular plant, a DC motor with angular position feedback.

The plant has analog transfer function (armature voltage input to rotation angle output), with parameters $\kappa = 0.90566$ and $\tau = 0.283$:

$$\theta/V = \frac{\kappa}{s(1+\tau s)}$$

Daniel Petersen is Professor Emeritus, Department of Electrical and Computer Engineering, The University of New Mexico, Albuquerque, NM 87131.

For a sampling period $T = 0.01$ s with piecewise-constant input, the discrete-time state-variable representation is [1, eq. (28)], with constants as calculated from [1, eq. (27)]: $a_{12} = 0.009825$, $a_{22} = 0.9653$, $b_1 = 0.0001746$, $b_2 = 0.03472$:

$$\begin{bmatrix}\theta((k+1)T)\\ \dot\theta((k+1)T)\end{bmatrix} = \begin{bmatrix}1 & a_{12}\\ 0 & a_{22}\end{bmatrix}\begin{bmatrix}\theta(kT)\\ \dot\theta(kT)\end{bmatrix} + \begin{bmatrix}b_1\\ b_2\end{bmatrix}\kappa U(kT$$

The measured output, motor shaft angle θ, is thus related to armature voltage V in the z domain by:

$$(z-1)(z-a_{22})\theta = b_1\kappa\,(z+(a_{12}b_2-b_1a_{22})/b_1)\,V.$$

Thus, the transfer function θ/V, has poles at $z = 1$, 0.9653 and a zero at $z = -0.9885$.

The end results of the optimization procedure in [1] are [1, tables I and II] (evidently interchanged in labeling) for "five-rule" and "nine-rule" controllers with associated membership functions shown in [1, fig. 2]. Clearly, this provides a deterministic input/output map through interpolation between linear "rules"; these maps are shown in [1, figs. 3 and 4]. The response of each rule set to a 2-rad initial error is shown in [1, fig. 7]. For comparison, the response of the system incorporating a (proportional/derivative) "PD controller ... tuned to minimize rise time with less than 5% overshoot" is also shown, and as expected is measurably inferior.

Unfortunately, the authors of [1] do not show explicitly how they "tuned" the conventional PD controller, nor its parameters so obtained. The author of this comment, by using well-known design methods with no attempt at optimization, has selected PD parameters which not only outperform the authors' "tuned" reference system but their "optimized" fuzzy controllers as well; and

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