Control Systems

Root Locus & Pole Assignment

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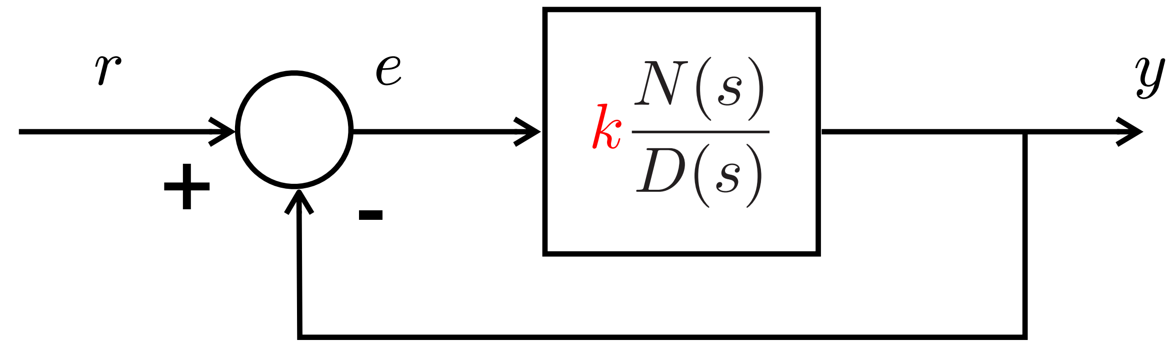
Outline

- root locus definition
- main rules for hand plotting
- root locus as a design tool
- other use of the root locus
- pole assignment

Root locus

Question: how do the closed-loop poles vary when a (**real**) gain in the open loop changes?



Hypothesis on $N(s)$ and $D(s)$

- monic polynomials
- coprime
- $m < n$ (excess of the number of poles w.r.t. zeros in the loop function $L(s)$)

loop function $$L(s) = k\frac{N(s)}{D(s)} = k\frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{j=1}^{n}(s - p_j)}$$ zero/pole representation

closed-loop system

- **zeros** = zeros of open loop (if we consider the complementary sensitivity)
- **poles** of the closed-loop system are the roots of the closed loop characteristic polynomial

root locus = location of the closed-loop poles in the s-plane as k varies from - ∞ to $+\infty$

Root locus

Considering, for example, the complementary sensitivity function $T(s)$

$$T(s) = \frac{y(s)}{r(s)} = \frac{L(s)}{1+L(s)} = \frac{k\dfrac{N(s)}{D(s)}}{1+k\dfrac{N(s)}{D(s)}}$$

- the **poles** of the closed-loop system are the roots of $1 + L(s) = 0$ which can be rewritten as

$$1 + k\frac{N(s)}{D(s)} = 0 \quad \Leftrightarrow \quad \boxed{D(s) + kN(s) = 0}$$

root locus equation

$$(\text{if } k \neq 0) \quad \frac{N(s)}{D(s)} = -\frac{1}{k} \quad \Leftrightarrow \quad N(s) + \frac{1}{k}D(s) = 0$$

the root locus equation $D(s) + kN(s) = 0$ is usually denoted as $\boxed{p(s,k) = 0}$

- since $N(s)$ and $D(s)$ are coprime, any closed loop transfer function will have the poles given by the root locus equation
- since the root locus equation $p(s,k) = 0$ is a polynomial of the same order than $D(s)$ then the **closed-loop** system will have as many poles as the open-loop one, that is n.

- the **zeros** of the closed-loop system $T(s)$ coincide with those of the open loop $L(s)$

Formal way to plot the root locus (we will use some simplified rules)

Let us define $$L(s) = k \frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{j=1}^{n}(s - p_j)}$$ m zeroes, n poles, with $m < n$

$$p(s,k) = \prod_{j=1}^{n}(s - p_j) + k \prod_{i=1}^{m}(s - z_i) = 0 \rightarrow \prod_{j=1}^{n}(s - p_j) = -k \prod_{i=1}^{m}(s - z_i)$$

complex numbers

$$\boxed{|k| = \frac{\prod_{j=1}^{n}|s - p_j|}{\prod_{i=1}^{m}|s - z_i|}}$$

magnitude condition

$$\sum_{j=1}^{n} \angle(s - p_j) - \sum_{i=1}^{m} \angle(s - z_i) = \pi + \angle k + 2h\pi \quad h \in \mathbf{Z}$$

$$\boxed{\sum_{j=1}^{n} \angle(s - p_j) - \sum_{i=1}^{m} \angle(s - z_i) = \begin{cases} (2h+1)\pi & \text{for} \quad k \geq 0 \\ 2h\pi & \text{for} \quad k \leq 0 \end{cases}}$$

phase condition

phase condition

from $1 + k\dfrac{N(s)}{D(s)} = 0$

positive locus (k positive) $\quad k\angle\left(\dfrac{N(s)}{D(s)}\right) = \angle(-1) = (2h+1)\pi$

negative locus (k negative) $\quad |k|\angle\left(\dfrac{N(s)}{D(s)}\right) = \angle(1) = 2h\pi$

the phase condition is used to draw the locus of the roots: we do not need to solve for the high-order polynomial roots, we just need to **verify** if a given point of the complex plane satisfies the phase condition and therefore corresponds to a root of $1 + k\,N(s)/D(s) = 0$ for some real value of k (this value is found by using the magnitude condition)

there are however some **guidelines** for drawing rapidly, by hand, a sketch of the root locus

Since the closed-loop system has the same number n of poles as the open-loop, each of n the closed-loop poles will move along a **branch** of the root locus as k varies from 0 to $+\infty$ (similarly for the negative locus as k varies from - ∞ to 0)

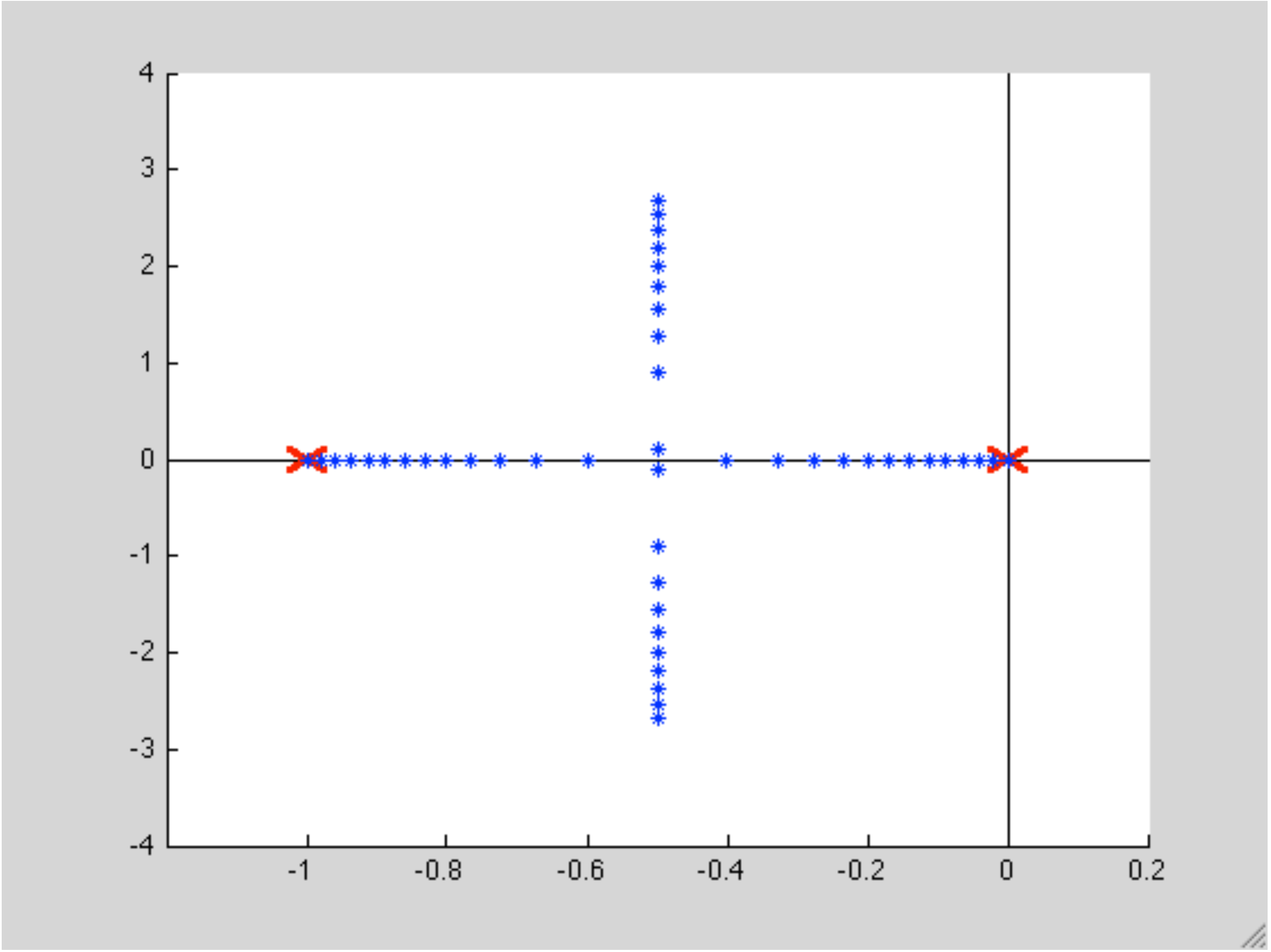
$\longrightarrow$ the positive root locus has n branches (same for negative locus)

The coefficients of the root locus equation are real

$\longrightarrow$ the root locus is **symmetric** w.r.t. the real axis we have to learn to visualize **how all the poles move simultaneously** in the complex plane as k increases

see the animation (not available in the PDF file) for the positive root locus of

$$1 + k\frac{N(s)}{D(s)} = 1 + k\frac{1}{s(s+1)}$$

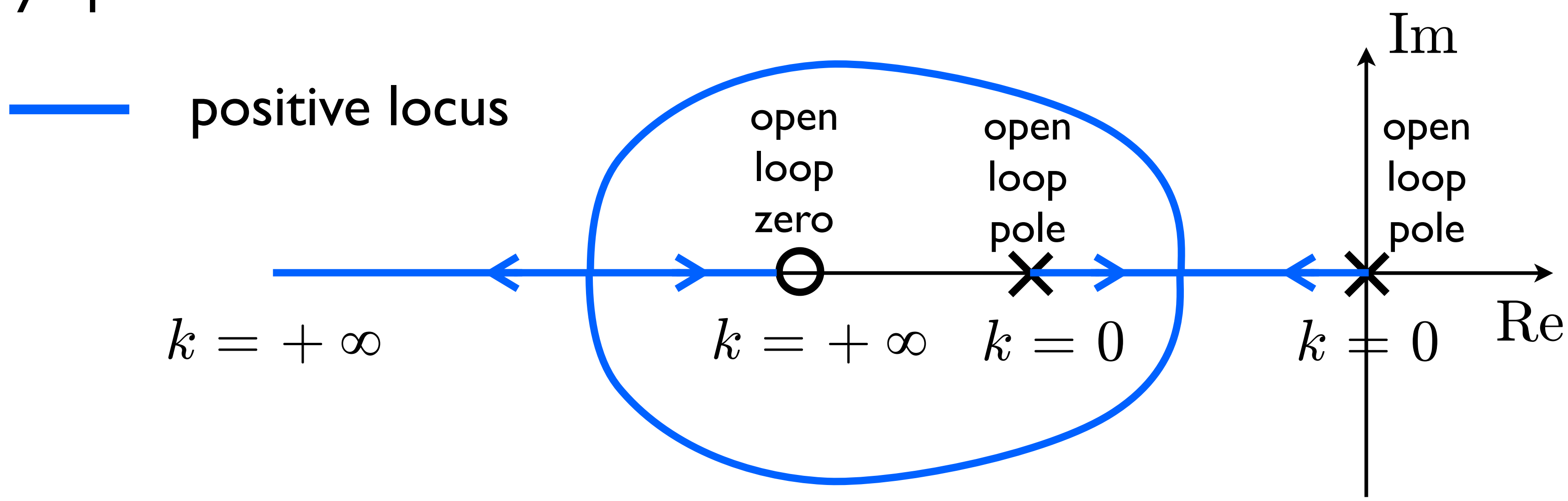


a point s^* belongs to the root locus $p(s,k) = 0$ (or equivalently s^* is a pole of the closed-loop system for some value of $k = k^*$) if and only if there exists a **real** value k^* such that

$$p(s^*,k^*) = 0$$

Rule 1 (positive locus)

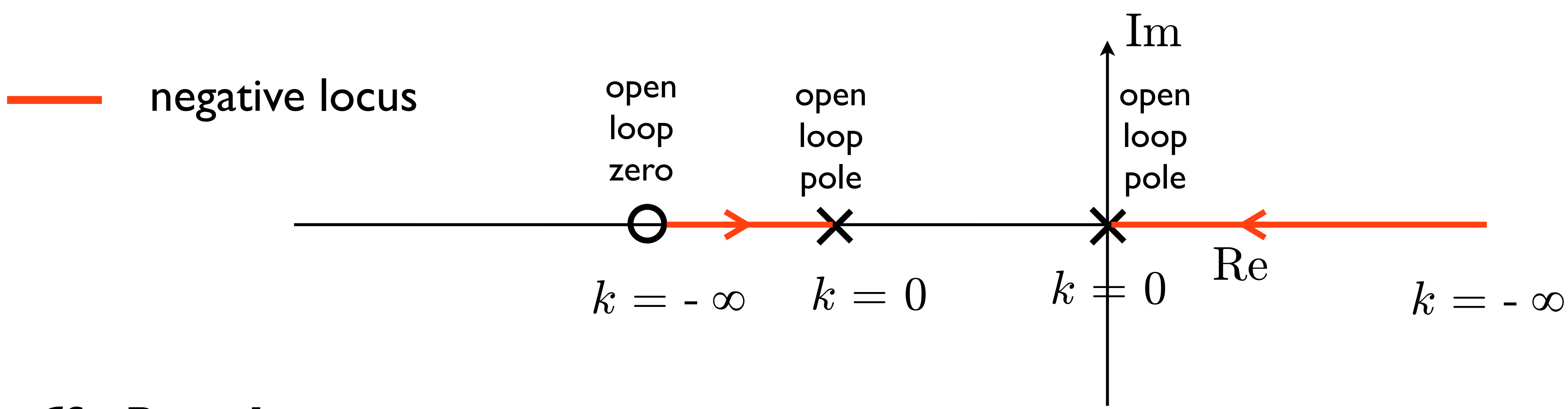
The n branches of the positive locus start at the open-loop poles and, when k tends to $+\infty$, m branches tend to the finite m open-loop zeros while $(n - m)$ tend to infinity along $(n - m)$ asymptotes



$$F(s) = \frac{s+3}{s(s+2)}$$

Rule 1bis (negative locus)

For the n branches of the negative locus, as k varies from $-\infty$ to 0, m branches start from the m finite open-loop zeros while the other $(n - m)$ branches come from infinity along $(n - m)$ asymptotes. All branches end, for $k = 0$ in the open-loop poles

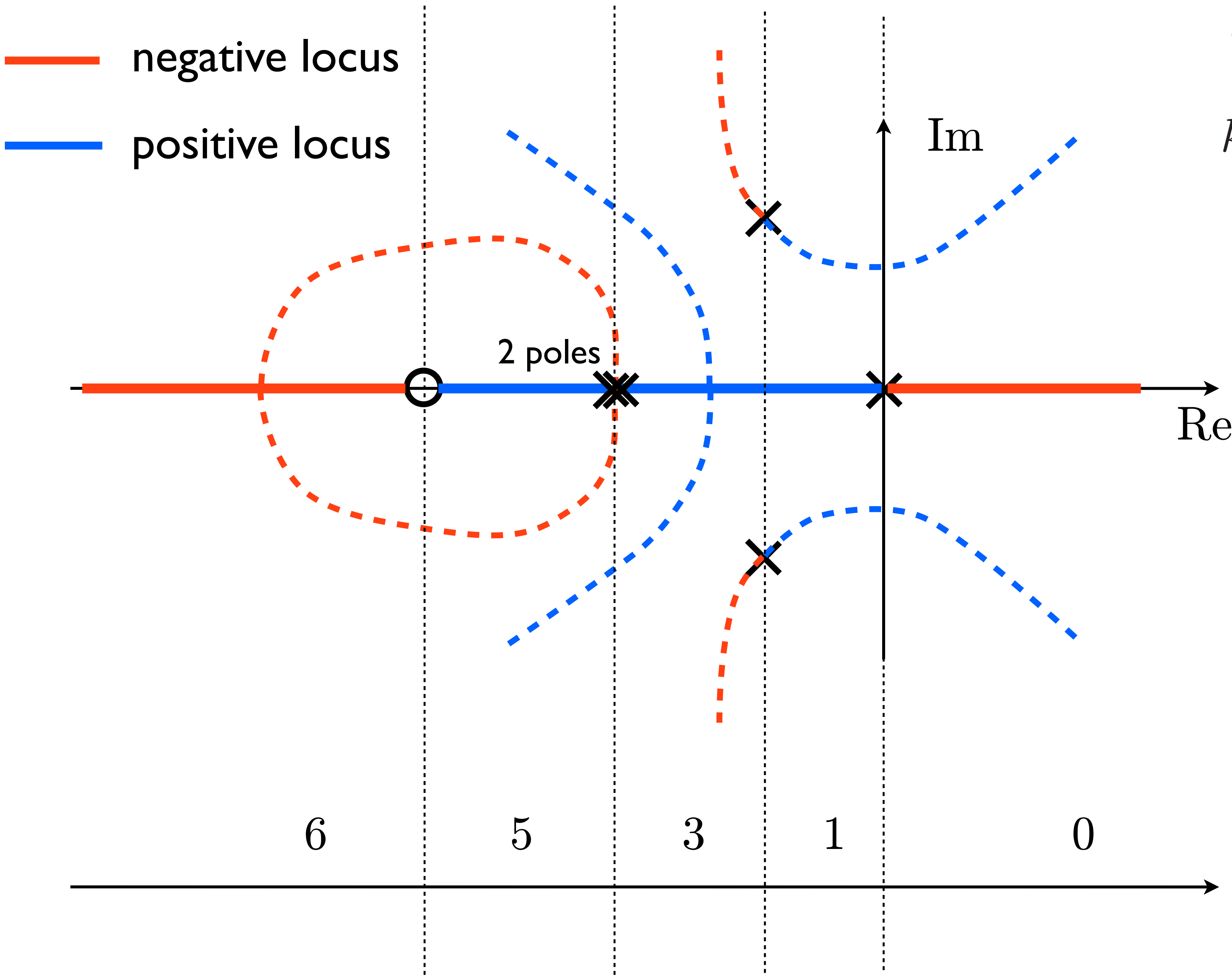


Rule 2

Every point on the real axis belongs to either the positive or negative locus.

A point on the real axis that leaves on its **right** an **odd** number of poles and zeros counted with their multiplicity belongs to the **positive locus**.

All the other points belong to the negative locus.



example

$$k\frac{s+4}{s(s+0.5+j)(s+0.5-j)(s+2)^2}$$

from the magnitude condition, if $s = r$ is real then $k = -D(r)/N(r)$ with $N(s)$ & $D(s)$ having real coefficients, then k real so there alwats exists a k real which gives the closed loop pole $s = r$

# of open-loop poles/zeros left on the right

Rule 3

The $(n - m)$ asymptotes are centered in a center of asymptotes

$$s_0 = \frac{\sum_{j=1}^{n} p_j - \sum_{i=1}^{m} z_i}{n - m}$$

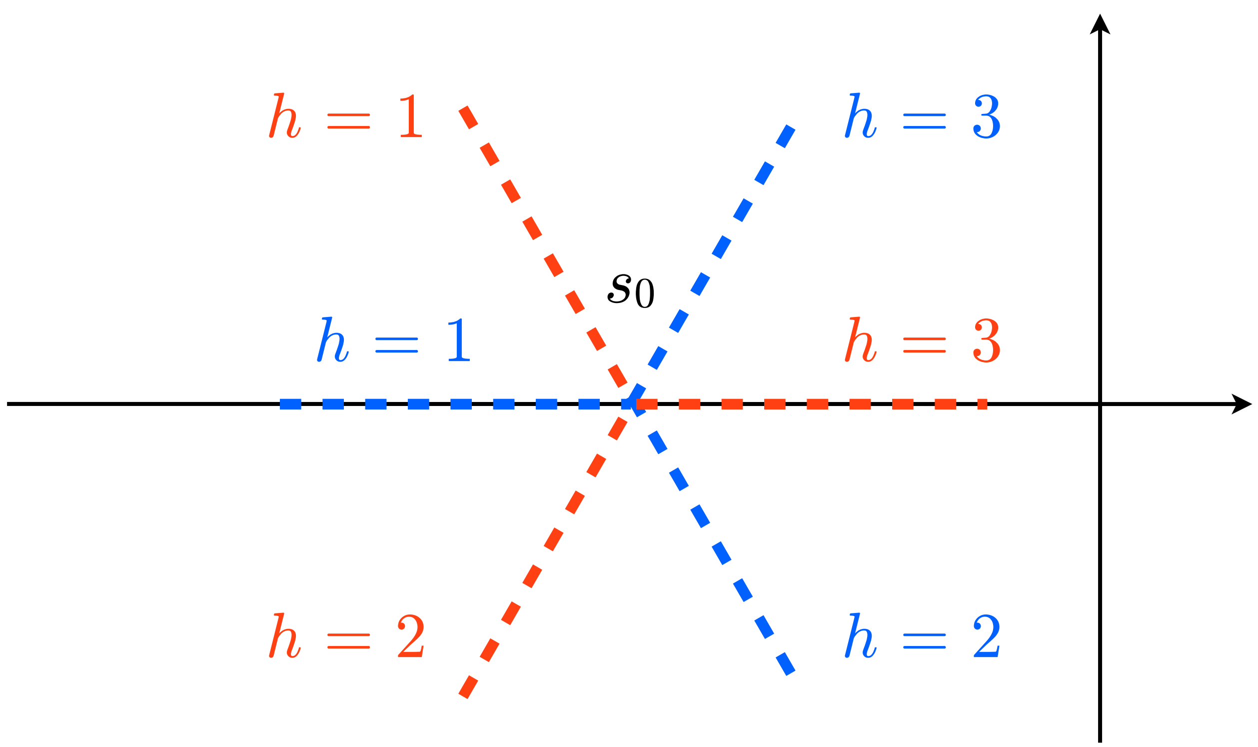
center of asymptotes

and form angles of

$$\begin{cases} \dfrac{(2h+1)\pi}{n-m} & \text{for positive locus} \\ \dfrac{2h\pi}{n-m} & \text{for negative locus} \end{cases} \qquad h = 1, 2, \ldots, n - m$$

example

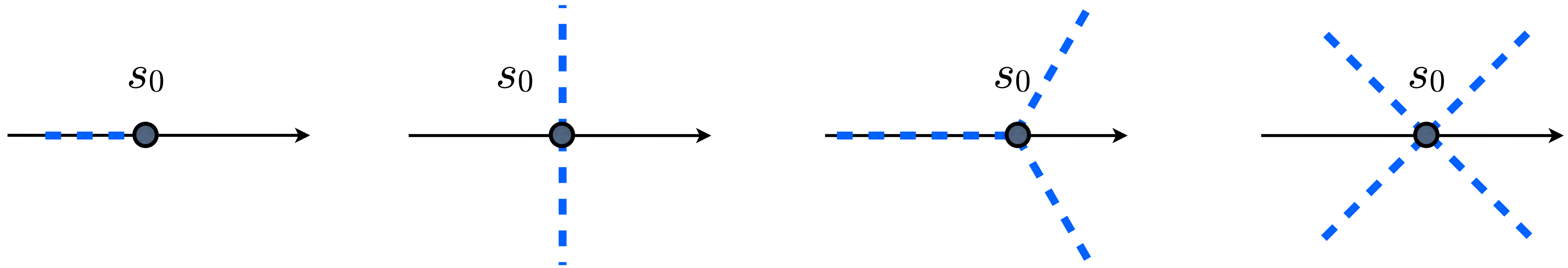
$n - m = 3$



	Pos	Neg
$h = 1$	$3\pi/3$	$2\pi/3$
$h = 2$	$5\pi/3$	$4\pi/3$
$h = 3$	$7\pi/3$	$6\pi/3$

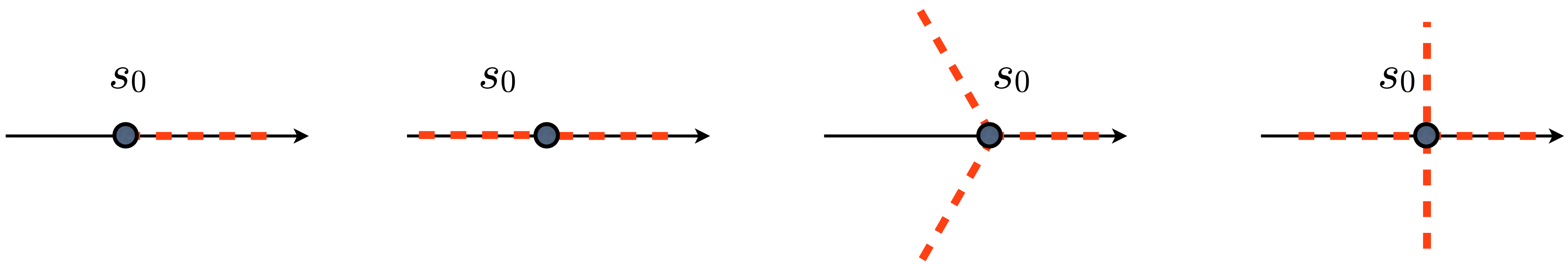
we have the following cases

positive locus



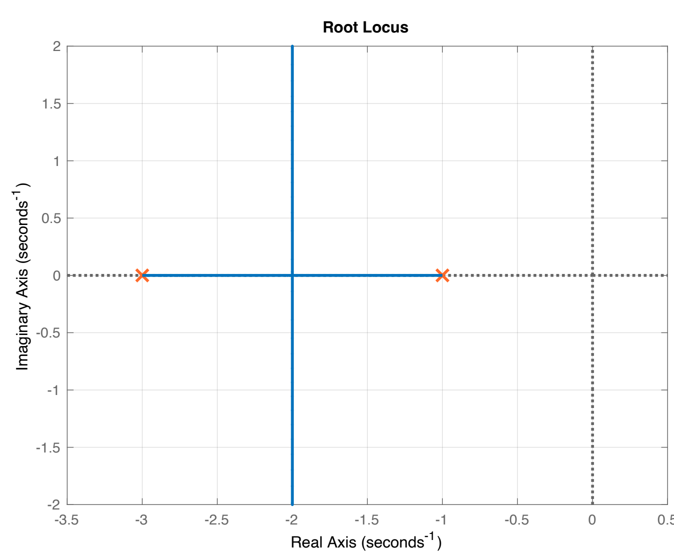
n - $m = 1$ $\quad$ n - $m = 2$ $\quad$ n - $m = 3$ $\quad$ n - $m = 4$

negative locus

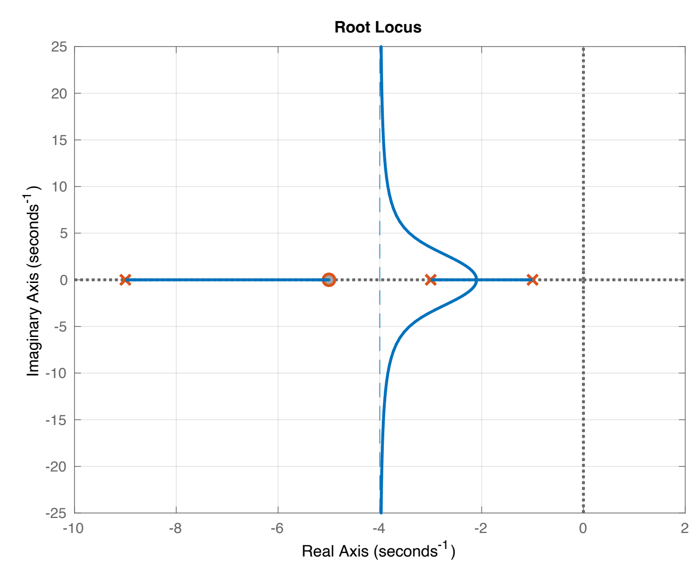


N.B. the asymptotes are **not necessarily branches of the root locus**

$$\frac{K}{(s+1)(s+3)}$$



$$\frac{K(s+5)}{(s+1)(s+3)(s+9)}$$



the points of the locus are either

- regular points (solutions of $p(s,k) = 0$) or
- **singular points** (or **breakaway**/**break-in** points) solutions of

$$\begin{cases} p(s,k) = 0 \\ \dfrac{\partial p(s,k)}{\partial s} = 0 \end{cases} \quad \longleftrightarrow \quad \begin{cases} \displaystyle\prod_{j=1}^{n}(s-p_j) + k\prod_{i=1}^{m}(s-z_i) = 0 \\ \displaystyle\frac{\partial}{\partial s}\prod_{j=1}^{n}(s-p_j) + k\frac{\partial}{\partial s}\prod_{i=1}^{m}(s-z_i) = 0 \end{cases}$$

being $k = -\dfrac{\displaystyle\prod_{j=1}^{n}(s-p_j)}{\displaystyle\prod_{i=1}^{m}(s-z_i)}$ from locus equation, substituted in the second gives

$$\boxed{\prod_{i=1}^{m}(s-z_i)\frac{\partial}{\partial s}\prod_{j=1}^{n}(s-p_j) - \prod_{j=1}^{n}(s-p_j)\frac{\partial}{\partial s}\prod_{i=1}^{m}(s-z_i) = 0}$$

equation of order $n + m$ - 1 for the **candidates singular points** (we may have solutions corresponding to complex k which are therefore not points of the root locus) a **candidate singular s^* point** is a true singular point if the corresponding value k^* is **real**

$$k^* = -\frac{\displaystyle\prod_{j=1}^{n}(s^* - p_j)}{\displaystyle\prod_{i=1}^{m}(s^* - z_i)}$$

clearly candidate singular points which are real valued are for sure singular point

the singular point s^* will be a solution of the locus equation $p\,(s^*,k^*) = 0$ with **multiplicity** μ greater equal to 2

$$p(s,k^*) = (s - s^*)^{\mu}\, p'(s,k^*) \qquad \mu \geq 2$$

There are some situations where finding singular points is easier

every open-loop pole/zero with multiplicity greater than 1 is a singular point of the root locus

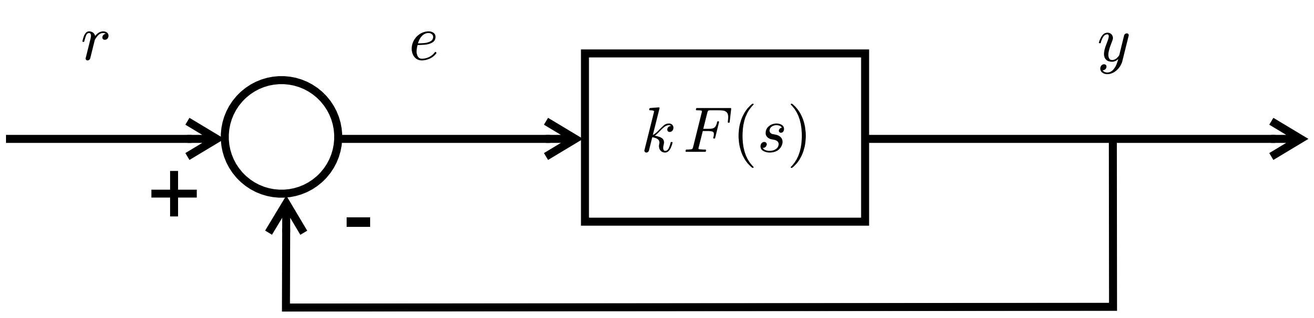
Proof: from the candidate singular points equation

Rule 4

Let s^* be a singular point with multiplicity μ, then:

- 2μ branches merge in the singular point and are alternatively convergent and divergent.
- these branches divide the plane in equal parts.
- if the singular point is a multiple pole or zero of the open-loop system, then the branches also alternate as positive and negative branches.

example $\quad F(s) = \dfrac{s+3}{s(s+2)}$



$$p(s,k) = s(s+2) + k(s+3) = s^2 + (2+k)s + 3k$$

$$\frac{\partial}{\partial s}p(s,k) = 2s + (2+k)$$

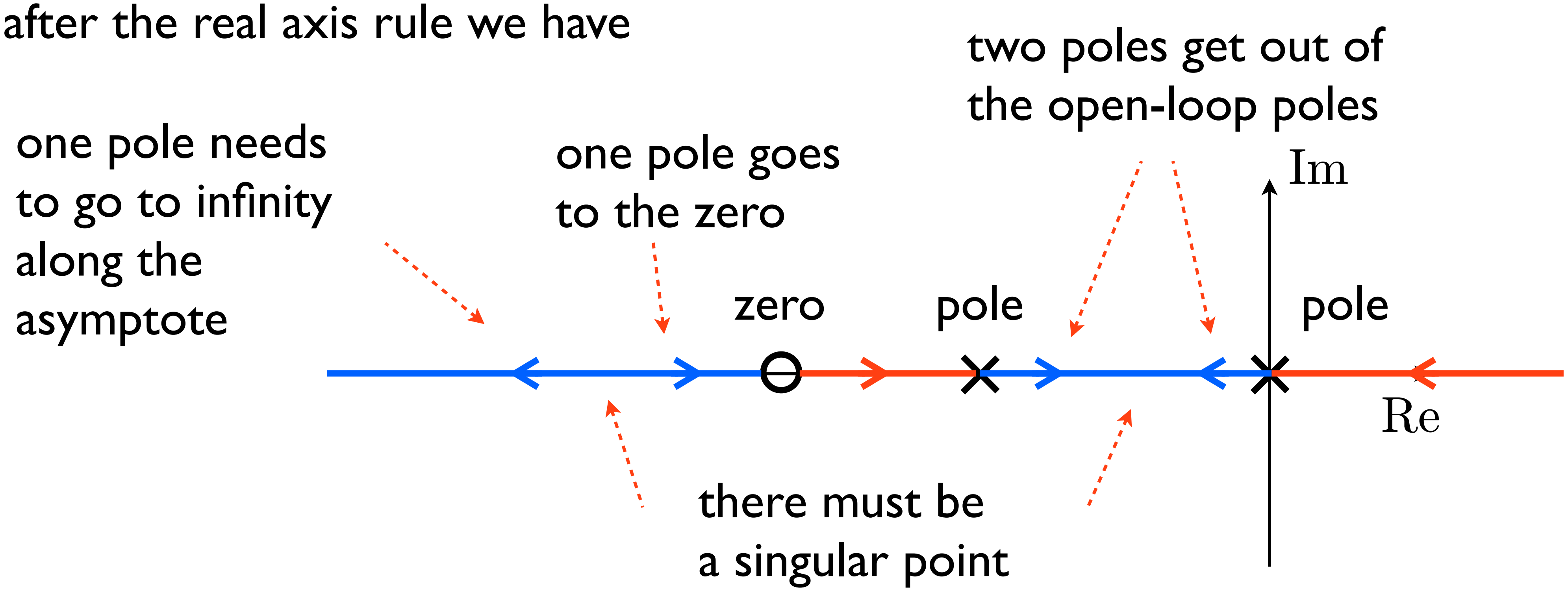
candidates $\quad (s+3)\dfrac{\partial}{\partial s}s(s+2) - s(s+2)\dfrac{\partial}{\partial s}(s+3) = s^2 + 6s + 6$

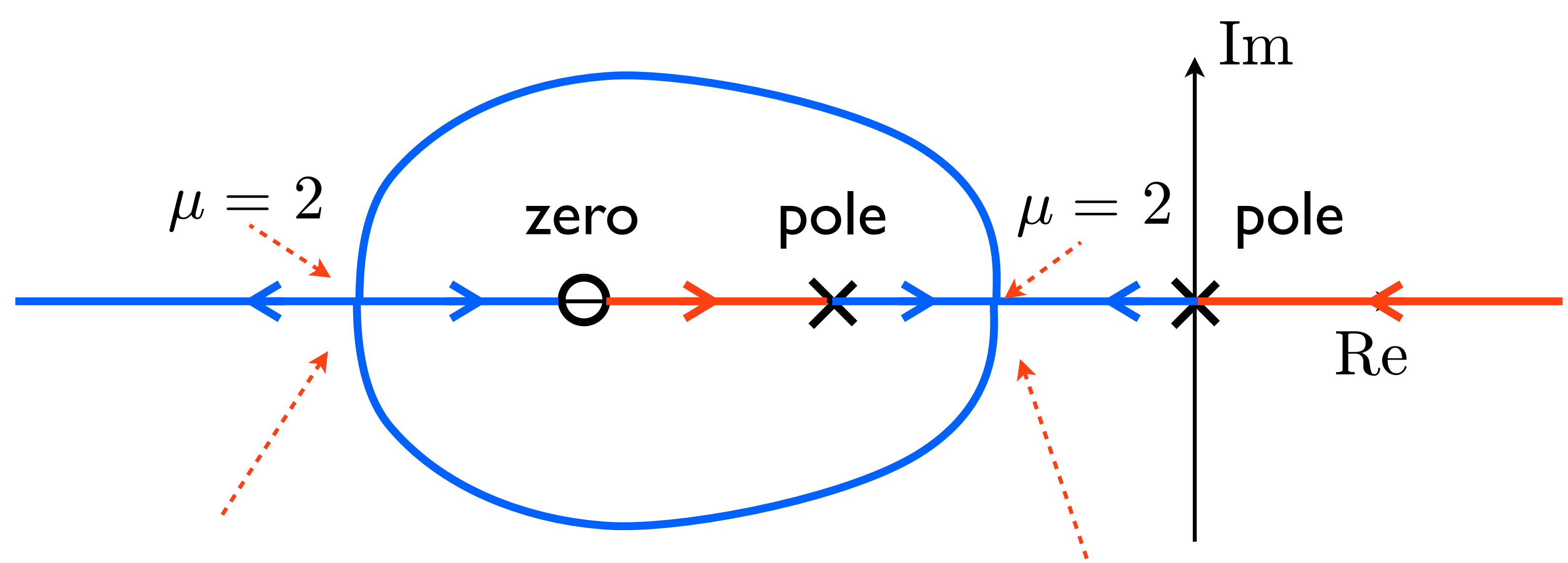
$s_1{}^* =$ -4.73

$s_2{}^* =$ -1.27

real values → real values of k^* → singular points was it necessary to compute the singular points?

after the real axis rule we have





at these singular points of multiplicity $\mu = 2$ the entire round angle is divided into 2 x $\mu = 4$ equal angles of $90°$ each alternative formula to determine the candidates breakaway/break-in (singular) points

$$\sum_{j=1}^{n} \frac{1}{s-p_j} - \sum_{i=1}^{m} \frac{1}{s-z_i} = 0$$

- this formula will **not** give us the singular points corresponding to **repeated poles or zeros** of the **open-loop** (obvious singular points)
- repeated poles and zeros in this formula need to be taken into account in the sum with their multiplicity

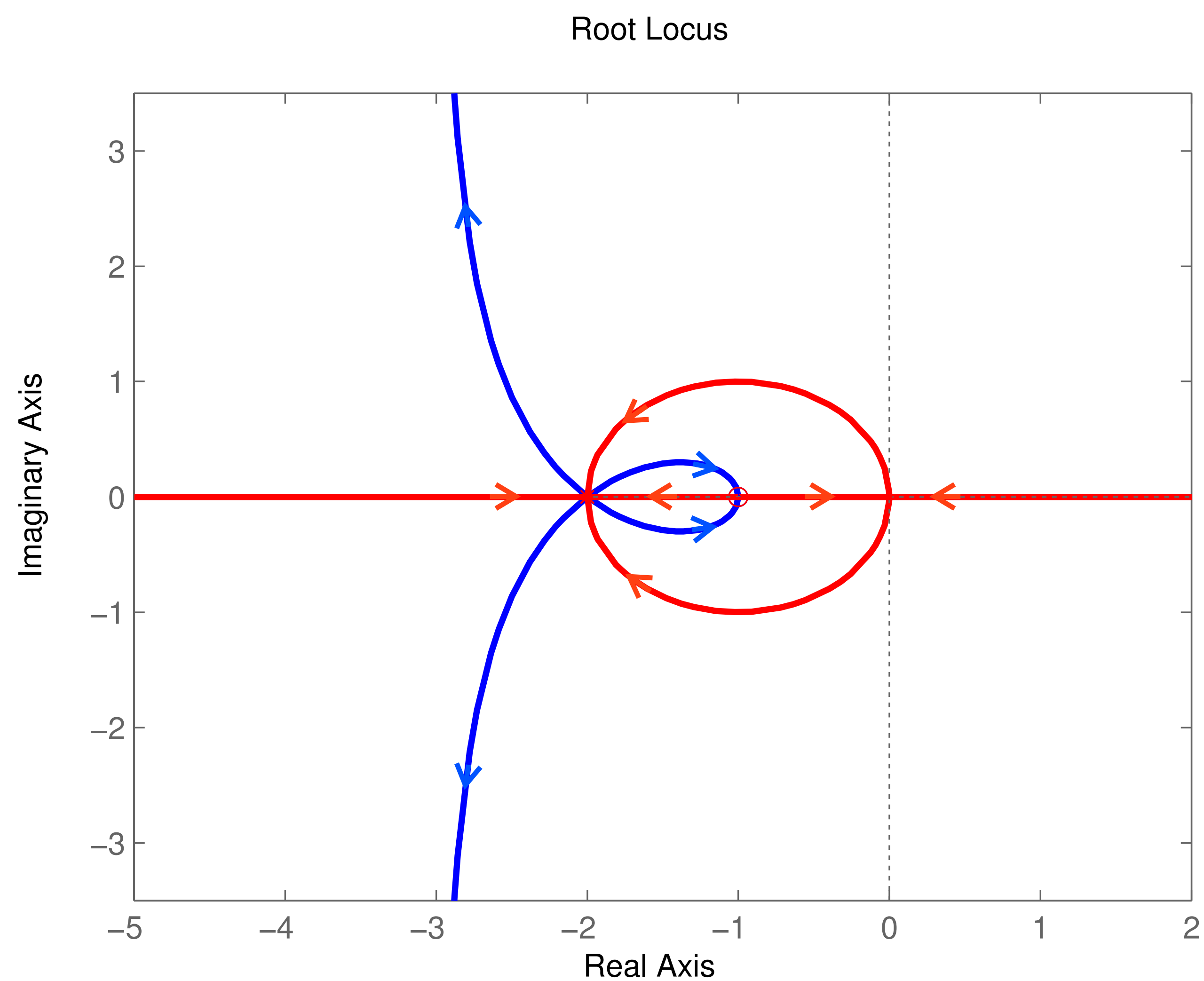
example $\quad F(s) = \dfrac{(s+1)^2}{(s+2)^4}$

$$\frac{4}{s+2} - \frac{2}{s+1} = \frac{2s}{(s+2)(s+1)}$$

$$\downarrow$$

$$s^* = 0$$

since it belongs to the real axis it is for sure a singular point



In order to determine **stability** (all the poles should be, for the same interval of values of k, in the open left half plane) it may be important to establish for which values of k some branches cross the imaginary axis. This can be achieved by determining for which values of k the elements of the first column of the Routh table become 0 since this is when a first column term changes sign and therefore a pole crosses the imaginary axis (remember that, when the table can be built from the basic definition, the number of sign changes in the first column is equal to the number of roots with positive real part).

$$F(s) = \frac{s+1}{s(s-2)(s+4)}$$

$$p(s,k) = s(s-2)(s+4) + k(s+1) = s^3 + 2s^2 + (k-8)s + k$$

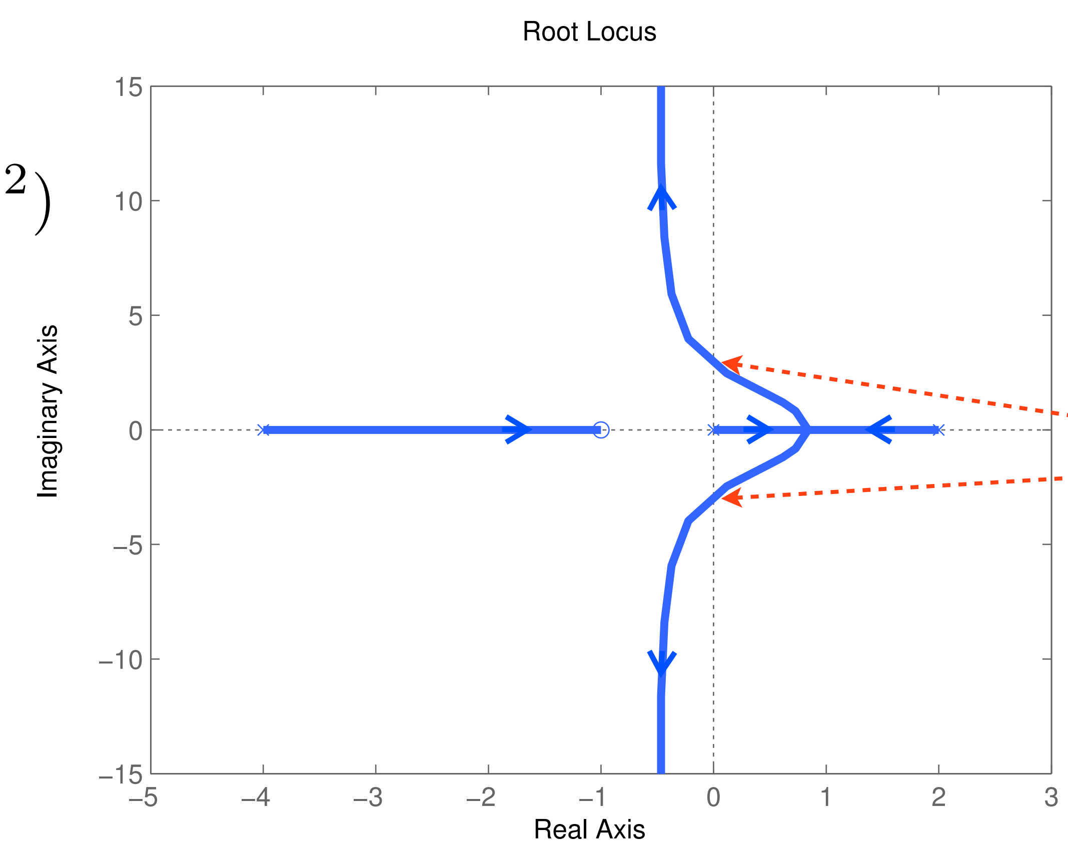
characteristic equation of the closed-loop system

for $k = 16$

$$p(s,16) = (s+p)(s^2+\omega^2)$$

$$p = 2$$

$$\omega^2 = 8$$



Routh table

$$\begin{array}{|cc} 1 & k-8 \\ 2 & k \\ k-16 & \\ k & \end{array}$$

RL as a design tool

the basic idea is based on the **positive root locus** behavior for **high values** of the **gain** k

- $(n - m)$ branches tend at infinity along $(n - m)$ asymptotes
- the remaining m branches tend to the m open loop zeros

therefore if the zeros are in the open left half-plane (i.e. have negative real part) and the asymptotes (for the positive root locus) always stay to the left of the imaginary axis then for sufficiently high values of the gain all the poles will be to the left of the imaginary axis that is a high gain will stabilize the system

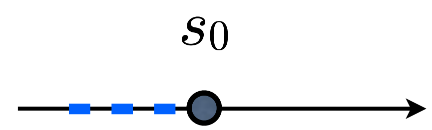
the asymptotes and the open-loop zeros attract the positive branches

A system with

- all its zeros, if any, in the open left half-plane or equivalently
- having no zeros with positive or null real part

is said to be **minimum phase**

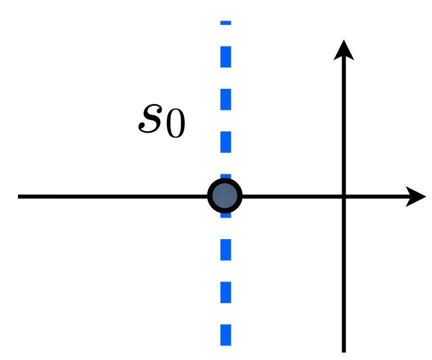
- we can have either n - $m = 1$ asymptote (positive locus)



center of asymptotes is not important since we look at the high-gain behavior

→ if the open-loop system is minimum phase and has (relative degree) n - $m = 1$ then for sufficiently high values of k the closed-loop system is for sure asymptotically stable

- or n - $m = 2$ asymptotes (positive locus)



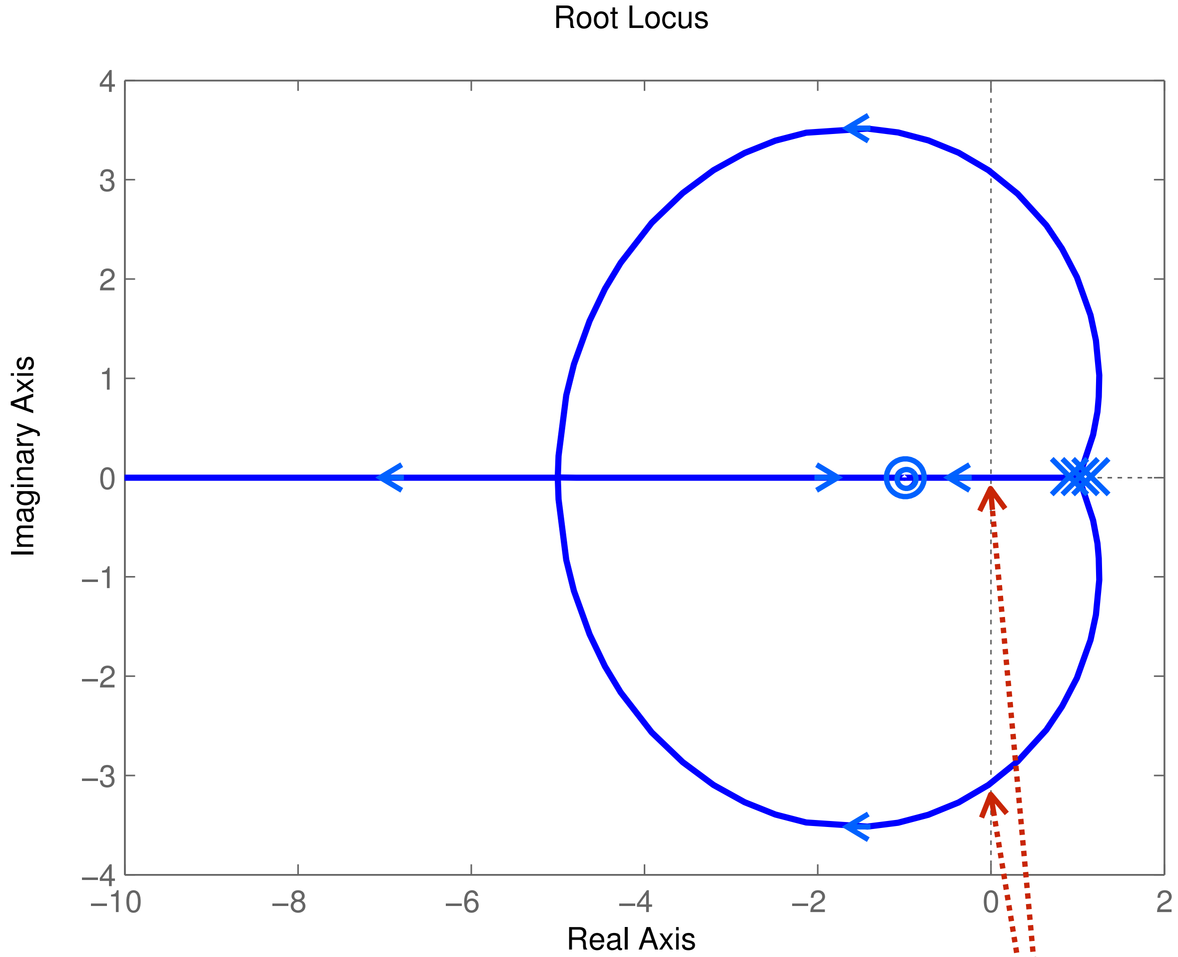
center of asymptotes has to be negative

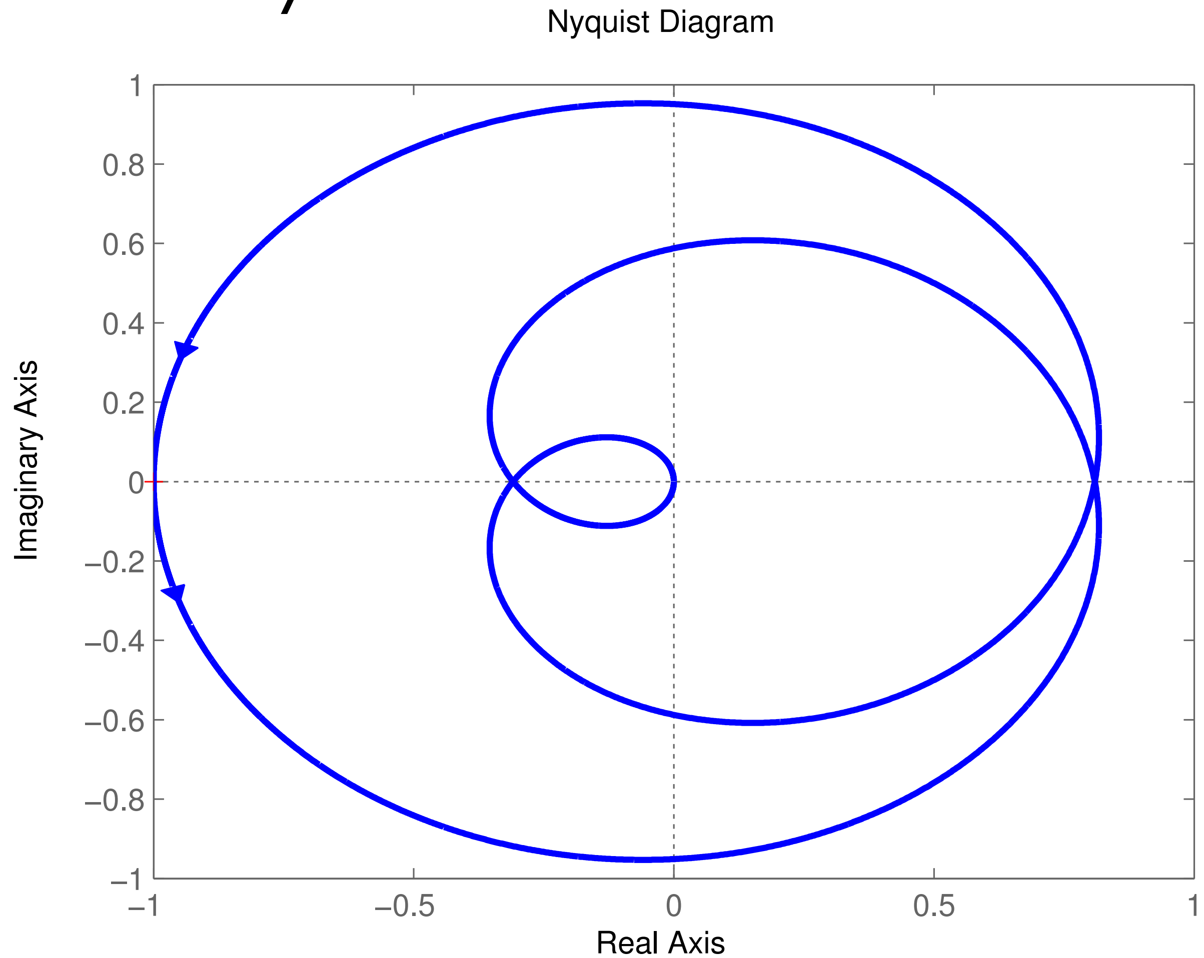
→ if the open-loop system is minimum phase and has (relative degree) n - $m = 2$ with a negative center of asymptotes then in the positive locus for sufficiently high values of k positive the closed-loop system is for sure asymptotically stable example

$$F(s) = \frac{(s+1)^2}{(s-1)^3}$$

$n_{F^+} = 3$ 3 poles with positive real part

A controller $C(s) = K_C$ with sufficiently high values of the gain K_C will certainly stabilize the closed loop system





positive root locus

critical values of K_C

Nyquist plot for $K_C = 1$ passes through point (-1,0) example (cont'd)

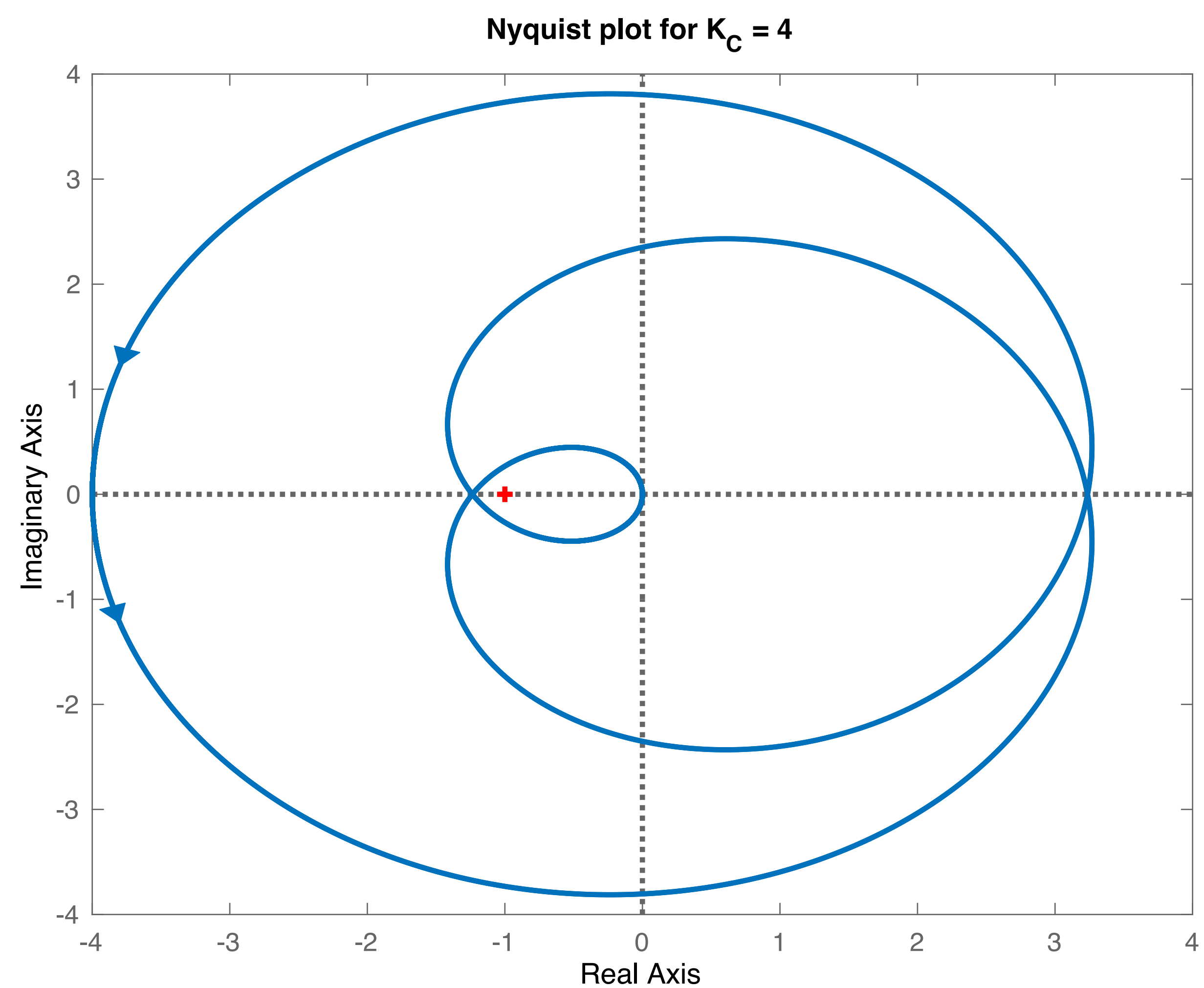
$K_C F(s)$ will give in closed loop the pole polynomial $p(s, K_C)$

$$p(s, K_C) = (s-1)^3 + K_C(s+1)^2 = s^3 + s^2(K_C - 3) + s(2K_C + 3) + K_C - 1$$

the necessary condition leads to $K_C > 1$ while the Routh table

1	$2K_C + 3$
K_C - 3	K_C - 1
$\dfrac{2(K_C^2 - 2K_C - 4)}{K_C - 3}$	
K_C - 1	

gives the necessary & sufficient condition
$K_C > 3.2361$



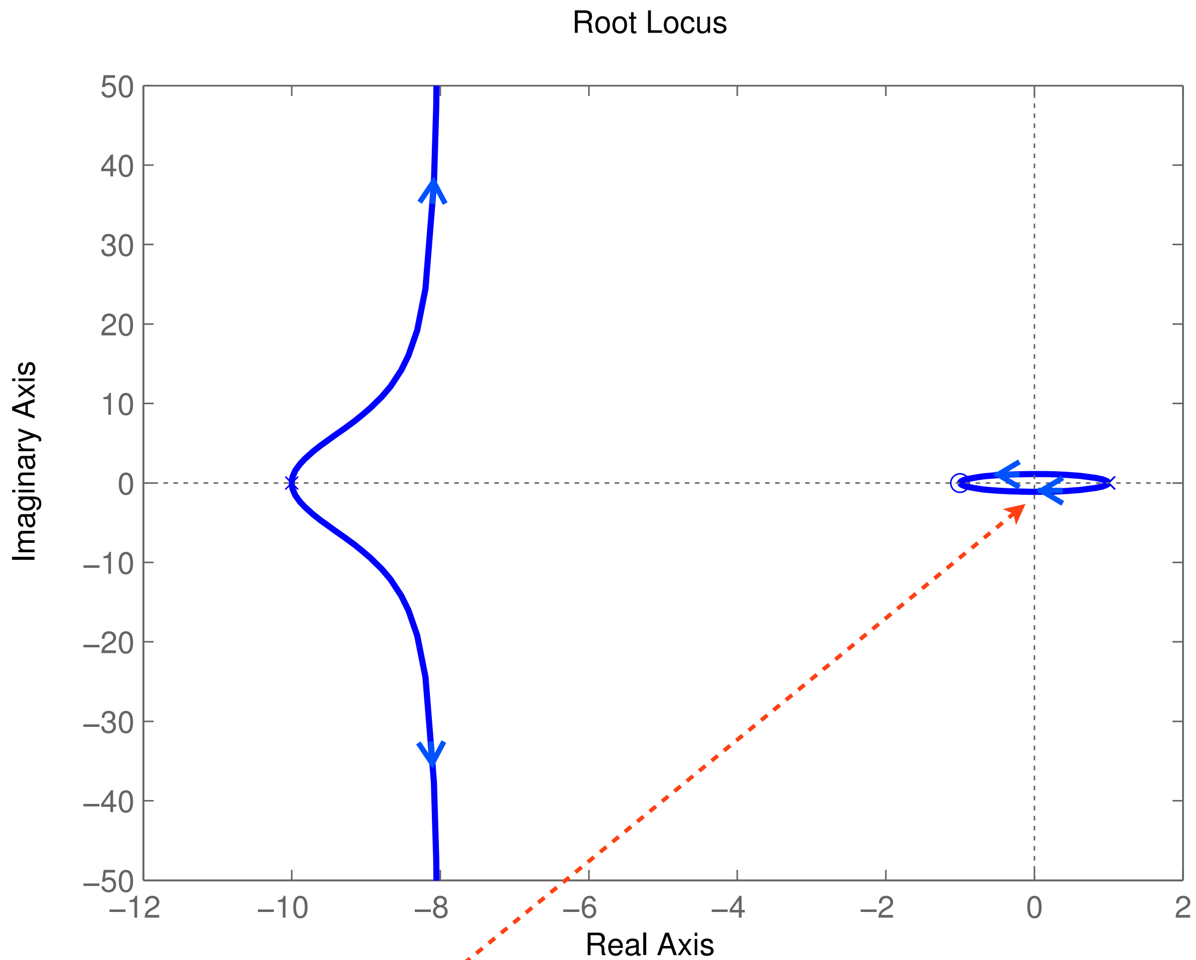
for $K_C > 3.2361$ we have $N_{cc} = n_F^+ = 3$

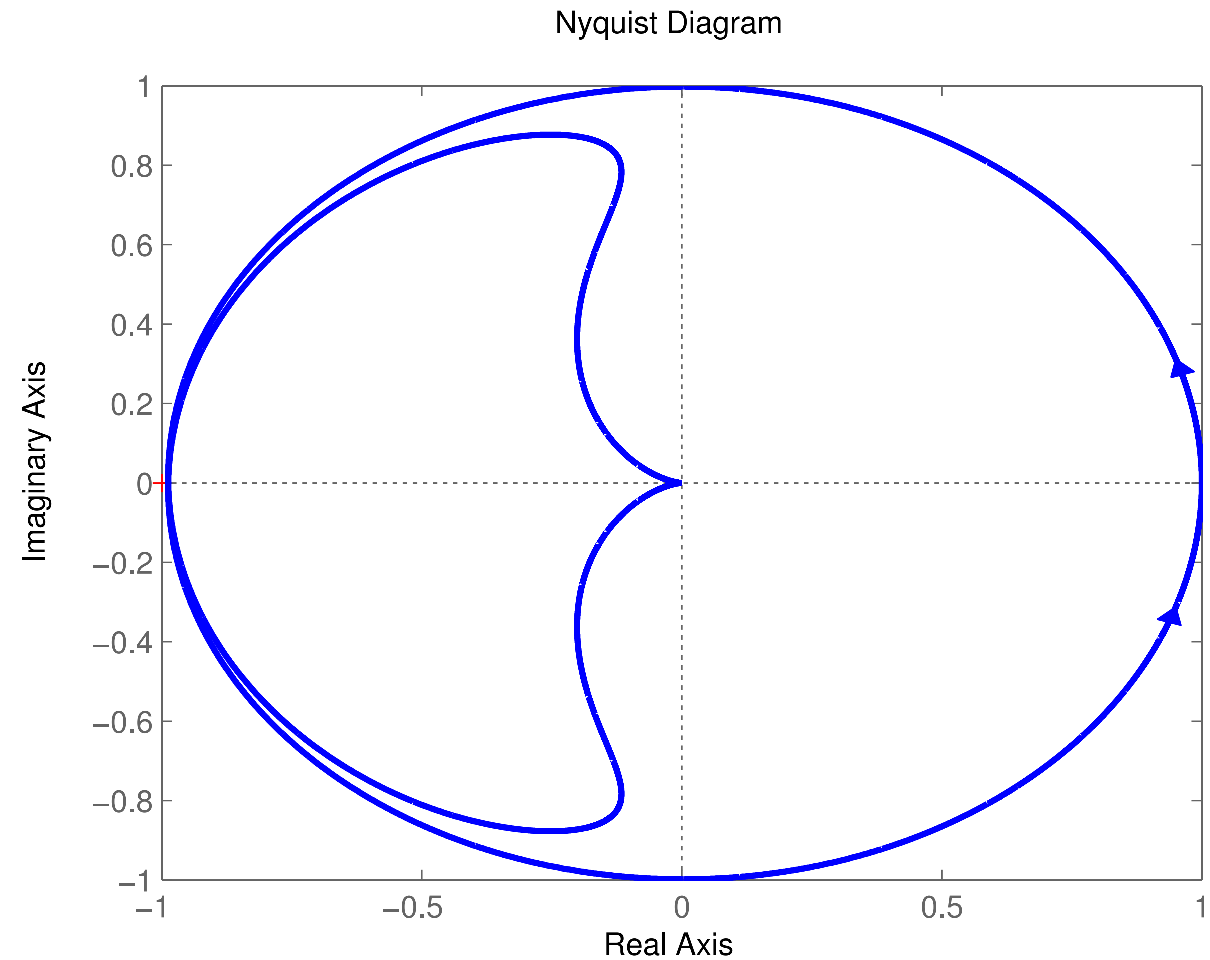
Nyquist stability criterion verified (here $K_C = 4$)

example

$$F(s) = \frac{(s+1)^2}{(s-1)^2(s+10)^2}$$

Nyquist plot of $100\ F(s)$
passes through the point (-1,0)





gain $K_C = 100$ corresponds to the Imaginary axis crossing

for $K_C > 100$ we have $N_{cc} = n_{F^+} = 2$

Nyquist stability criterion verified

what if n - $m = 2$ but the center of asymptotes is non-negative?

Let us assume we have $F(s)$ with n - $m = 2$ and center of asymptotes s_0 (non-negative) and see the effect of adding a zero/pole pair $(z_a,\, p_a)$ both negative

$$F(s)\,\frac{s - z_a}{s - p_a}$$

the new center of asymptotes is given by (with n - $m = 2$)

$$z_a < 0$$

$$p_a < 0$$

$$\underbrace{s_0'}_{\text{new center of asymptotes}} = \frac{\displaystyle\sum_{j=1}^{n} p_j + p_a - \sum_{i=1}^{m} z_i - z_a}{n - m} = \underbrace{s_0}_{\text{old center of asymptotes}} + \underbrace{\frac{p_a - z_a}{n - m}}_{\text{center of asymptotes variation}}$$

- we cannot just add a zero to obtain n - $m = 1$ since the controller would be improper; moreover a positive zero is not allowed to exploit this technique
- since n - m remains 2, we can choose the additional negative pole p_a and zero z_a such that the new center of asymptotes becomes negative
- once we have made the center of asymptotes negative with the addition of a pole and a zero, everything we said before applies

NB for the variation to be negative the pole needs to be to the left of the zero note that

$$\frac{s - z_a}{s - p_a} \quad \text{with } p_a < z_a < 0$$

can be rewritten as $$\frac{s - z_a}{s - p_a} = \frac{z_a - s}{p_a - s} = \frac{z_a}{p_a} \frac{(1 - s/z_a)}{(1 - s/p_a)} \qquad 0 < \frac{z_a}{p_a} < 1$$

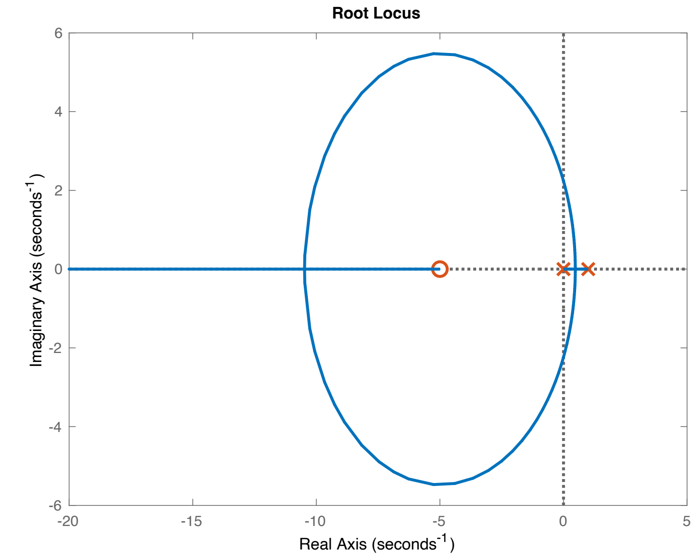
and therefore, being the cut-off frequency of the zero smaller than the one of the pole, the particular pole/zero pair is equivalent to

a positive gain smaller than 1 x a lead compensator

but the final controller will require also the choice of a sufficiently high gain $k^* > k_{crit}$ so the controller will be

$$C(s) = \text{Gain x Lead compensator}$$

case n - $m = 2$: variations



- the controller has already some poles (for example in $s = 0$) deriving from some steady-state specification and the modified plant has n - $m = 2$. Then we can add a negative zero and move from the case n - $m = 2$ to 1 and the corresponding considerations apply.

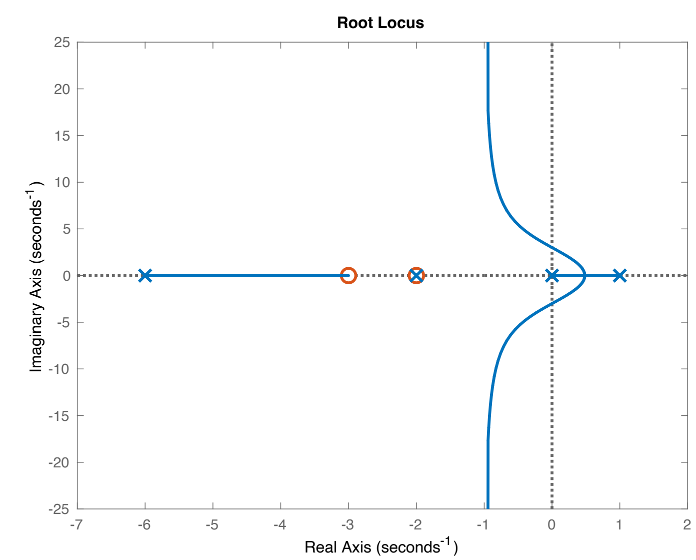
example:

$$P(s) = \frac{1}{s-1} \longrightarrow C(s) = \frac{K}{s} \longrightarrow \hat{P}(s) = \frac{K}{s(s-1)} \longrightarrow C(s) = \frac{K(s+5)}{s}$$

type 1 requirement:

modified plant

final controller: type 1 and stabilizing for $K > 1$



root locus with $p = 6$

- when adding a zero/pole (i.e. maintaining n - $m = 2$), one may choose to place the zero so to cancel a stable pole of the plant. This is possible if there are no further restrictions on the closed-loop eigenvalues (e.g., belonging to some specific region), however note that canceling a negative pole makes the center of asymptote increase (or even become positive).

example: $P(s) = \dfrac{s+3}{s(s-1)(s+2)}$ $\quad C(s) = K\dfrac{s+2}{s+p}$ makes the center of asymptotes negative for $p > 4$

what if $n - m > 2$ and the system is minimum phase?

The idea is:

- turn the $n - m > 2$ case into a $n - m = 2$ one by adding zeros $(s - z_k)$ to the controller
- solve the stabilization problem: if necessary move the center of asymptotes and choose K
- if necessary introduce high frequency dynamics $(1 + \tau_h s)$ to make the controller at least proper (note that we get back to the $n - m > 2$ case but now the gain has been chosen). The following result guarantees that, if properly chosen, these dynamics do not alter the closed-loop stability

Th.

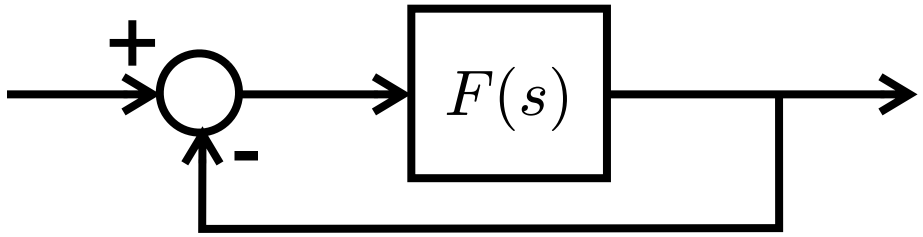
Consider an open-loop system $F(s)$ which results in an asymptotically stable unit feedback closed-loop system. Then there exists a sufficiently small $\tau_h > 0$ such that the closed-loop system having as open-loop

$$\left(\frac{1}{1+\tau_h s}\right) F(s)$$

remains asymptotically stable.

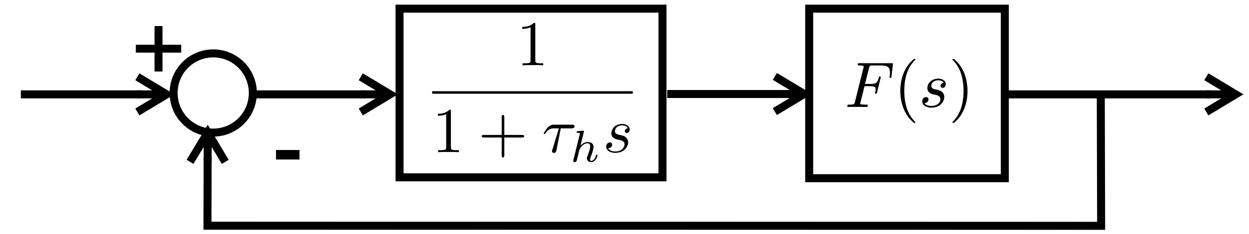
The previous theorem states that:

if the closed-loop system



is asymptotically stable

then the closed-loop system



is also asymptotically stable provided $\tau_h > 0$ is sufficiently small

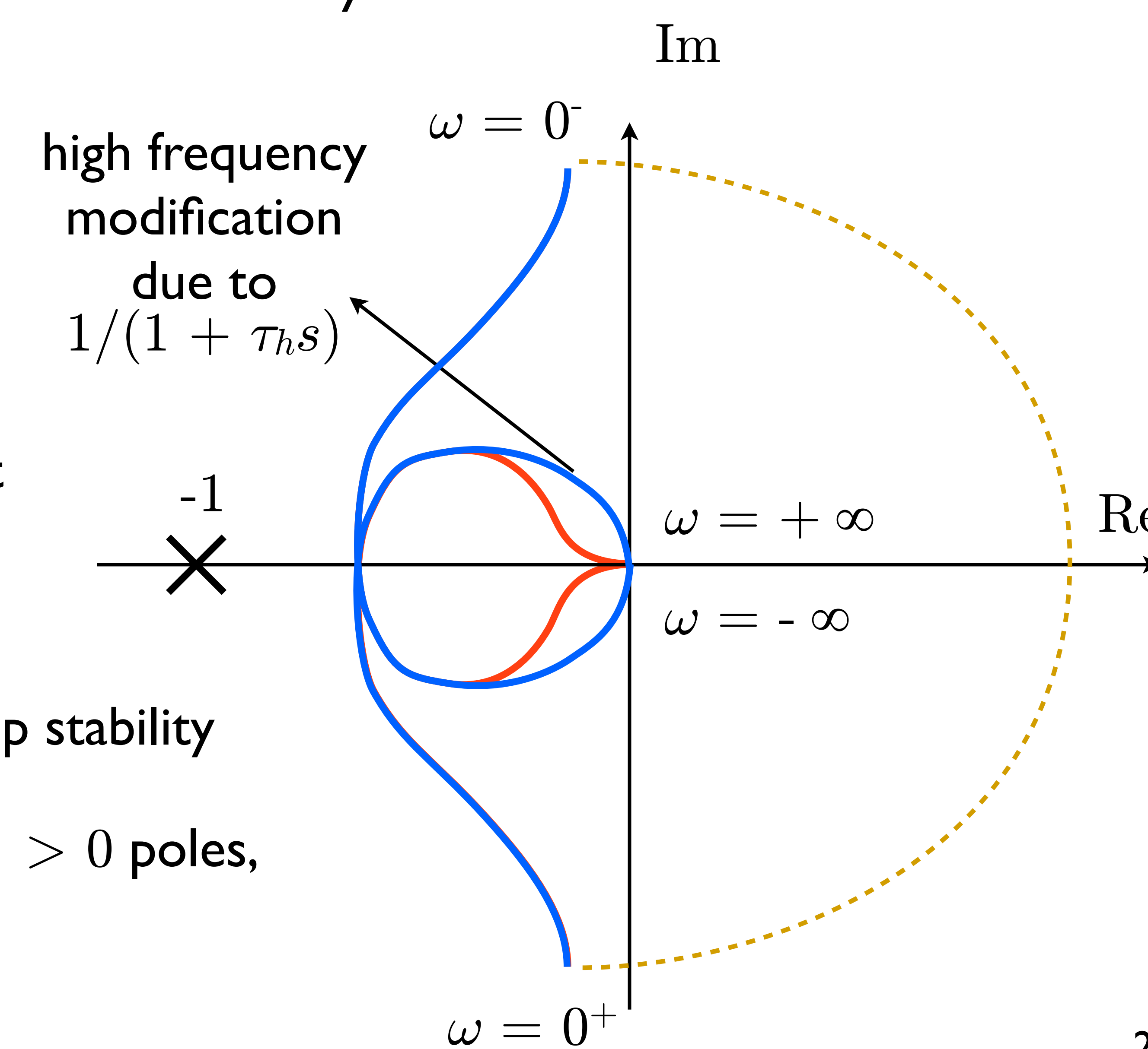
note that $1/(1+\tau_h s)$

- does not change the gain of the open-loop
- represents some high-frequency dynamics

informal proof: $\dfrac{1}{1+\tau_h s}$ alters the Bode plots only at high frequency → alters the Nyquist plot only at high frequency

↓

no effect on closed-loop stability



Nyquist plot example here for F with no $\mathrm{Re}[p_i] > 0$ poles, but the result is general.

Case n - $m = 3$ & minimum-phase system: possible algorithm

1) add a negative zero $(s$ - $z_k)$ so to obtain n - $m = 2$.

a) if $s_0 < 0$ then choose $k^* > k_{crit}$ to guarantee asymptotic stability of the closed-loop system. Go to point 2)

b) if $s_0 > 0$ then choose a pole/zero pair $p_a < z_a < 0$ to make the new center of the asymptotes s_0' negative. Go to case a)

2) if the resulting controller is improper, add a high-frequency dynamics term $(1 + \tau_h s)$ where the time constant τ_h can be chosen applying the Routh criterion in order to guarantee that the overall closed-loop system is asymptotically stable

Note that step 2) may not be necessary since we may apply this result once the static specifications have been met, i.e., we may just need to stabilize the extended plant $\hat{P}(s)$ and therefore the controller may already have an excess of poles w.r.t. the number of zeros so that adding the extra zero $(s$ - $z_k)$ would not make the controller improper

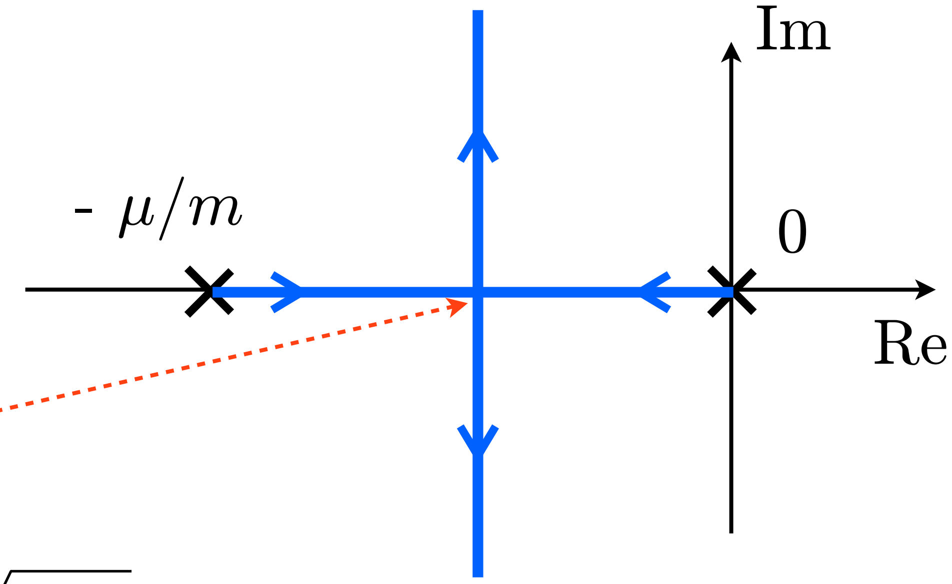
other use of the RL

The root locus can be applied to any polynomial with a single parameter k entering linearly

Example Mass-Spring-Damper: we want to explore the influence of some parameters on the system's dynamics represented by the following characteristic polynomial $ms^2 + \mu s + k$

- varying the spring stiffness $k \in [0, +\infty)$ $p(s,k) = (ms + \mu)s + k = D(s) + kN(s)$

$k = 0$ no spring, poles in $(0, \text{-}\mu/m)$

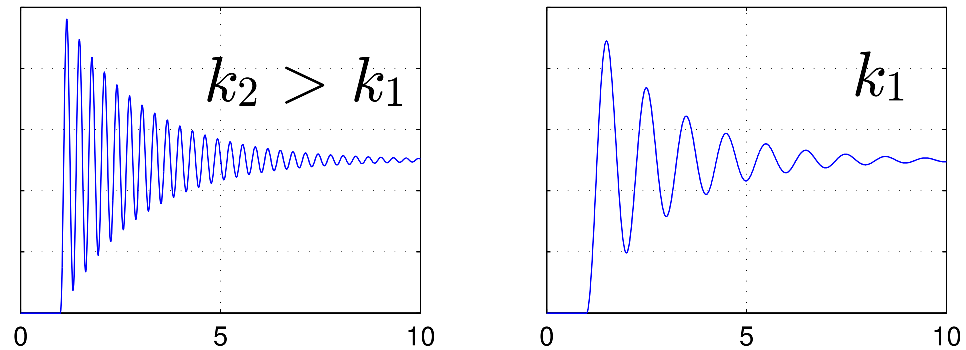


singular point

$$\frac{\partial p(s,k)}{\partial s} = 2ms + \mu = 0 \longrightarrow s^* = \text{-}\mu/2m$$

at $k^* = -(ms^* + \mu)s^* = \dfrac{\mu^2}{4m}$ i.e. $\mu = 2\sqrt{k^* m}$

as k increases (spring more and more stiff) poles are complex with constant real part while imaginary part becomes larger



step response - normalized plots
(the steady-state value changes with k)

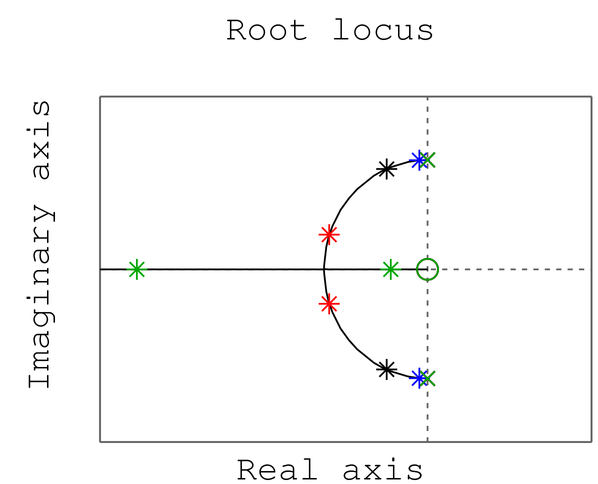
- varying the damping $\mu \in [0, +\infty)$ $p(s, \mu) = ms^2 + k + \mu s = D(s) + \mu N(s)$

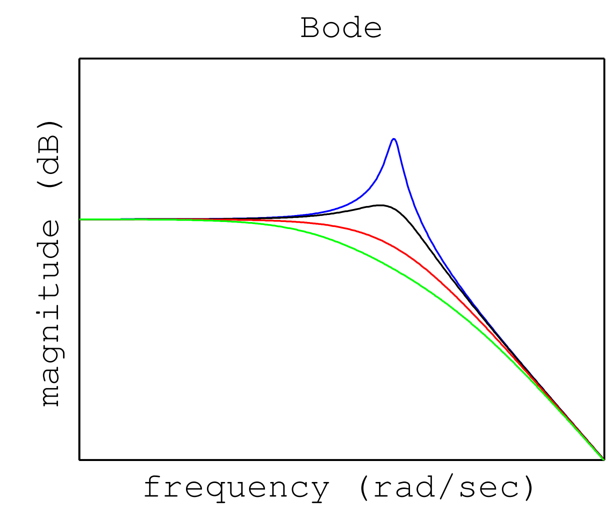
the MSD poles move, as μ increases,
as the positive root locus of → $\mu \dfrac{N(s)}{D(s)} = \mu \dfrac{s}{ms^2 + k}$

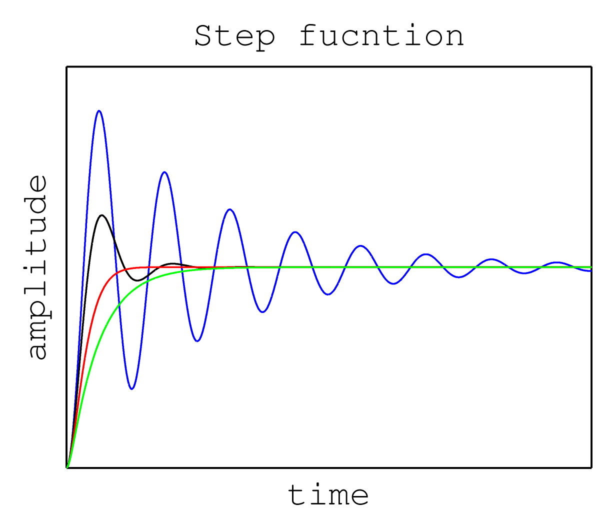
$\mu = 0$ no damper, pure imaginary poles

same singular point as before

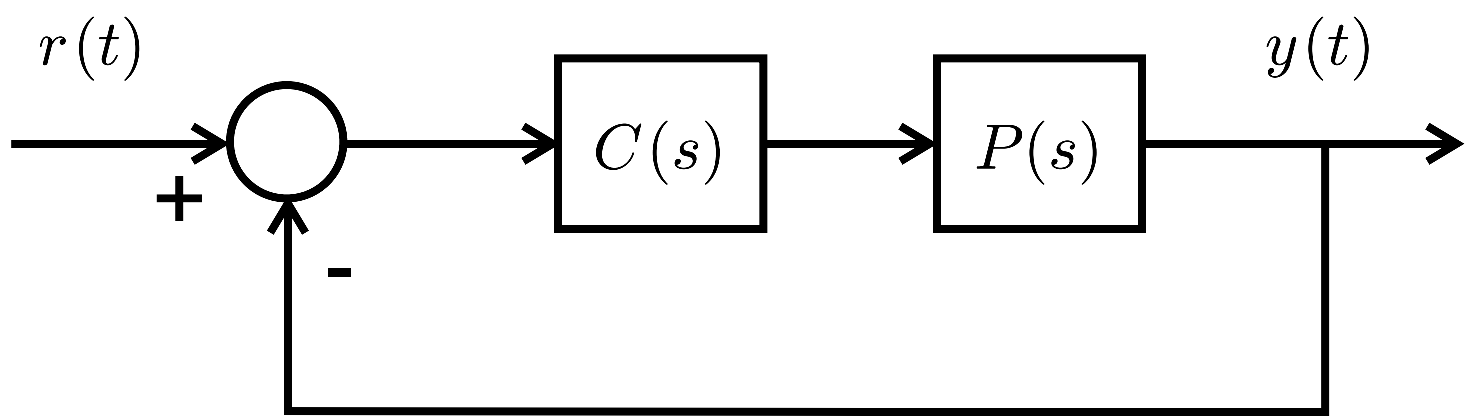
note how, as μ increases and the poles become real, how a dominant (slow) dynamics arises







Pole assignment



plant

$$P(s) = \frac{N_P(s)}{D_P(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

strictly proper $m < n$

monic (coefficient $= 1$)

controller as a function of unknowns parameters to be determined

$$C(s) = \frac{N_C(s)}{D_C(s)} = \frac{d_r s^r + d_{r-1} s^{r-1} + \cdots + d_1 s + d_0}{s^r + c_{r-1} s^{r-1} + \cdots + c_1 s + c_0}$$

proper

monic

$$\{\underbrace{d_r, d_{r-1}, \ldots, d_1, d_0}_{r+1}, \underbrace{c_{r-1}, \ldots, c_1, c_0}_{r}\}$$

$2r+1$ **unknown coefficients**

closed-loop characteristic polynomial

$$d_{CL}(s) = D_P(s)D_C(s) + N_P(s)N_C(s) \longrightarrow \text{degree } n+r$$

degree $\underbrace{n \quad r}_{\text{also monic}} \quad m \quad r$

we want to assign all the closed-loop poles to be $\{p_1^*, p_2^*, \ldots, p_{n+r}^*\}$ **desired closed-loop poles**

which can be seen as the solutions of an $n+r$ order polynomial

$$d_{CL}^*(s) = (s - p_1^*)(s - p_2^*)\ldots(s - p_{n+r}^*)$$

The problem can be stated as:
we need to determine the $2r+1$ unknowns c_i and d_j ($i = 1, ..., r+1$; $j = 1, ..., r$) such that

$$D_P(s)D_C(s) + N_P(s)N_C(s) = d_{CL}^*(s)$$

Diophantine equation

Result

if $N_P(s)$ and $D_P(s)$ are coprime
and $r = n - 1$ → we can always solve and have a **unique** solution

↕

we can arbitrarily assign the $2n - 1$ closed-loop poles

Remarks

closed-loop system zeros:

- the zeros depend upon where we consider the input entering in the control system and which output we decide to monitor. For example in a unit feedback system, the output disturbance to controlled output transfer function $S(s)$ and the reference to controlled output one $T(s)$ will have the same poles (if there are no hidden dynamics in the open-loop system) but different zeros
- the closed-loop zeros of $T(s)$ coincide with the open-loop ones when the open-loop system has no hidden dynamics

$$T(s) = \frac{N_T(s)}{D_T(s)} = \frac{F(s)}{1 + F(s)} = \frac{N_F(s)}{D_F(s) + N_F(s)}$$

$$N_F(s) = N_C(s) N_P(s)$$

- since the choice of the controller $C(s)$ is uniquely determined by the previous algorithm, the **controller zeros** are a consequence of the pole assignment technique and so are the closed-loop zeros

- note that if we choose as desired closed-loop poles some open loop zeros (necessarily of the plant since the controller has not been chosen yet) then at closed loop we have a cancellation (being the closed loop zeros equal to the open loop ones). Moreover a closed loop cancellation (for finite values of the open loop gain) can be originated only by an open loop cancellation, which being $N_P(s)$ and $D_P(s)$ coprime, is generated by the series interconnection plant/controller. Therefore if we choose as desired closed loop poles some open loop zeros, these will necessarily be also poles of the controller

- from the above comment it is clear that we are **not** going to choose a desired closed-loop pole coincident with a non-minimum phase zero of the open-loop.

example

plant $P(s) = \dfrac{(s-1)}{s(s-2)}$ $\quad n = 2 \quad \longrightarrow \quad C(s) = \dfrac{as+b}{s+c}$

$d^*_{CL}(s) = (s+1)^3 = s^3 + 3s^2 + 3s + 1 \quad \longleftarrow$ desired closed-loop poles: 3 in -1

$$d_{CL}(s) = s(s-2)(s+c) + (s-1)(as+b) = s^3 + (c-2+a)s^2 + (b-a-2c)s - b$$

equating

$$\begin{cases} a+c-2 &= 3 \\ -a+b-2c &= 3 \\ -b &= 1 \end{cases} \longrightarrow \begin{cases} a &= 14 \\ b &= -1 \\ c &= -9 \end{cases}$$

$$C(s) = \frac{14s-1}{s-9} = 14\frac{(s-\frac{1}{14})}{(s-9)}$$

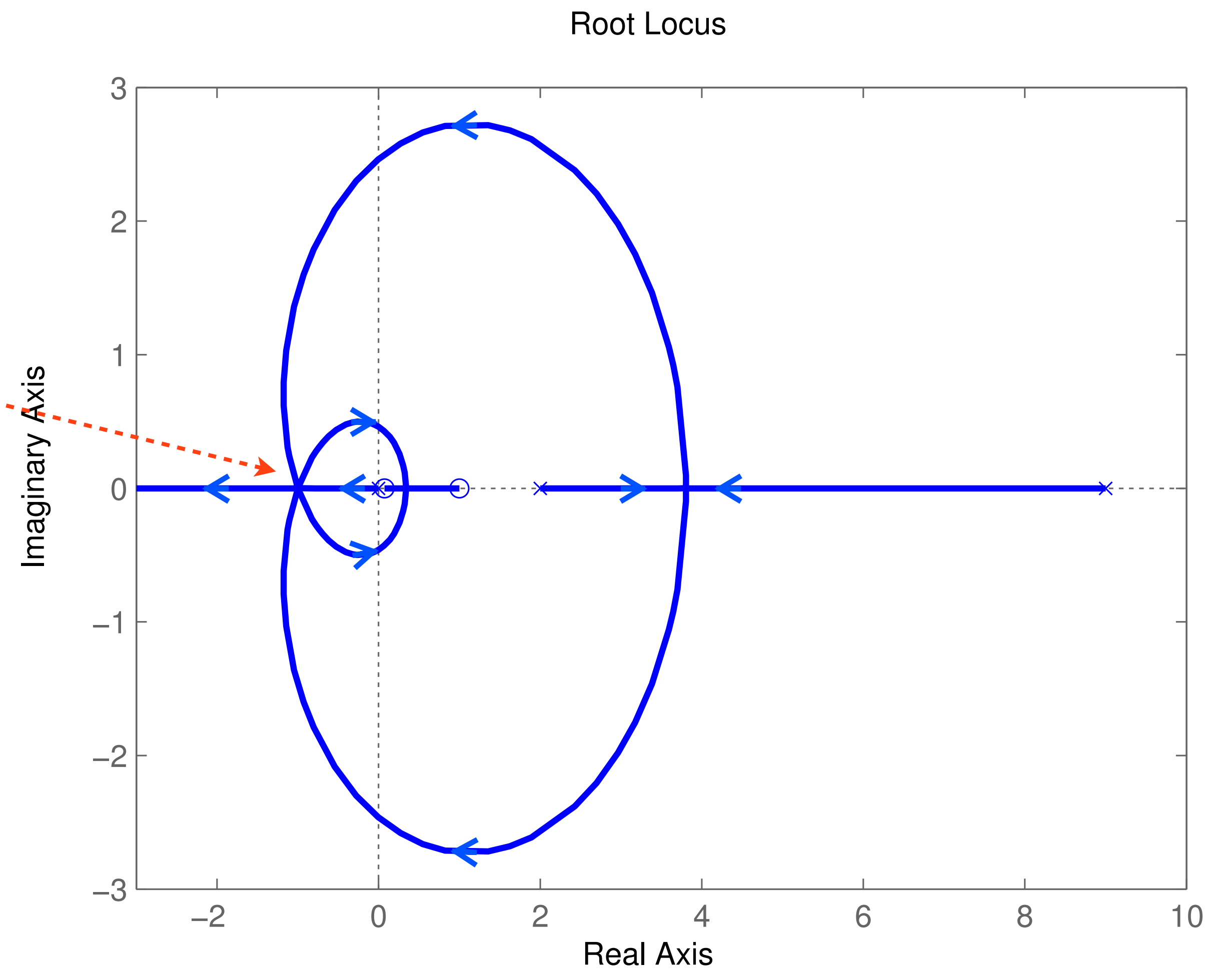
closed-loop complementary sensitivity $\quad T(s) = \dfrac{C(s)P(s)}{1+C(s)P(s)} = \dfrac{14(s-1)(s-\frac{1}{14})}{(s+1)^3}$

note that

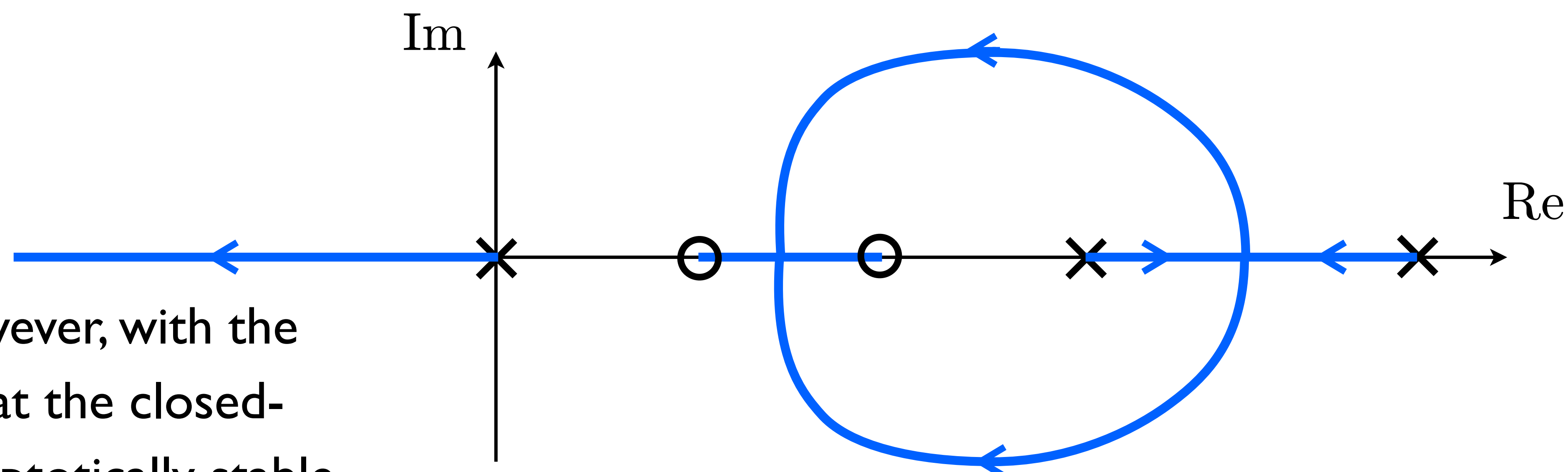
- the plant has a real positive pole so the open-loop loop shaping technique cannot be used
- the plant has a real positive zero (so it is non-minimum phase) and therefore the high-gain principle seen in the root locus cannot be applied
- the resulting controller, in this example, is unstable and non-minimum phase: we have no control over the final structure of the fixed dimensional ($r = n$ - 1) controller

root locus of $F(s) = k\dfrac{(s-\frac{1}{14})}{(s-9)}\,\dfrac{(s-1)}{s(s-2)}$

for $k = 14$ we obtain the three closed-loop poles at the desired location

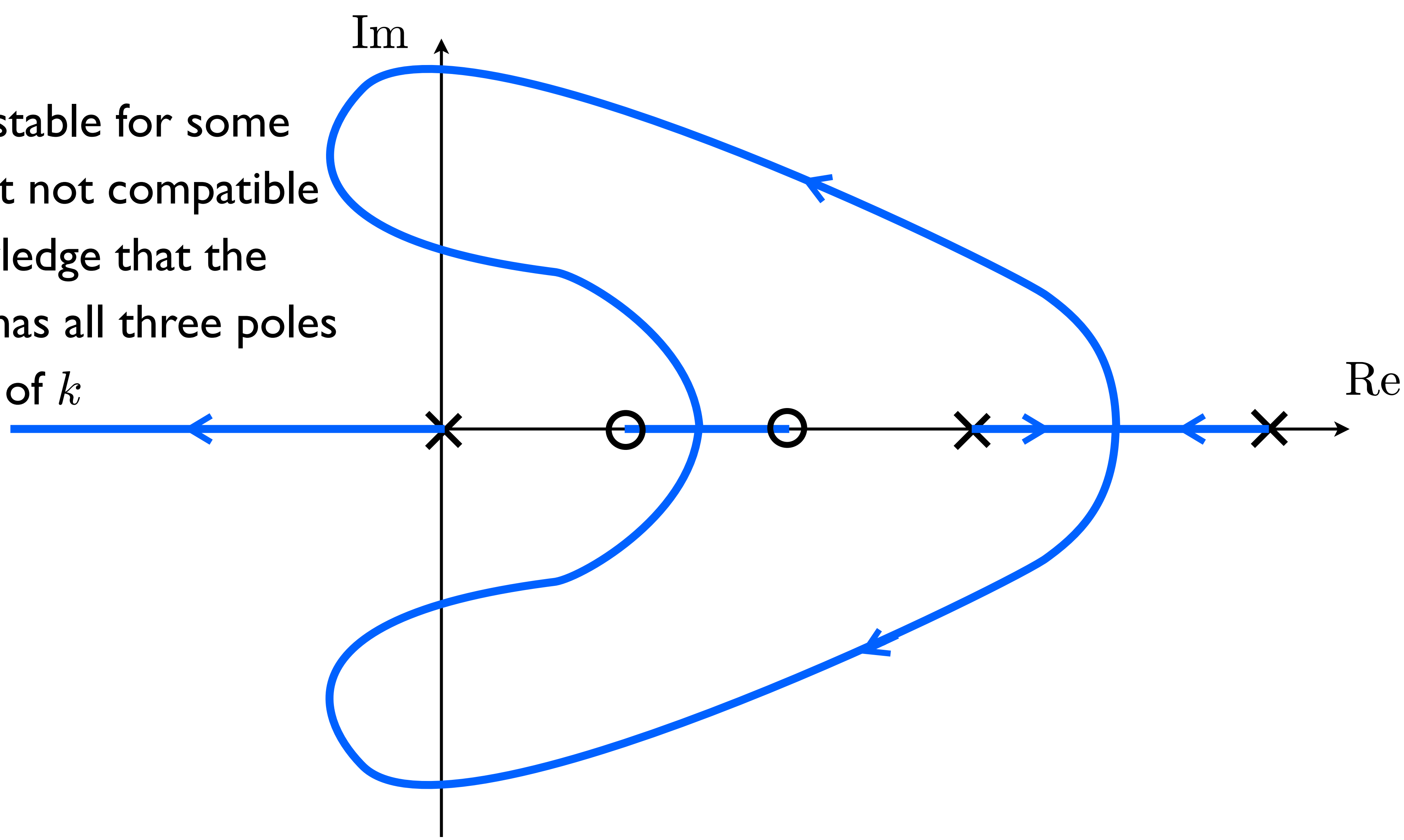


note that using the basic tracing rules we can also have the following compatible positive root locus



not compatible, however, with the extra knowledge that the closed-loop system is asymptotically stable

closed-loop system stable for some values of the gain but not compatible with the extra knowledge that the closed loop system has all three poles in -1 for some value of k



Vocabulary

English	Italiano
root locus	luogo delle radici
singular point (breakaway/break-in)	punto singolare
locus branch	ramo del luogo
minimum phase	a fase minima
center of asymptotes	centro degli asintoti