Control Systems

Bode diagrams

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Outline

- Bode's canonical form for the frequency response
- Magnitude and phase in the complex plane
- The decibels (dB)
- Logarithmic scale for the abscissa
- Bode's plots for the different contributions

Frequency response

The **steady state response** of an asymptotically stable system $P(s)$ to a sinusoidal input $u(t) = \sin \bar{\omega} t$ is given by

$$y_{\rm ss}(t) = |P(j\bar{\omega})| \sin(\bar{\omega} t + \angle P(j\bar{\omega}))$$

amplification/attenuation depends on the system at the frequency of the input

same frequency as input

$P(j\omega)$ is the restriction of the transfer function to the imaginary axis $P(j\omega)|_{s\,=\,j\omega}$

Frequency response $P(j\omega)$

$$P(j\omega) \begin{cases} |P(j\omega)| & \textbf{gain curve} \text{ (or magnitude)} \\ \angle P(j\omega) & \textbf{phase curve} \end{cases}$$

for $\omega \in \mathbb{R}^+ = [0, +\infty)$

a complex number p can always be represented in terms of its magnitude and phase $(|p|, \angle p)$

$$p = |p| e^{j \angle p}$$

Bode canonical form

pole/zero representation of the transfer function

$$F(s) = K' \frac{1}{s^m} \frac{\prod_k (s - z_k) \prod_\ell (s^2 + 2\zeta_\ell \omega_{n\ell} s + \omega_{n\ell}^2)}{\prod_i (s - p_i) \prod_z (s^2 + 2\zeta_z \omega_{nz} s + \omega_{nz}^2)}$$

with m such that

- $m = 0$ if no pole or zero in $s = 0$
- $m < 0$ if m zeros in $s = 0$
- $m > 0$ if m poles in $s = 0$

remarks

- numerator and denominator are by hypothesis coprime
- denominator is monic
- K' is not the system gain

Bode canonical form

- the terms $(s - z_k)$ and $(s - p_i)$ are relative to
  - real zeros (in $s = z_k$)
  - real poles (in $s = p_i$)

- the terms $(s^2 + 2\zeta_\ell \omega_{n\ell} s + \omega_{n\ell}^2)$ and $(s^2 + 2\zeta_z \omega_{nz} s + \omega_{nz}^2)$ are relative to
  - complex conjugate zeros (in $s = \alpha_\ell \pm j\beta_\ell$)
  - complex conjugate poles (in $s = \alpha_z \pm j\beta_z$)

with

- natural frequency $\omega_{n*} = \sqrt{\alpha_*^2 + \beta_*^2}$
- damping coefficient $\zeta_* = -\alpha_*/\omega_{n*} = -\alpha_*/\sqrt{\alpha_*^2 + \beta_*^2}$

Bode canonical form

factoring out the constant terms

$$s - z_k = -z_k(1 - 1/z_k s) = -z_k(1 + \tau_k s) \quad \text{with} \quad \tau_k = -1/z_k$$
$$s - p_i = -p_i(1 - 1/p_i s) = -p_i(1 + \tau_i s) \quad \text{with} \quad \tau_i = -1/p_i$$

with τ_i and τ_k being time constants

$$F(s) = K' \frac{1}{s^m} \frac{\prod_k(-z_k)\prod_\ell(\omega_{n\ell}^2)\prod_k(1+\tau_k s)\prod_\ell(1+2\zeta_\ell/\omega_{n\ell}s + s^2/\omega_{n\ell}^2)}{\prod_i(-p_i)\prod_z(\omega_{nz}^2)\prod_i(1+\tau_i s)\prod_z(1+2\zeta_z/\omega_{nz}s + s^2/\omega_{nz}^2)}$$

defining $\quad K = K' \frac{\prod_k(-z_k)\prod_\ell(\omega_{n\ell}^2)}{\prod_i(-p_i)\prod_z(\omega_{nz}^2)}$

$$K = [s^m F(s)]_{s=0} \quad \text{for any} \quad m \gtreqless 0$$

how to compute K

$$F(s) = K \frac{1}{s^m} \frac{\prod_k(1+\tau_k s)\prod_\ell(1+2\zeta_\ell/\omega_{n\ell}s + s^2/\omega_{n\ell}^2)}{\prod_i(1+\tau_i s)\prod_z(1+2\zeta_z/\omega_{nz}s + s^2/\omega_{nz}^2)}$$

Bode canonical form

Gains

$$F(s) = K \frac{1}{s^m} \boxed{\frac{\prod_k (1+\tau_k s) \prod_\ell (1 + 2\zeta_\ell/\omega_{n\ell} s + s^2/\omega_{n\ell}^2)}{\prod_i (1+\tau_i s) \prod_z (1 + 2\zeta_z/\omega_{nz} s + s^2/\omega_{nz}^2)}} = 1 \text{ for } s = 0$$

generalized gain $\quad K = \left[s^m F(s)\right]_{s=0} \quad \text{for any} \quad m \gtreqless 0$

Note that

- for a system with no poles in $s = 0$ (i.e. m negative or zero) we have defined as **dc-gain** (or static gain)

$$K_s = F(s)\Big|_{s=0} = F(0)$$

  if $m < 0$ (zeros in $s = 0$) we have $F(0) = 0$

- static and generalized gain coincide only when $m = 0$

$$K = K_s \quad \Leftrightarrow \quad m = 0$$

- for an asymptotically stable system, the step response tends to the static gain $K_s = F(0)$

Bode canonical form

Examples

- $$F(s) = \frac{s-1}{2s^2+6s+4} = \frac{s-1}{2(s+1)(s+2)} = -\frac{1}{4}\frac{1-s}{(1+s)(1+s/2)}$$

  $$K = -\frac{1}{4} = K_s$$

- $$F(s) = \frac{s(s-1)}{2(s+1)^2(s+2)} = -\frac{1}{4}\frac{s(1-s)}{(1+s)^2(1+s/2)}$$

  $$K = -\frac{1}{4} \qquad K_s = 0$$

- $$F(s) = \frac{s-1}{2s(s+1)(s+2)} = -\frac{1}{4}\frac{1-s}{s(1+s)(1+s/2)}$$

  $$K = -\frac{1}{4} \qquad \nexists \quad K_s$$

Bode canonical form

frequency response

$$F(j\omega) = K \frac{1}{(j\omega)^m} \frac{\prod_k (1 + j\omega\tau_k) \prod_\ell (1 + 2\zeta_\ell j\omega/\omega_{n\ell} + (j\omega)^2/\omega_{n\ell}^2)}{\prod_i (1 + j\omega\tau_i) \prod_z (1 + 2\zeta_z j\omega/\omega_{nz} + (j\omega)^2/\omega_{nz}^2)}$$

has 4 **elementary factors**

1. constant K (generalized gain)
2. monomial $j\omega$ (zero or pole in $s = 0$)
3. binomial $1 + j\omega\tau$ (non-zero real zero or pole)
4. trinomial $1 + 2\zeta(j\omega)/\omega_n + (j\omega)^2/{\omega_n}^2$ (complex conjugate pairs of zeros or poles)

so first check which kind of root you have and then factor it out

Bode diagrams

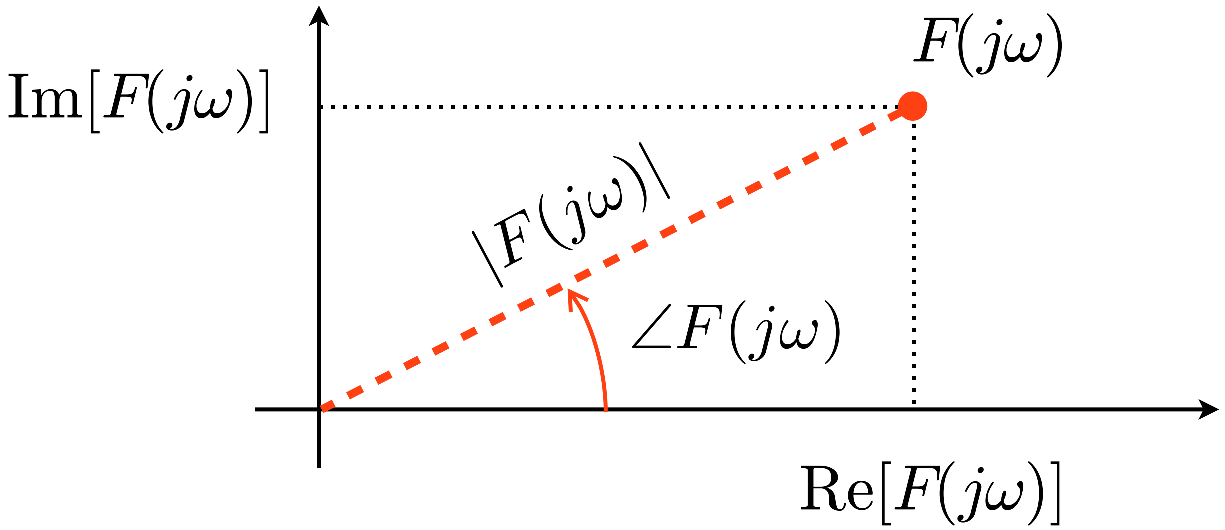
for any real value of the angular frequency ω the frequency response $F(j\omega)$ is a complex number

$|F(j\omega)|$ magnitude of the frequency response as a function of the angular frequency ω

$\angle F(j\omega)$ angle or phase of the frequency response as a function of the angular frequency ω

$$F(j\omega) = \mathrm{Re}[F(j\omega)] + j\mathrm{Im}[F(j\omega)]$$

$$F(j\omega) = |F(j\omega)|e^{j\angle F(j\omega)}$$



$$|F(j\omega)| = \sqrt{\mathrm{Re}[F(j\omega)]^2 + \mathrm{Im}[F(j\omega)]^2}$$

$$\angle F(j\omega) = \mathrm{atan2}(\mathrm{Im}[F(j\omega)], \mathrm{Re}[F(j\omega)])$$

Phase

$$\mathrm{Phase}[F.G] = \mathrm{Phase}[F] + \mathrm{Phase}[G]$$

the phase of a product is the sum of the phases

and therefore

$$\mathrm{Phase}\left[\frac{F}{G}\right] = \mathrm{Phase}[F] - \mathrm{Phase}[G]$$

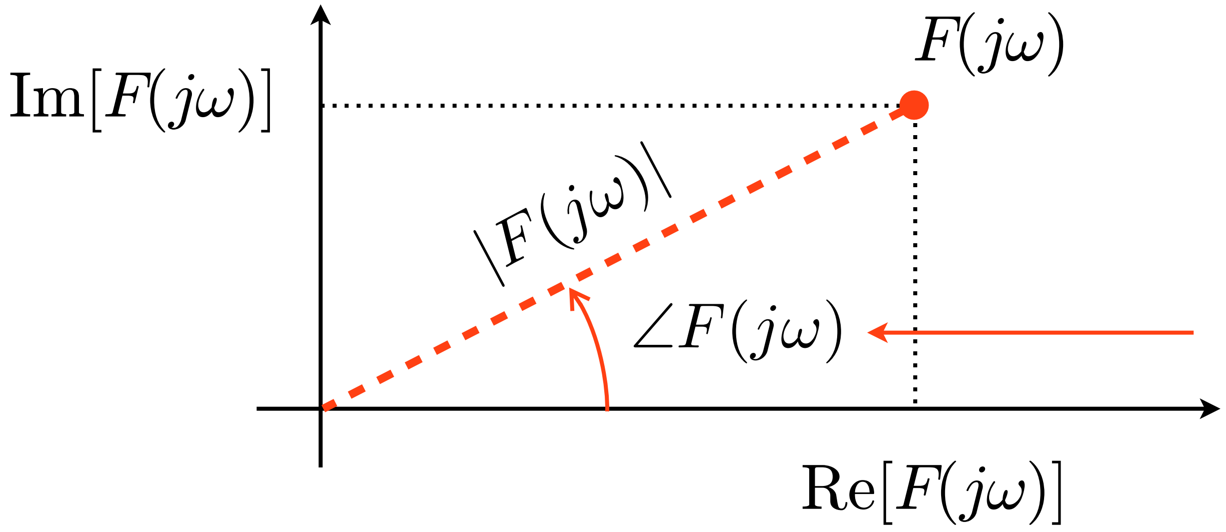
the phase of a ratio is the difference of the phases

since

$$\mathrm{Phase}\left[\tfrac{1}{G}\right] = -\mathrm{Phase}[G]$$

very useful since we can find the contribution to the phase of each term and then just do an algebraic sum

Phase

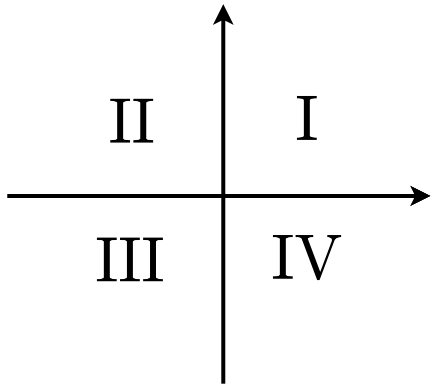


principal argument takes on values in (- π, π] and is implemented by the function with two arguments atan2

for

$P = \alpha + j\beta$

$$\mathrm{atan2}(\beta, \alpha) = \begin{cases} \arctan\left(\frac{\beta}{\alpha}\right) & \text{if} \quad \alpha > 0 & \text{(I \& IV quadrant)} \\ \arctan\left(\frac{\beta}{\alpha}\right) + \pi & \text{if} \quad \beta \geq 0 \ \text{ and } \ \alpha < 0 & \text{(II quadrant)} \\ \arctan\left(\frac{\beta}{\alpha}\right) - \pi & \text{if} \quad \beta < 0 \ \text{ and } \ \alpha < 0 & \text{(III quadrant)} \\ \frac{\pi}{2}\,\mathrm{sign}(\beta) & \text{if} \quad \alpha = 0 \ \text{ and } \ \beta \neq 0 & \\ \text{undefined} & \text{if} \quad \alpha = 0 \ \text{ and } \ \beta = 0 & \end{cases}$$



Magnitude

in order to have the same useful property we need to go through some logarithmic function

$$|F(j\omega)|_{\mathrm{dB}} = 20 \log_{10} |F(j\omega)|$$

decibels (dB)

same nice properties

as phase

$$|F.G|_{\mathrm{dB}} = |F|_{\mathrm{dB}} + |G|_{\mathrm{dB}}$$

$$\left|\frac{1}{F}\right|_{\mathrm{dB}} = -\,|F|_{\mathrm{dB}}$$

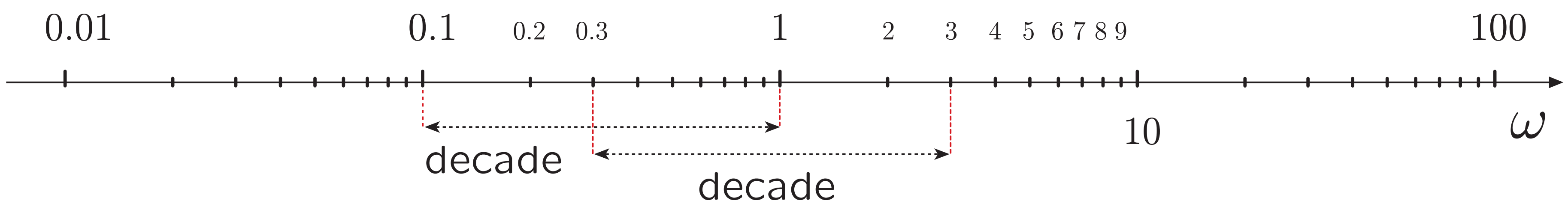
$$|1|_{\mathrm{dB}} = 0\,\mathrm{dB}$$

$$|0.1|_{\mathrm{dB}} = -20\,\mathrm{dB} \qquad |\sqrt{2}|_{\mathrm{dB}} \approx 3\,\mathrm{dB} \qquad |F|_{\mathrm{dB}} \nearrow +\infty \quad \text{if} \quad |F| \nearrow \infty$$

$$|10|_{\mathrm{dB}} = 20\,\mathrm{dB} \qquad |100|_{\mathrm{dB}} = 40\,\mathrm{dB} \qquad |F|_{\mathrm{dB}} \searrow -\infty \quad \text{if} \quad |F| \searrow 0$$

Logarithmic scale

we use a logarithmic (log_{10}) scale for the abscissa (angular frequency ω)

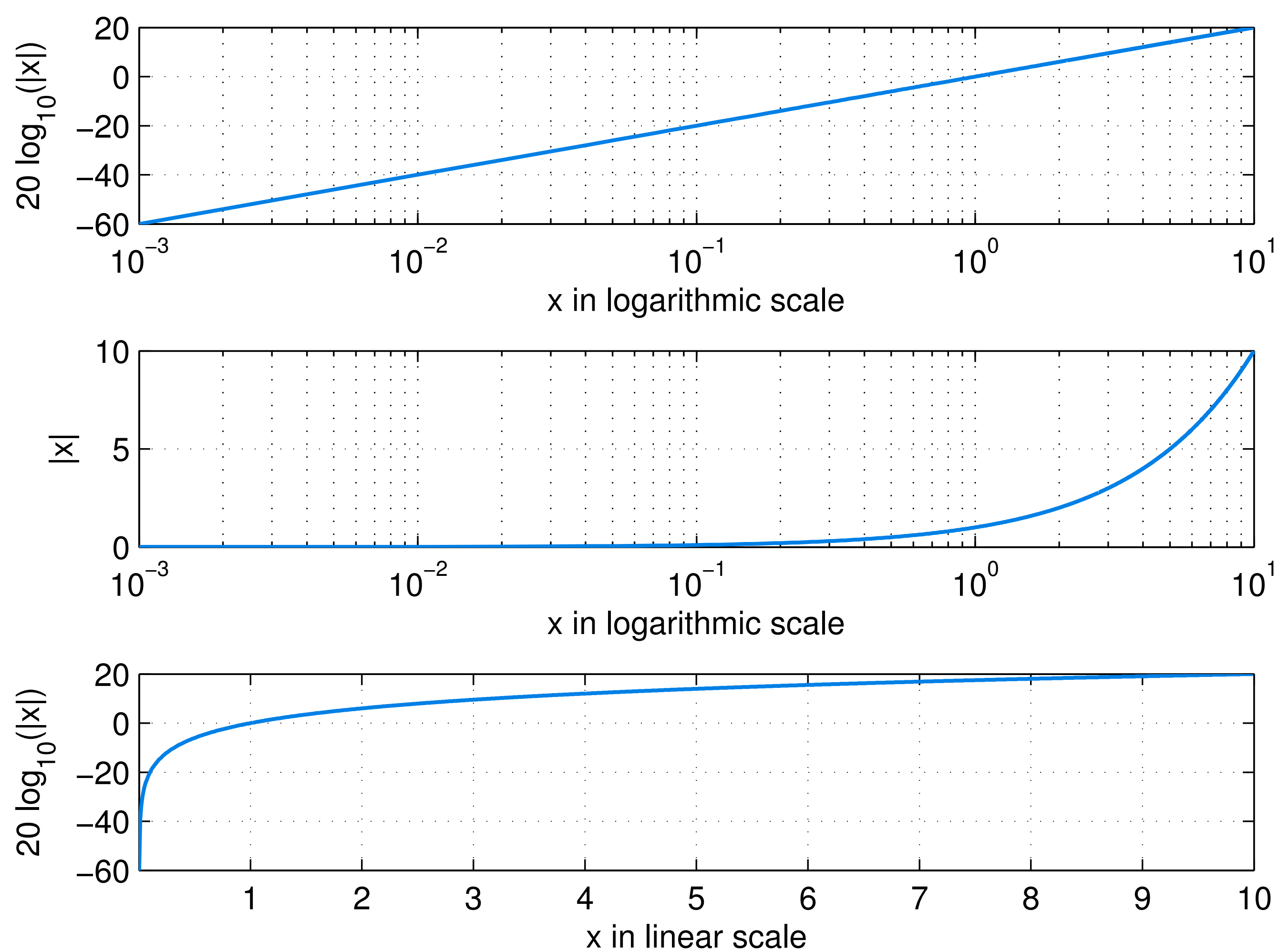


a **decade** corresponds to multiplication by 10

$\log_{10}(\omega)$ becomes a straight line if ω is in a logarithmic scale

↓

very useful when we add different contributions



Logarithmic scale

advantages

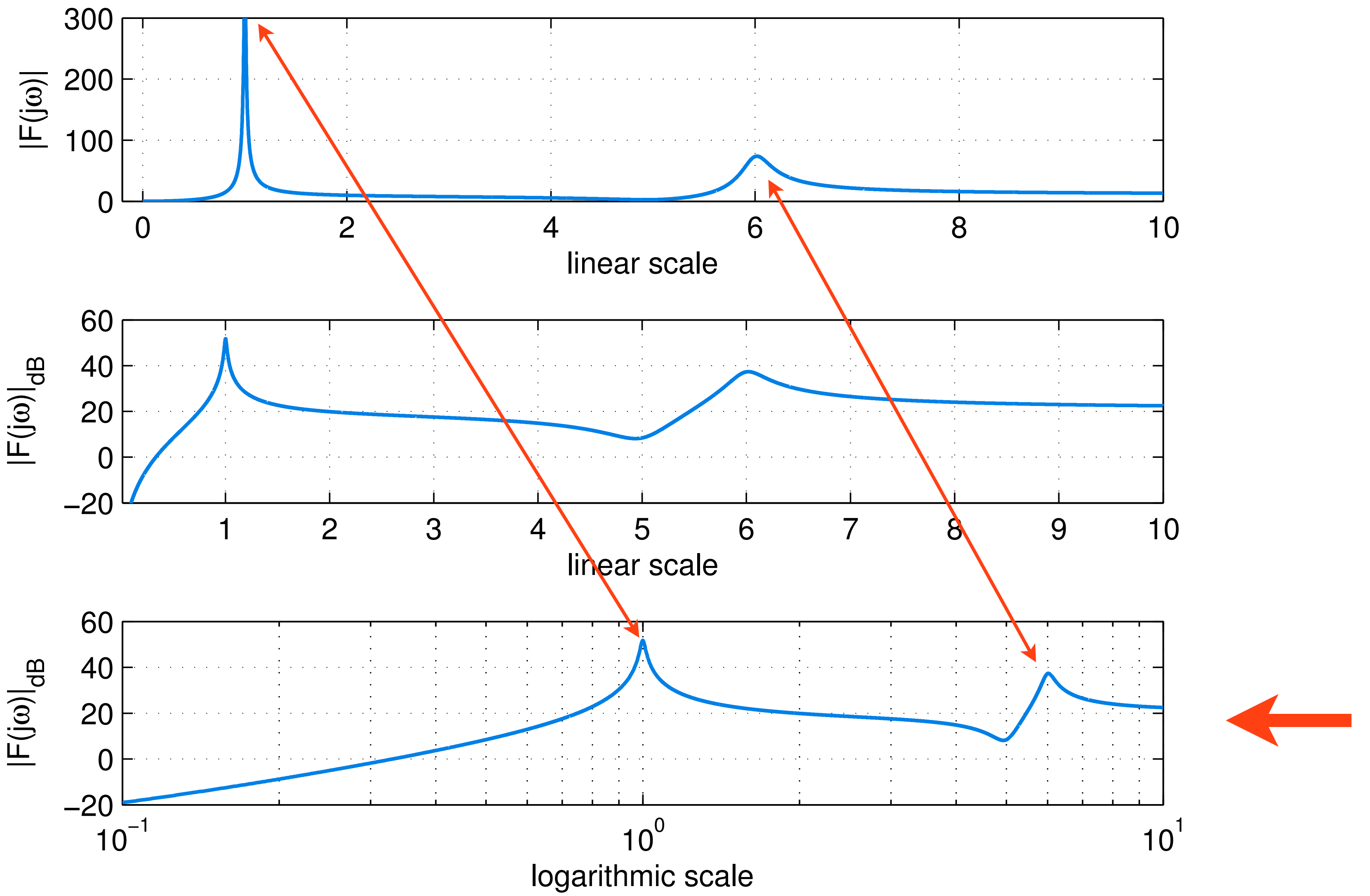
- quantities can vary in large range (both ω and magnitude)
- easy to build the magnitude plot in dB of a frequency response given in its Bode canonical form from the magnitudes of the single terms
- easy to represent series of systems

same data

different scales

for abscissa and

ordinates



this is the scale we are going to use

Bode diagrams

$|F(j\omega)|_{dB}$ magnitude in dB of the frequency response as a function of the angular frequency ω with logarithmic scale for ω

$\angle F(j\omega)$ angle or phase of the frequency response as a function of the angular frequency ω with logarithmic scale for ω

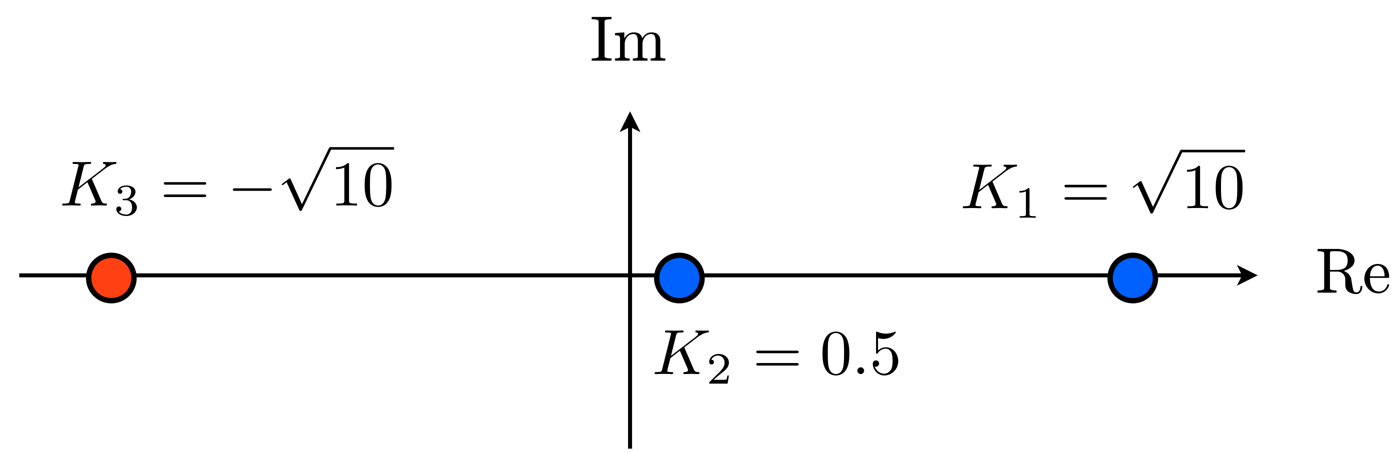
we need to find the magnitude (in dB) and phase for the 4 elementary factors

1. constant K (generalized gain)
2. monomial $j\omega$ (zero or pole in $s = 0$)
3. binomial $1 + j\omega\tau$ (non-zero real zero or pole)
4. trinomial $1 + 2\zeta(j\omega)/\omega_n + (j\omega)^2/\omega_n{}^2$ (complex conjugate pairs of zeros or poles)

for $\omega \in \mathbb{R}^+ = [0, +\infty)$

Constant

K



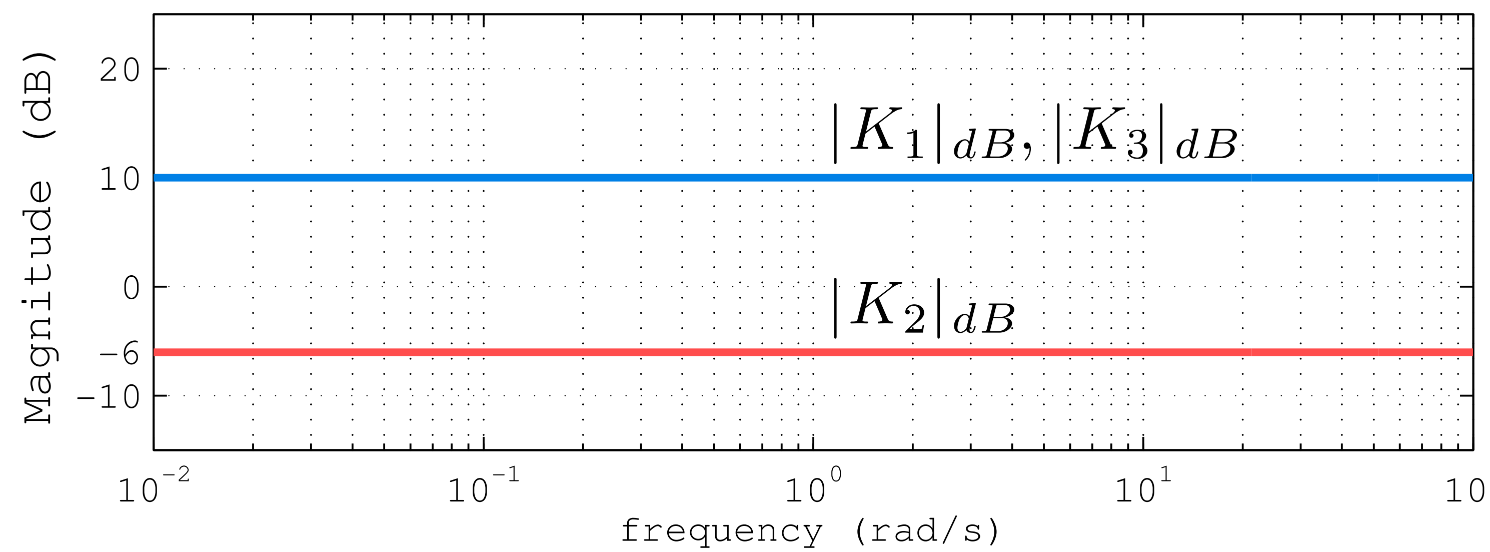
$$|\sqrt{10}|_{dB} = 10\,\text{dB}$$

$$|-\sqrt{10}|_{dB} = 10\,\text{dB}$$

$$|0.5|_{dB} \approx -6\,\text{dB}$$

magnitude

$$20\log_{10}|K|$$



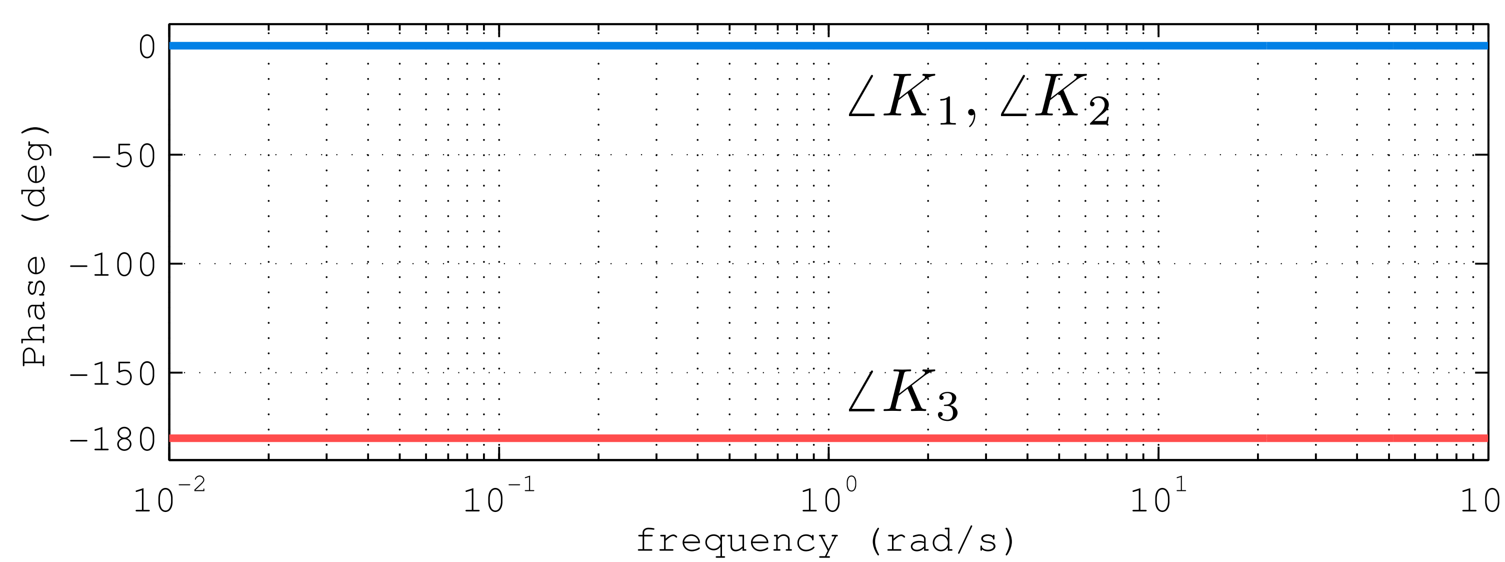
$$\angle\sqrt{10} = 0^\circ$$

$$\angle-\sqrt{10} = -180^\circ = -\pi$$

$$\angle 0.5 = 0^\circ$$

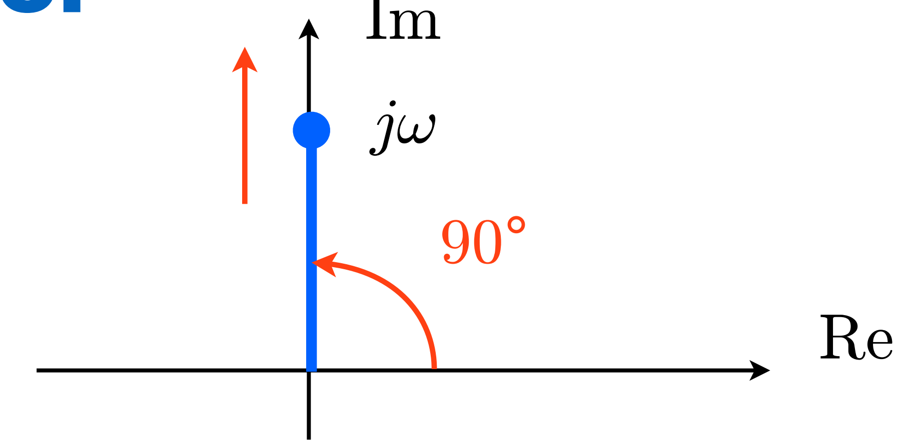
phase

$$\angle K$$



Monomial - Numerator

$j\omega$

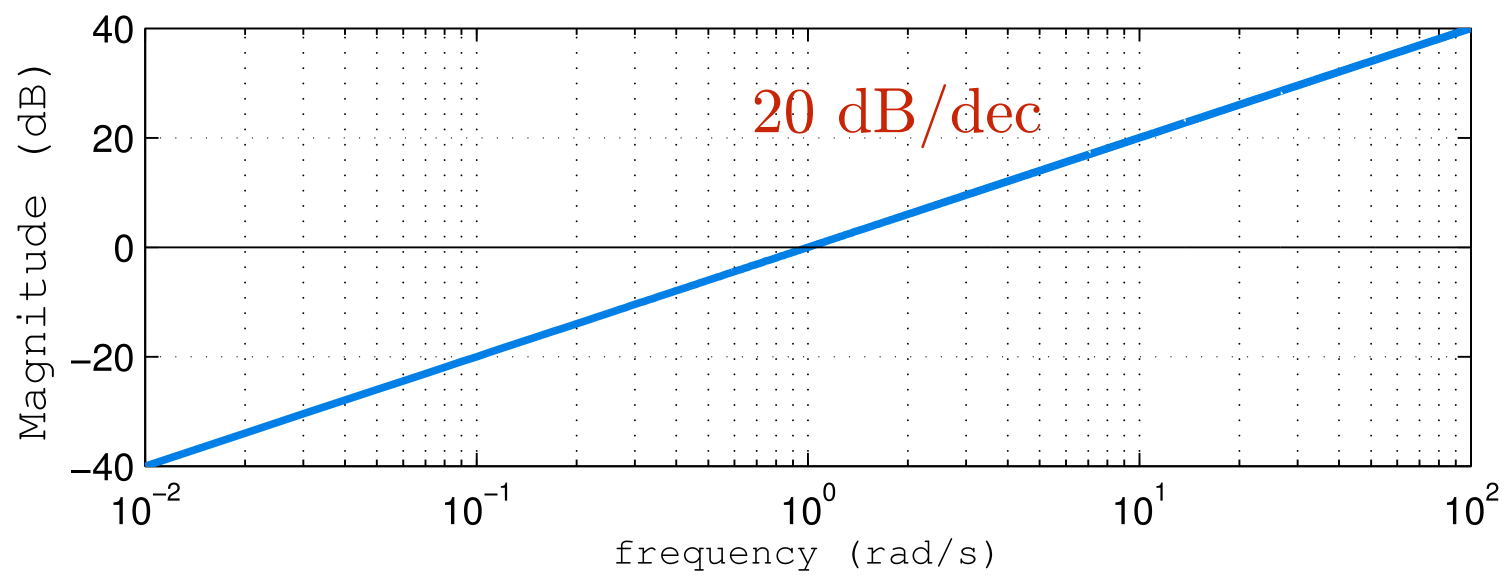


$$|j\omega|_{dB} = 20 \log_{10} \omega$$

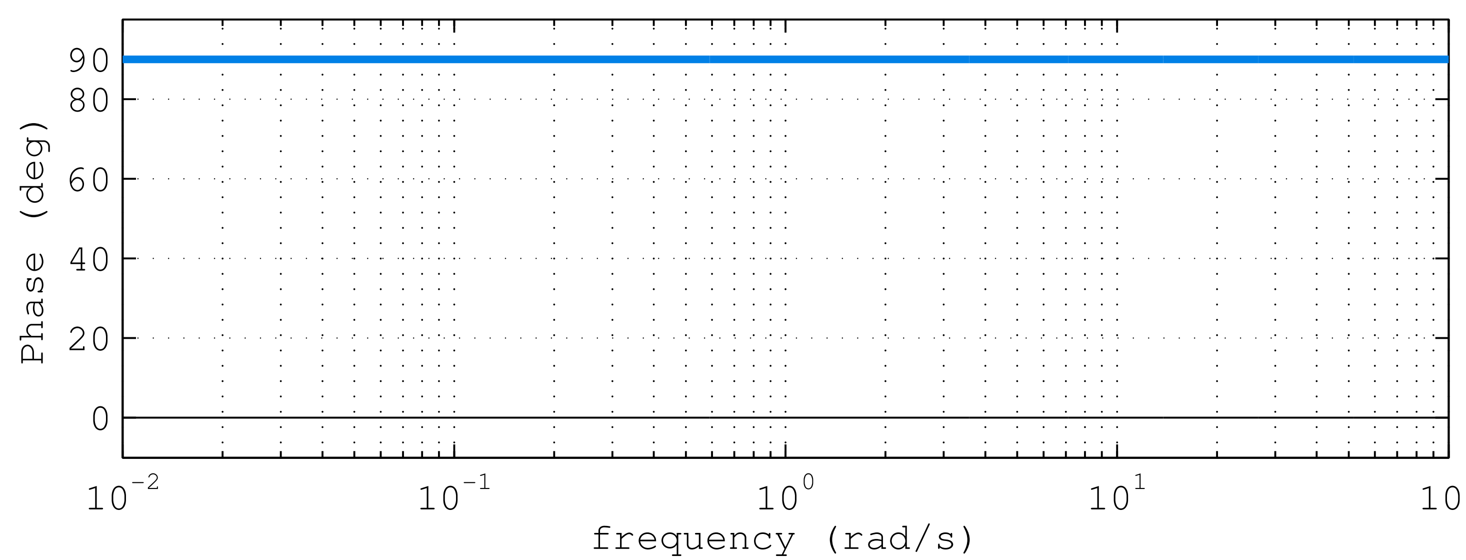
log scale

$$|j\omega|_{dB} = 20x$$

magnitude



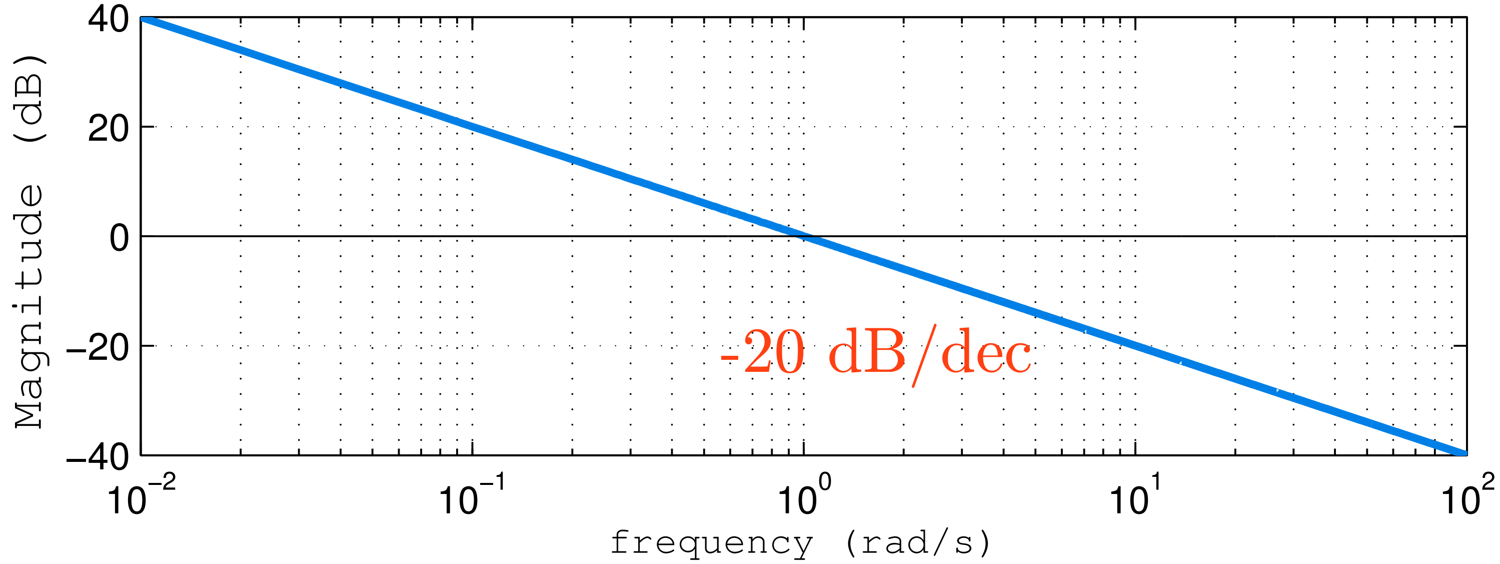
phase



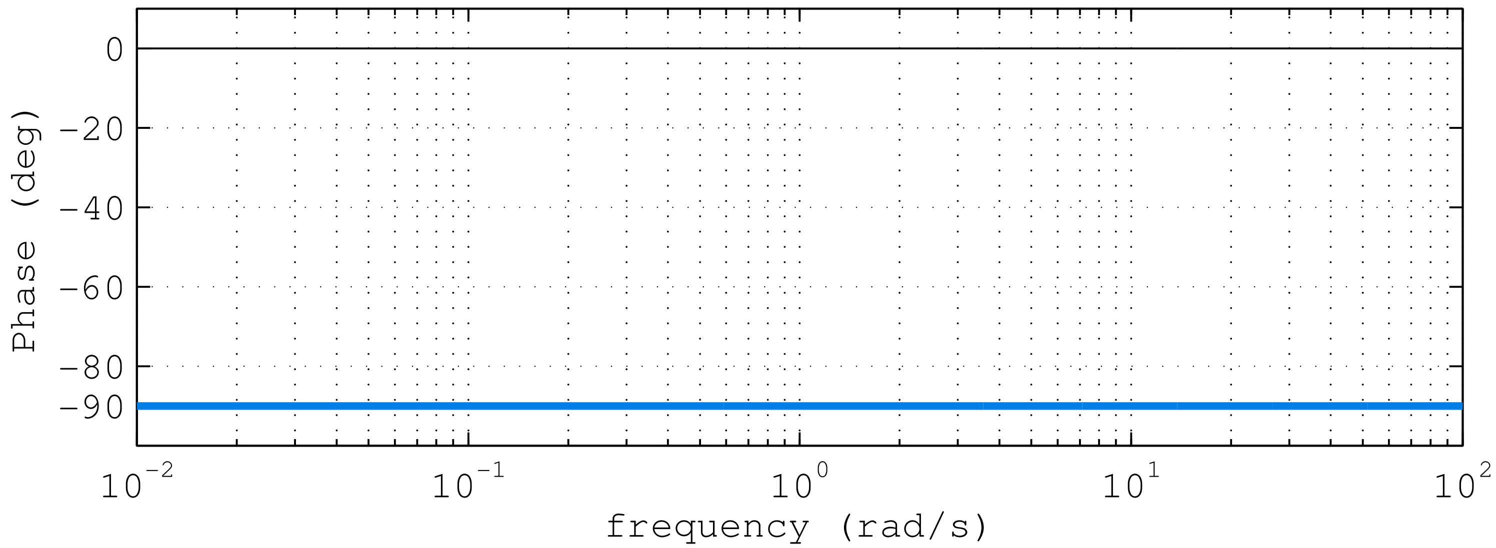
Monomial - Denominator

from properties of log and phase

magnitude



phase



Binomial - Numerator $1 + j\omega\tau$

magnitude $$|1 + j\omega\tau|_{dB} = 20 \log_{10} \sqrt{1 + \omega^2\tau^2}$$

approximation wrt the **cutoff frequency** $1/|\tau|$ (or **corner frequency**)

$$\sqrt{1 + \omega^2\tau^2} \approx \begin{cases} 1 & \text{if} \quad \omega \ll 1/|\tau| \\ \sqrt{\omega^2\tau^2} & \text{if} \quad \omega \gg 1/|\tau| \end{cases}$$

and therefore

$$|1 + j\omega\tau|_{dB} \approx \begin{cases} 0\,dB & \text{if} \quad \omega \ll 1/|\tau| \\ 20\log_{10}\omega + 20\log_{10}|\tau| & \text{if} \quad \omega \gg 1/|\tau| \end{cases}$$

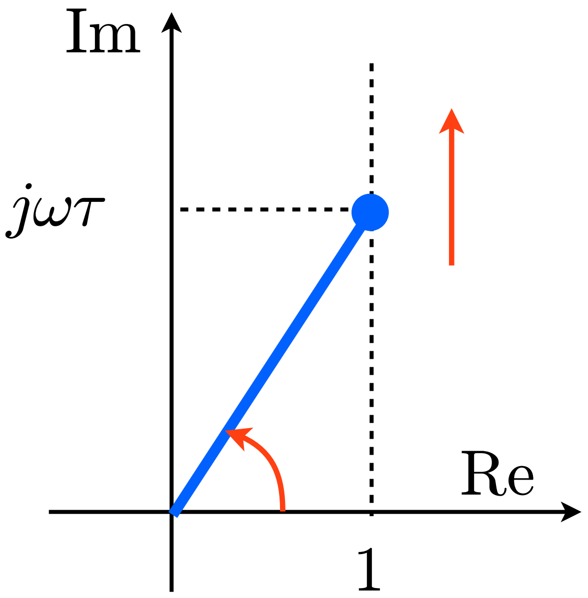
at the cutoff frequency $\omega^* = 1/|\tau|$ $\quad |1 + j\tau/|\tau||_{dB} = 20\log_{10}\sqrt{2} \approx 3\,dB$

two half-lines approximation: 0 dB until the cutoff frequency, $+$ 20dB/decade after

Binomial - Numerator

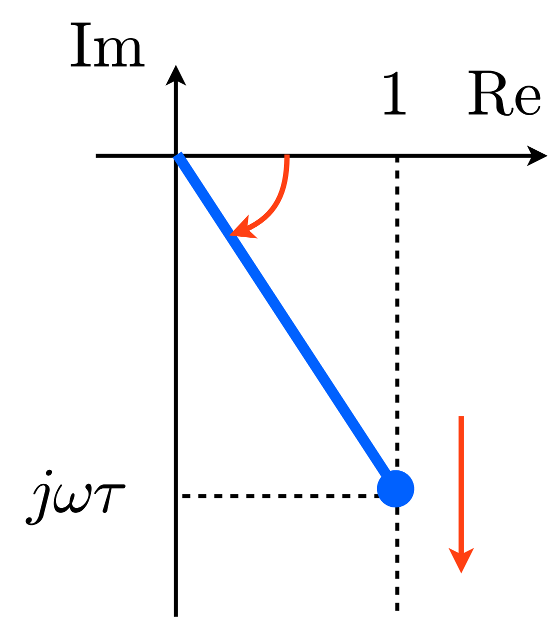
$1 + j\omega\tau$

phase depends on the sign of τ



$\tau > 0$

$\tau < 0$



see how the phase changes as ω increases

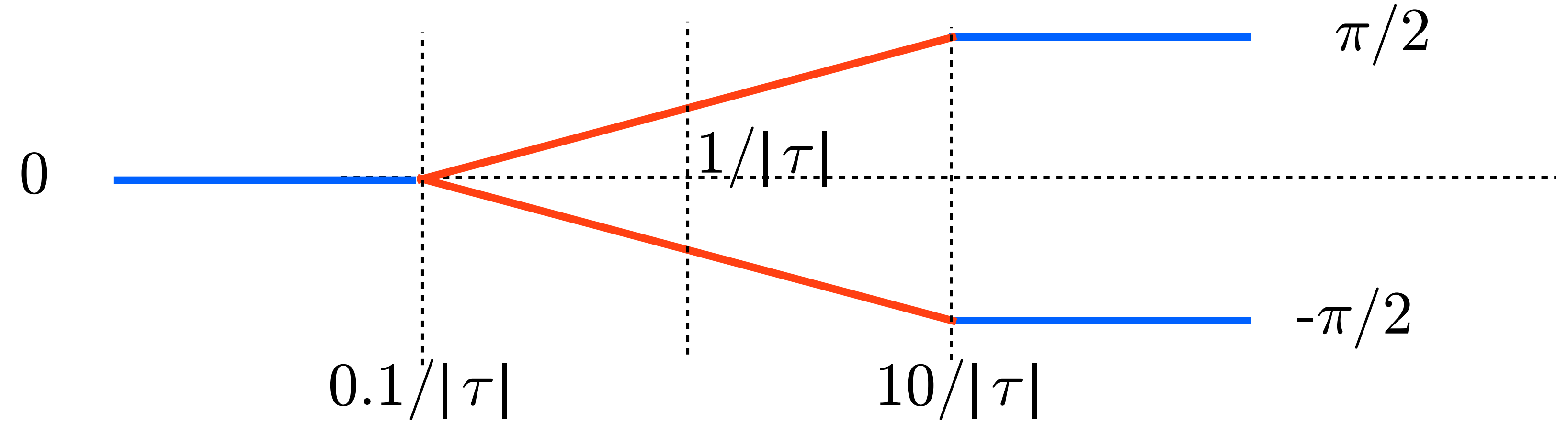
Binomial - Numerator

$1 + j\omega\tau$ **phase** depends on the sign of τ

case $\tau > 0$ $$\angle(1+j\omega\tau) \approx \begin{cases} 0 & \text{if} \quad \omega \ll 1/|\tau| \\ \frac{\pi}{2} & \text{if} \quad \omega \gg 1/|\tau| \quad \text{and} \quad \tau > 0 \end{cases}$$

case $\tau < 0$ $$\angle(1+j\omega\tau) \approx \begin{cases} 0 & \text{if} \quad \omega \ll 1/|\tau| \\ -\frac{\pi}{2} & \text{if} \quad \omega \gg 1/|\tau| \quad \text{and} \quad \tau < 0 \end{cases}$$

the two asymptotes are connected by a segment starting a decade before ($0.1/|\tau|$) the cutoff frequency and ending a decade after ($10/|\tau|$). The approximation is a broken line.

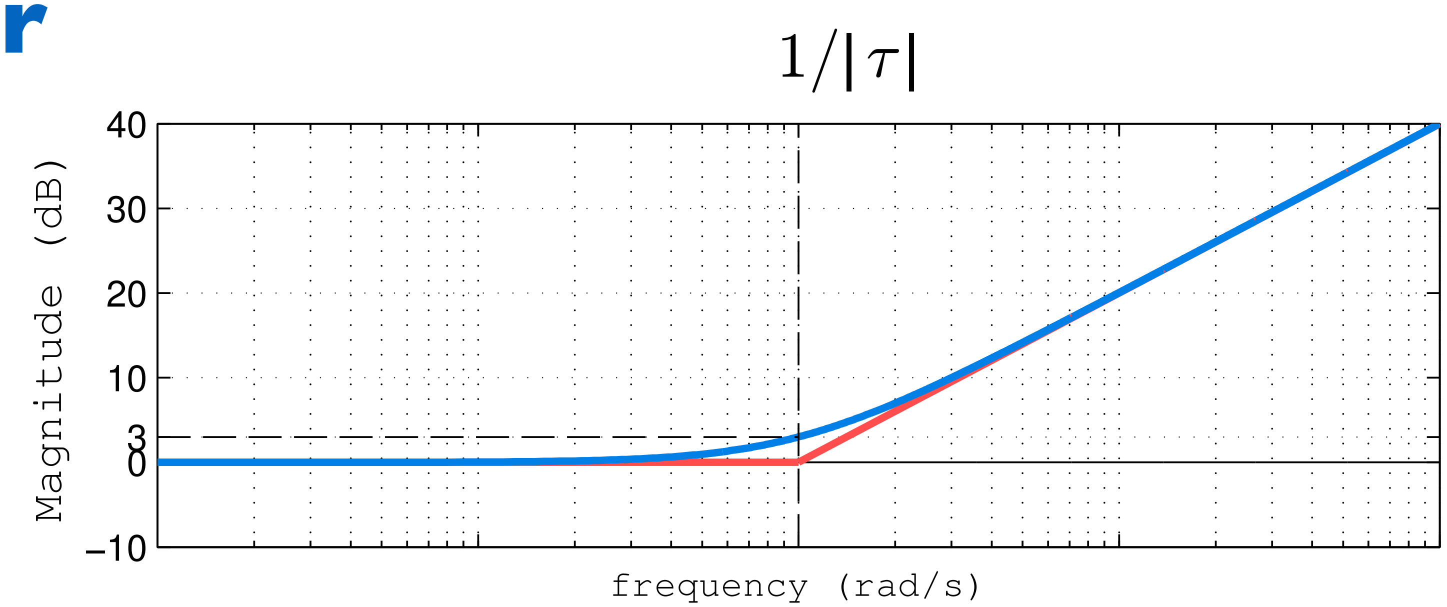


at the cutoff frequency $\omega^* = 1/|\tau|$ $$\angle(1 + j\tau/|\tau|) = \begin{cases} \frac{\pi}{4} & \text{if} \quad \tau > 0 \\ -\frac{\pi}{4} & \text{if} \quad \tau < 0 \end{cases}$$

Binomial - numerator

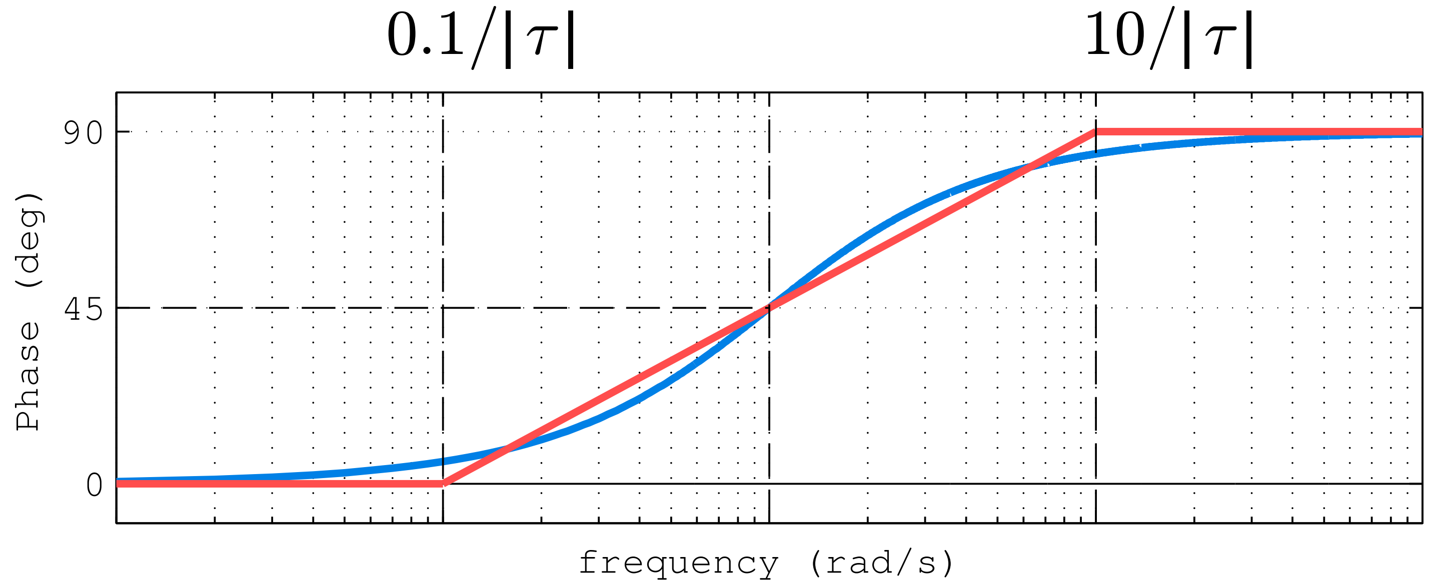
$1 + j\omega\tau$

magnitude



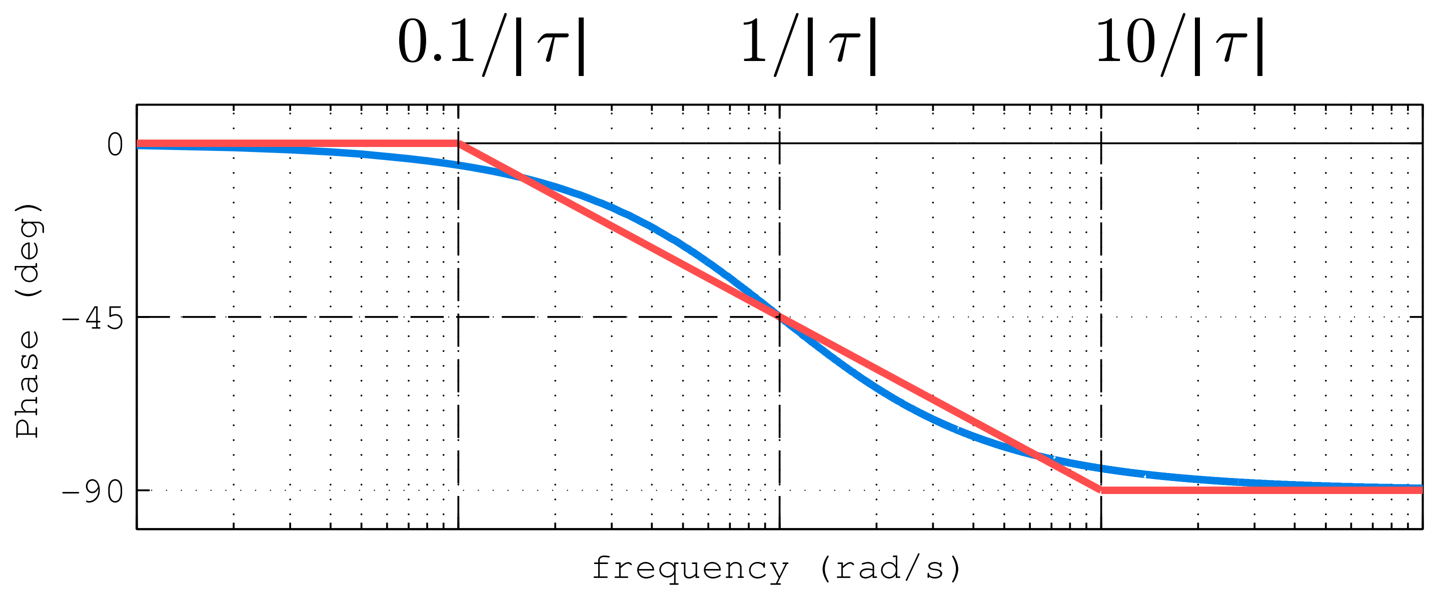
$\tau > 0$

phase



$\tau < 0$

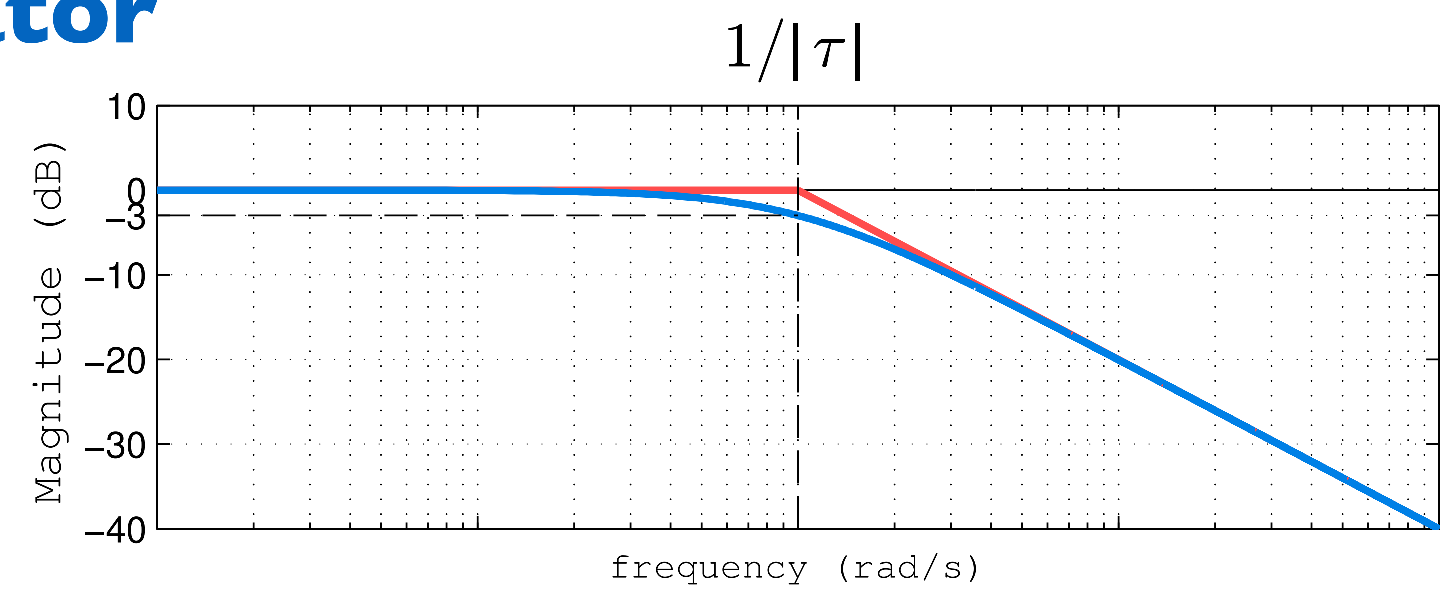
phase



Binomial - denominator

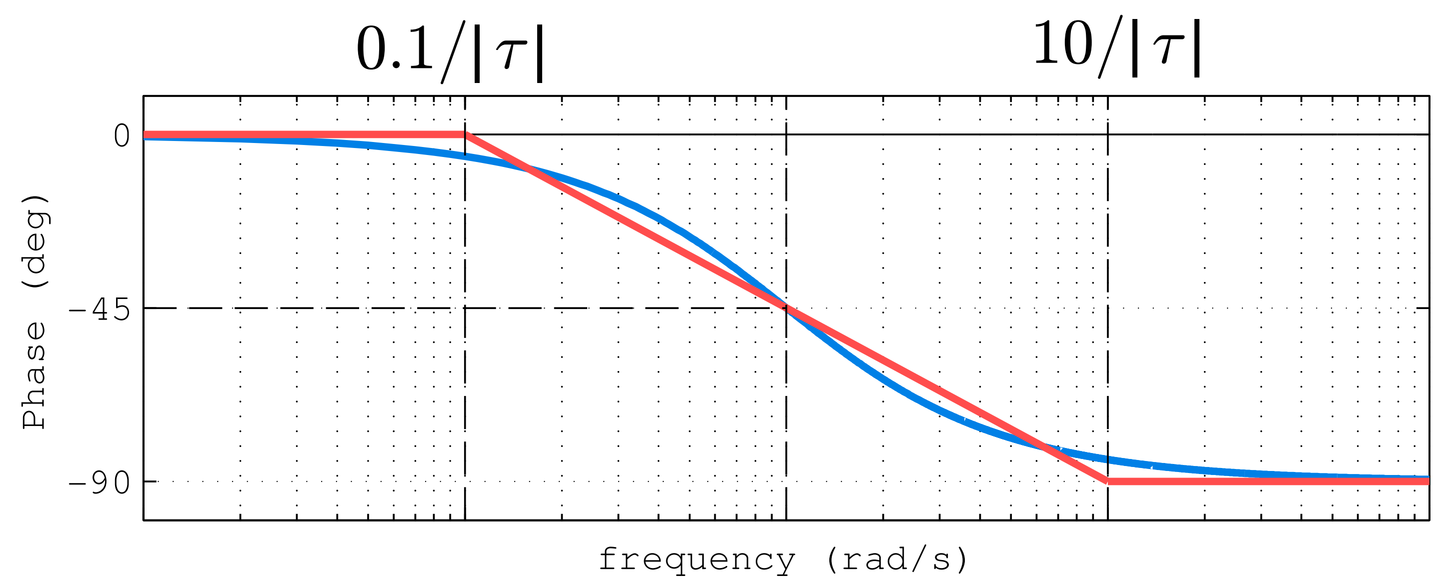
$1 \; / (1 + j\omega\tau)$

magnitude



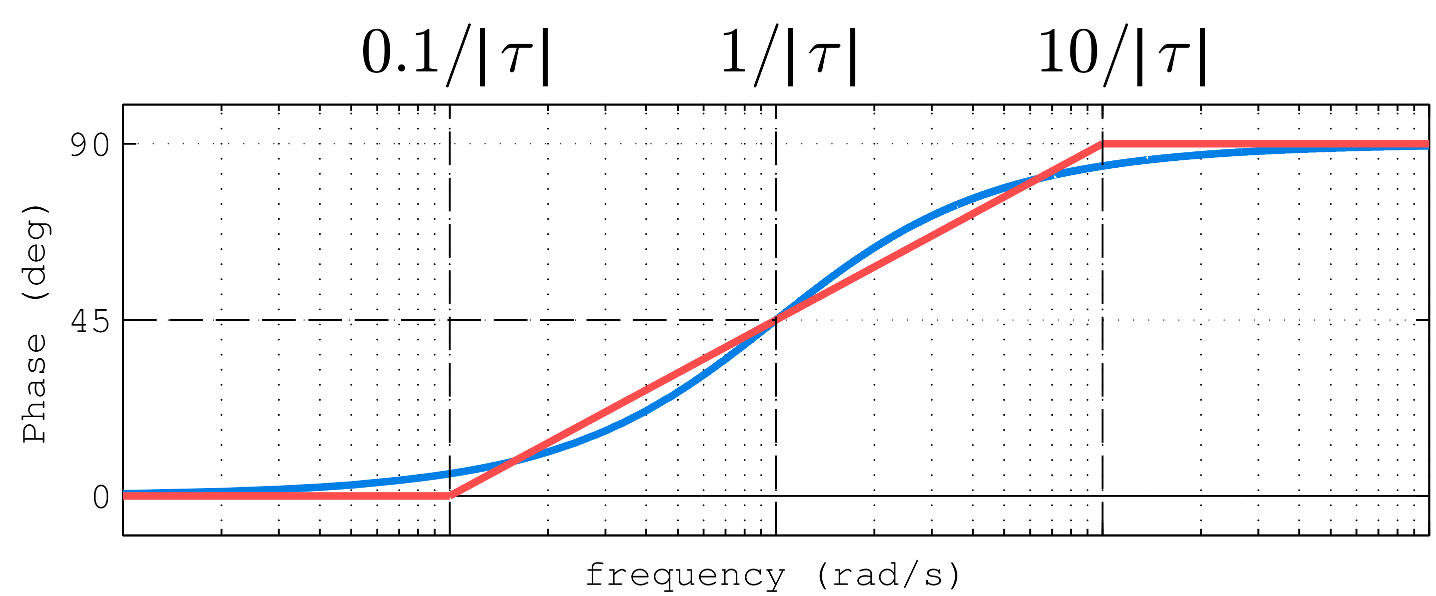
$\tau > 0$

phase



$\tau < 0$

phase



Trinomial

magnitude

$$\left|1 + 2\frac{\zeta}{\omega_n}(j\omega) + \frac{(j\omega)^2}{\omega_n^2}\right| = \left|1 - \frac{\omega^2}{\omega_n^2} + j2\zeta\frac{\omega}{\omega_n}\right|$$

$$= \sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(4\zeta^2\frac{\omega^2}{\omega_n^2}\right)}$$

approximation wrt ω_n

$$|\mathrm{TRINOMIAL}| \approx \begin{cases} 1 & \text{if} \quad \omega \ll \omega_n \\ \sqrt{\left(\frac{\omega^2}{\omega_n^2}\right)^2} = \frac{\omega^2}{\omega_n^2} & \text{if} \quad \omega \gg \omega_n \end{cases}$$

$$|\mathrm{TRINOMIAL}|_{dB} \approx \begin{cases} 0\, dB & \text{if} \quad \omega \ll \omega_n \\ 40\log_{10}\omega - 20\log_{10}\omega_n^2 & \text{if} \quad \omega \gg \omega_n \end{cases}$$

Trinomial

in $\omega = \omega_n$ the magnitude $|\,\mathrm{TRINOMIAL}\,|$ is equal to $2\,|\zeta|$

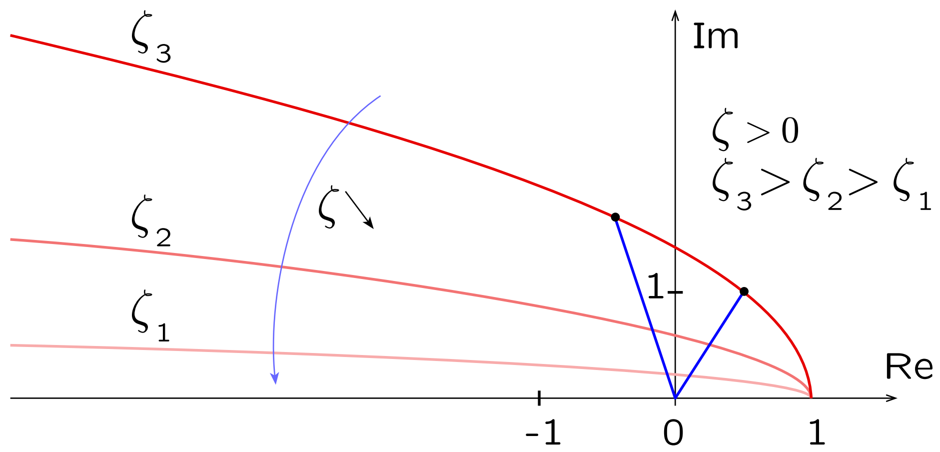
$\lvert\zeta\rvert$	0	0.5	$1/\sqrt{2} \approx 0.707$	1
$\lvert TRIN\rvert_{dB}$ in ω_n	$-\infty$	0 dB	3 dB	6 dB

large variation of the magnitude in $\omega = \omega_n$ depending upon the value of the damping coefficient ζ

no approximation around the natural frequency ω_n

Trinomial

How does a generic complex root varies in the plane as a function of ω



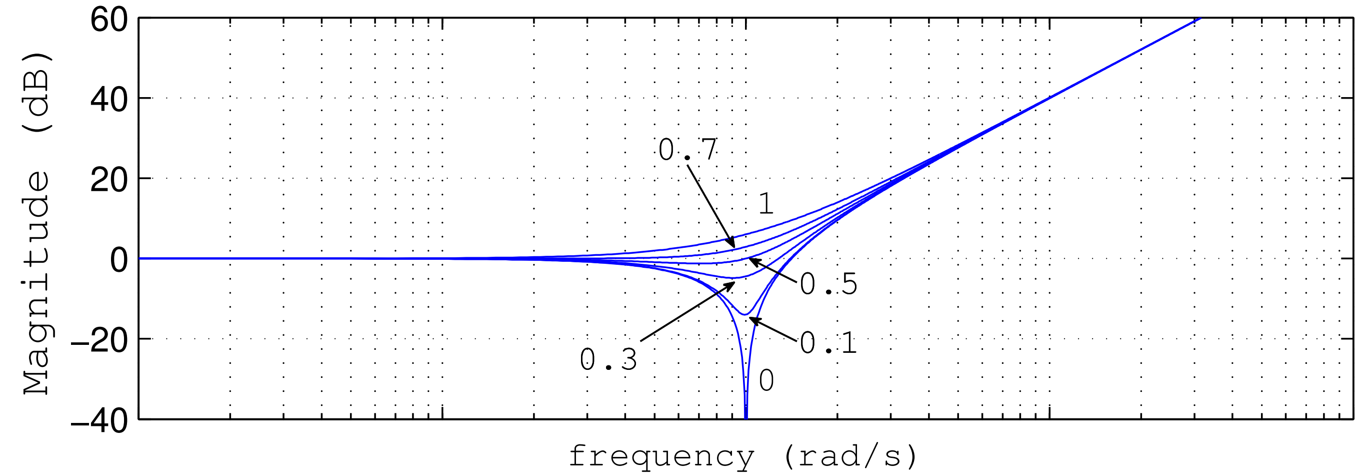
Phase

$$\angle\left(1 + 2\frac{\zeta}{\omega_n}(j\omega) + \frac{(j\omega)^2}{\omega_n^2}\right) = \begin{cases} 0 & \text{if} \quad \omega \ll \omega_n \\ \pi & \text{if} \quad \omega \gg \omega_n \quad \text{and} \quad \zeta \geq 0 \\ -\pi & \text{if} \quad \omega \gg \omega_n \quad \text{and} \quad \zeta < 0 \end{cases}$$

transition between 0 and π (or - π) is symmetric wrt ω_n and becomes more abrupt as $|\zeta|$ becomes smaller. When $\zeta = 0$ the phase has a discontinuity in ω_n

Trinomial - numerator

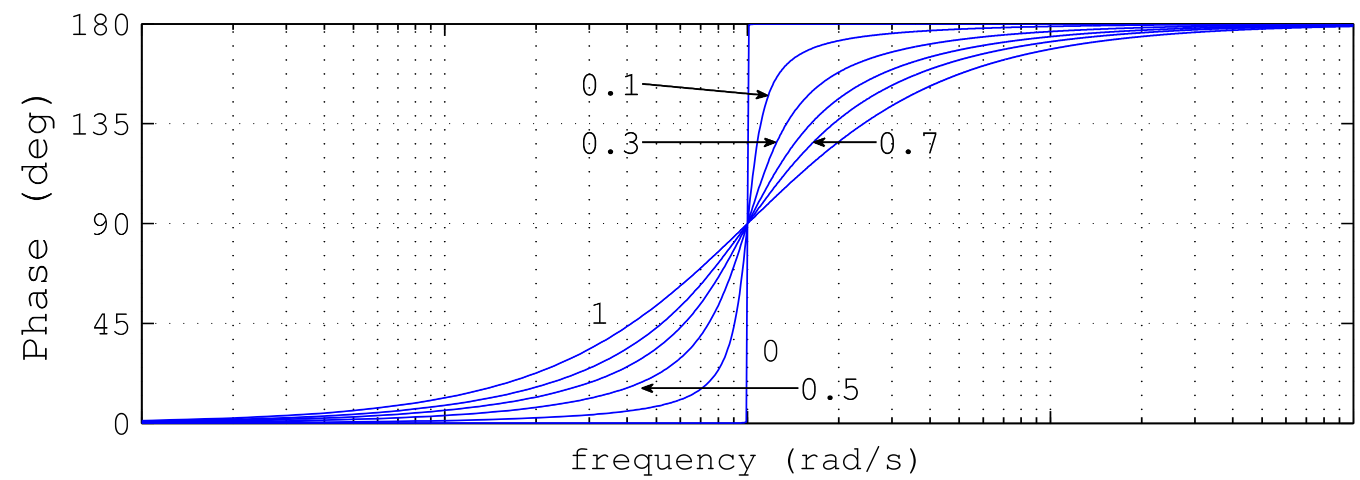
magnitude



$0.1\ \omega_n$ ω_n $10\ \omega_n$

phase

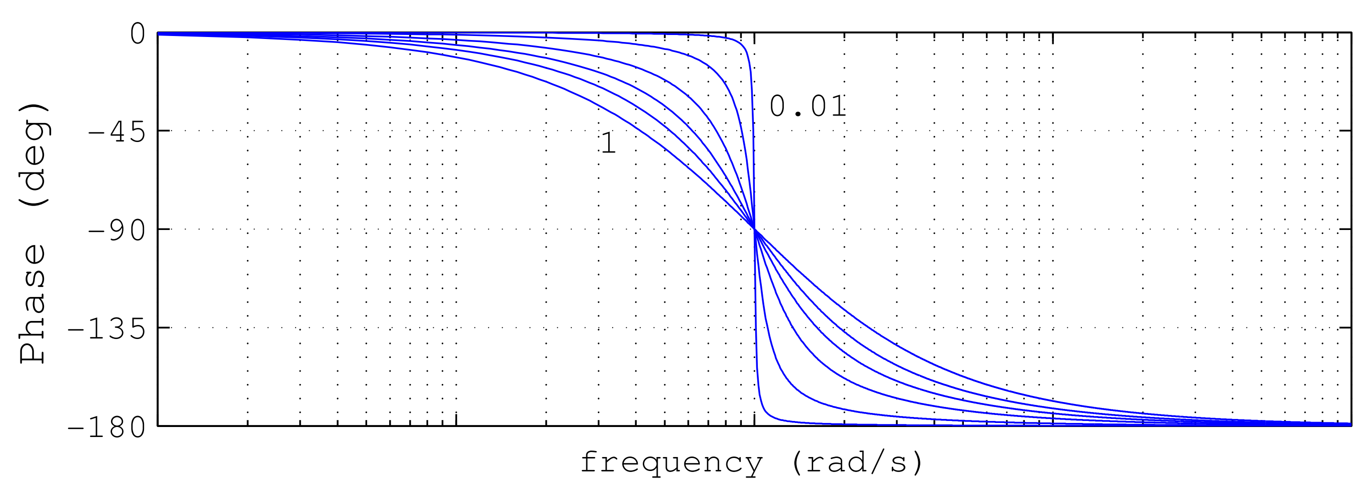
$\zeta \geq 0$



$0.1\ \omega_n$ ω_n $10\ \omega_n$

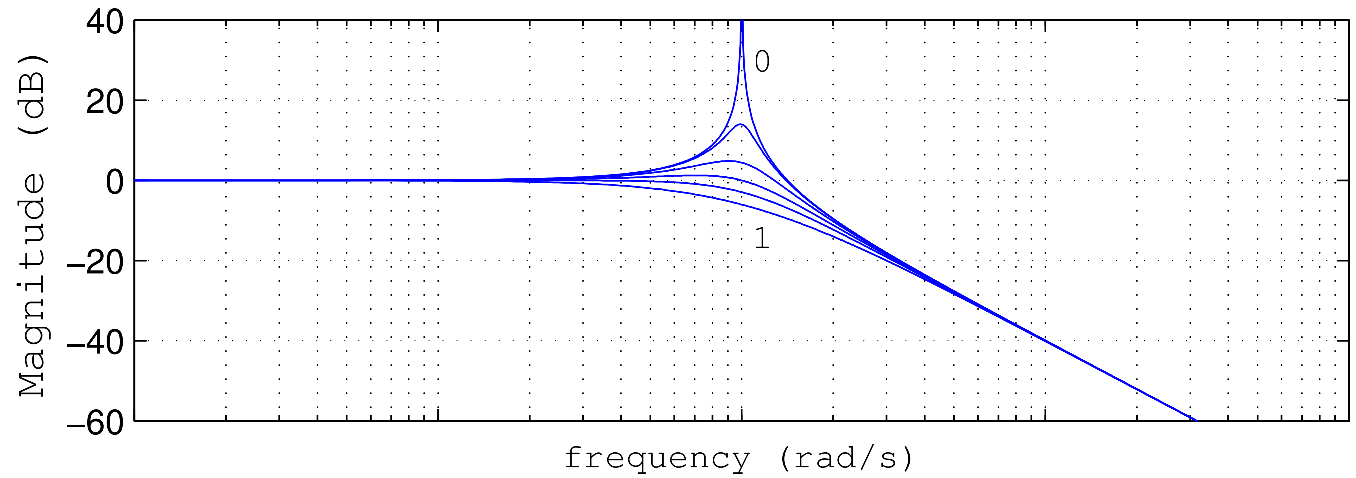
phase

$\zeta < 0$



Trinomial - denominator

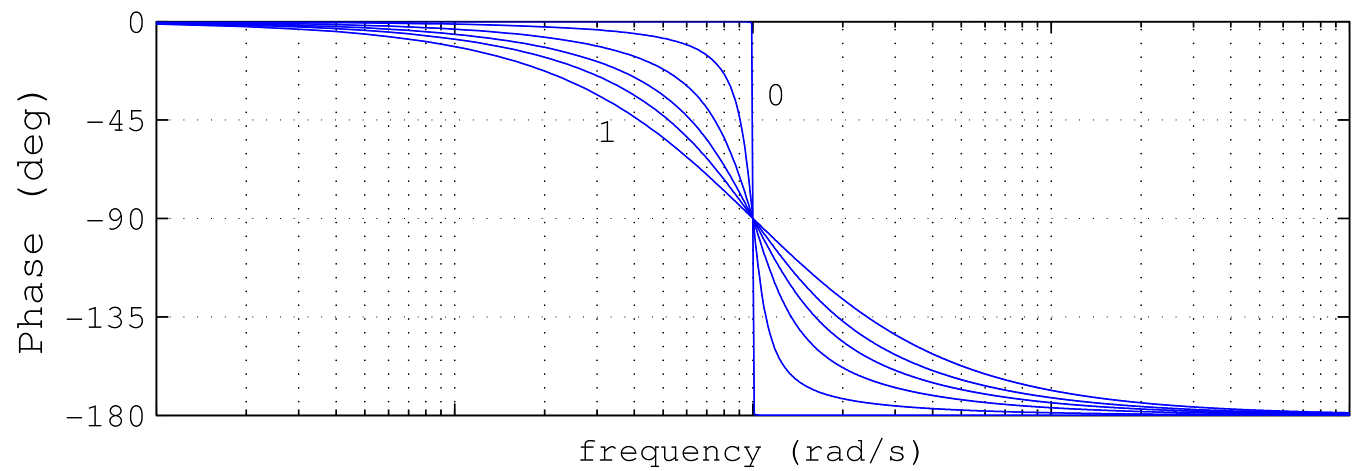
magnitude



$0.1\ \omega_n$ ω_n $10\ \omega_n$

phase

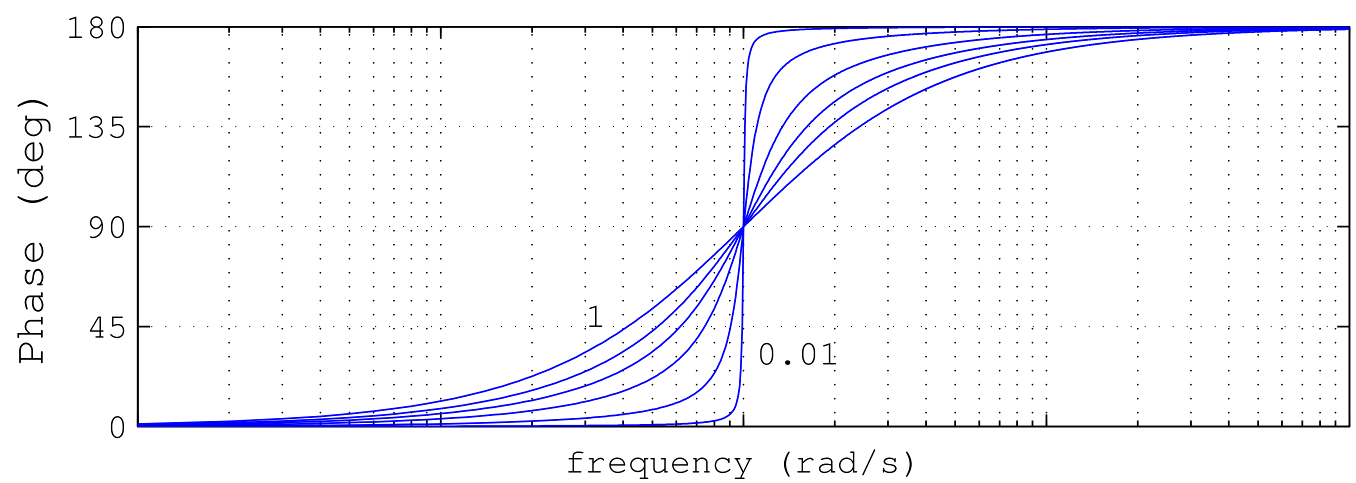
$\zeta \geq 0$



$0.1\ \omega_n$ ω_n $10\ \omega_n$

phase

$\zeta < 0$



Trinomial

When $|\zeta| = 1$ the trinomial reduces to a product of two identical binomials (real roots)

$$\text{roots} = \begin{cases} -\omega_n & \text{if} \quad \zeta = 1 \\ \omega_n & \text{if} \quad \zeta = -1 \end{cases}$$

$$\left(1 + 2\frac{\zeta}{\omega_n}s + \frac{s^2}{\omega_n^2}\right)_{\zeta=\pm 1} = \left(1 \pm \frac{s}{\omega_n}\right)^2$$

and therefore the magnitude and phase coincides with that of a double binomial with corner frequency

$$\frac{1}{|\tau|} = \omega_n$$

that is in $\omega = \omega_n$ when $|\zeta| = 1$

$$\begin{aligned} 2 \times (3\,\text{dB}) &= 6\,\text{dB} \quad (\text{numerator}) \\ 2 \times (-3\,\text{dB}) &= -6\,\text{dB} \quad (\text{denominator}) \end{aligned}$$

example: MSD system with critical value for the damping ($\mu^2 = 4km$)

Trinomial

if $|\zeta| < 1/\sqrt{2} \approx 0.707$ the magnitude of a trinomial factor at the denominator has a peak

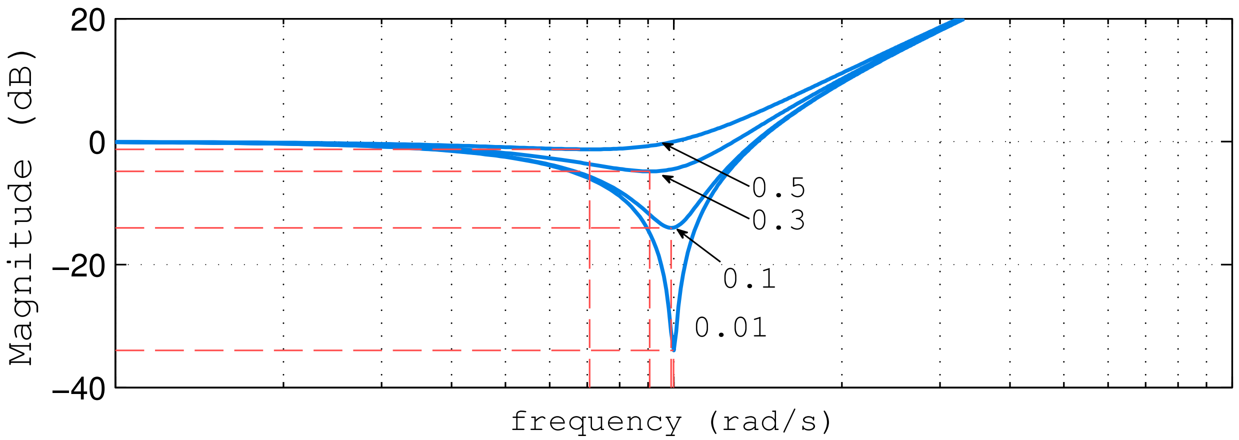
$$|F(j\omega_r)| = \frac{1}{2|\zeta|\sqrt{1-\zeta^2}}$$

resonance peak

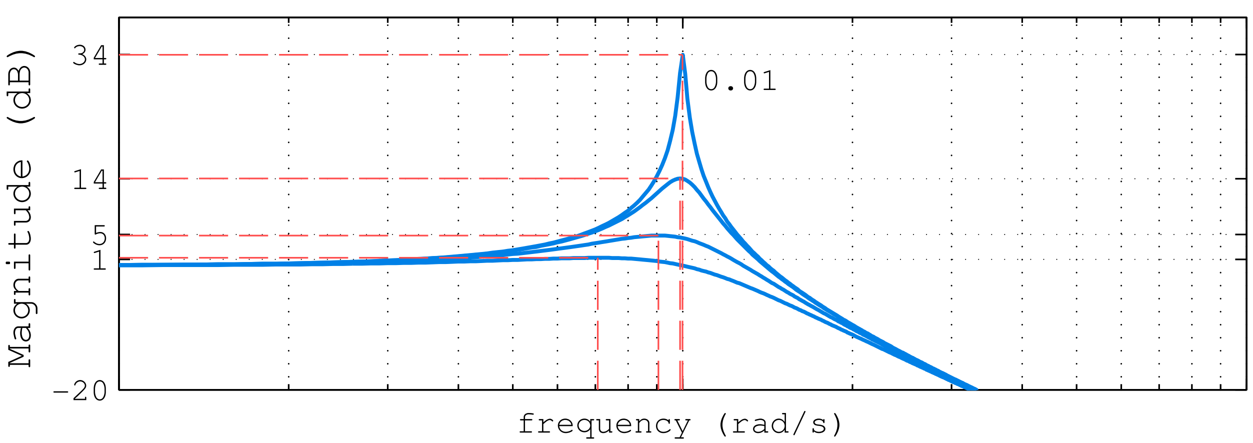
at the **resonance frequency**

$$\omega_r = \omega_n\sqrt{1-2\zeta^2}$$

(similarly for the **anti-resonance peak**)



anti-resonance peak



resonance peak