Control Systems

Response characteristics

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outline

- study a particular response, the step response, to a unit step as input
- final value theorem
- characterize the asymptotic and transient response on the step response
  - position of poles in the complex plane
  - definition of quantities on the step response in the time domain
- define the long term, asymptotic, behavior (steady-state) of a response and understand when it exists
- compute the steady-state response for different test inputs

step response

The forced response to a **step input** is referred as **step response**

- in s

$$u(t) = \delta_{-1}(t) \longrightarrow U(s) = \frac{1}{s} \longrightarrow X_s(s) = H(s)U(s) = H(s)\frac{1}{s} \quad \text{state}$$

$$x_0 = 0$$

$$Y_s(s) = W(s)U(s) = W(s)\frac{1}{s} \quad \text{output}$$

- in t, integral property of the Laplace transform

$$x_s(t) = \int_0^t h(\tau)d\tau$$

$$\boxed{y_s(t) = \int_0^t w(\tau)d\tau}$$

state step response

the (output) step response is equal to the integral of the (output) impulse response

step response

Let S be asymptotically stable

$$W(s) = \frac{N(s)}{\prod_{i=1}^{n}(s-p_i)}$$

distinct poles
(for ease of exposition)

- explicit response from s from t

$$Y_s(s) = W(s)U(s) = \frac{N(s)}{\prod_{i=1}^{n}(s-p_i)}\frac{1}{s} \xrightarrow[\text{fraction expansion}]{\text{distinct denominator roots}} Y_s(s) = \underbrace{\frac{R_0}{s}}_{\text{input contribution}} + \underbrace{\sum_{i=1}^{n}\frac{R_i}{s-p_i}}_{\text{system contribution}}$$

step response

(distinct poles)

$$\boxed{y_s(t) = \left(R_0 + \sum_{i=1}^{n} R_i e^{p_i t}\right)\delta_{-1}(t)}$$

these terms decay since
S is asymptotically stable $\mathrm{Re}[p_i] < 0$

step response

- explicit response directly in t (alternative solution)

forced response (convolution integral + direct term)

$$
\begin{aligned}
y_s(t) &= \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t) \\
&= C\int_0^t e^{A(t-\tau)}Bd\tau + D \\
&= C\int_0^t e^{A\sigma}Bd\sigma + D \\
&= C(\left[A^{-1}e^{A\sigma}B\right]_{\sigma=0}^{\sigma=t} + D \\
&= \underbrace{CA^{-1}e^{At}B}_{\text{(transient)}} + \underbrace{D - CA^{-1}B}_{\substack{\text{constant}\\ \text{(steady-state)}}}
\end{aligned}
$$

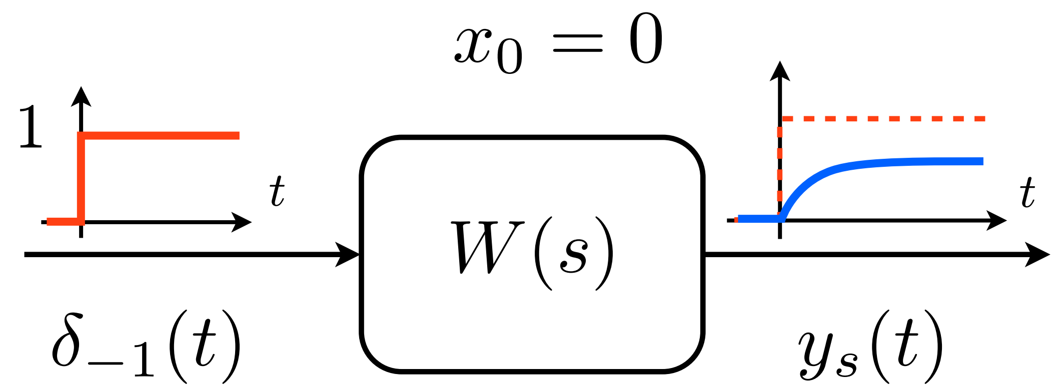
$u(t) = \delta_{\text{-}1}(t)$

$\sigma = t$ - τ

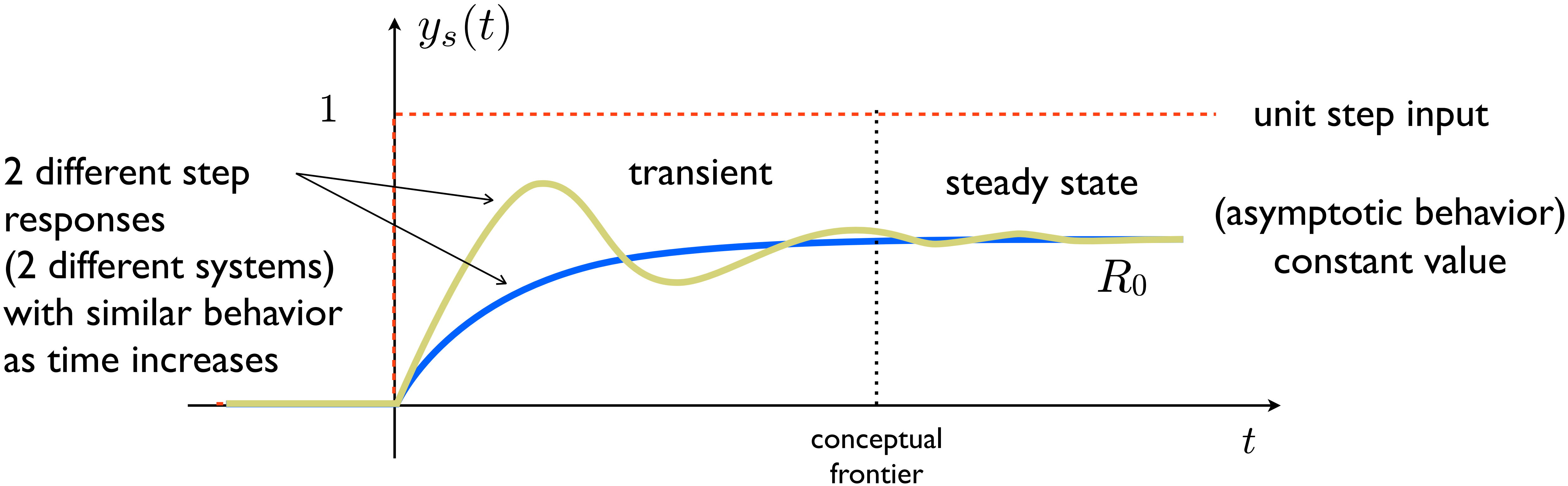
step response

for an asymptotically stable system

$$y_s(t) = \left(R_0 + \sum_{i=1}^{n} R_i e^{p_i t} \right) \delta_{-1}(t) \qquad \Longrightarrow$$



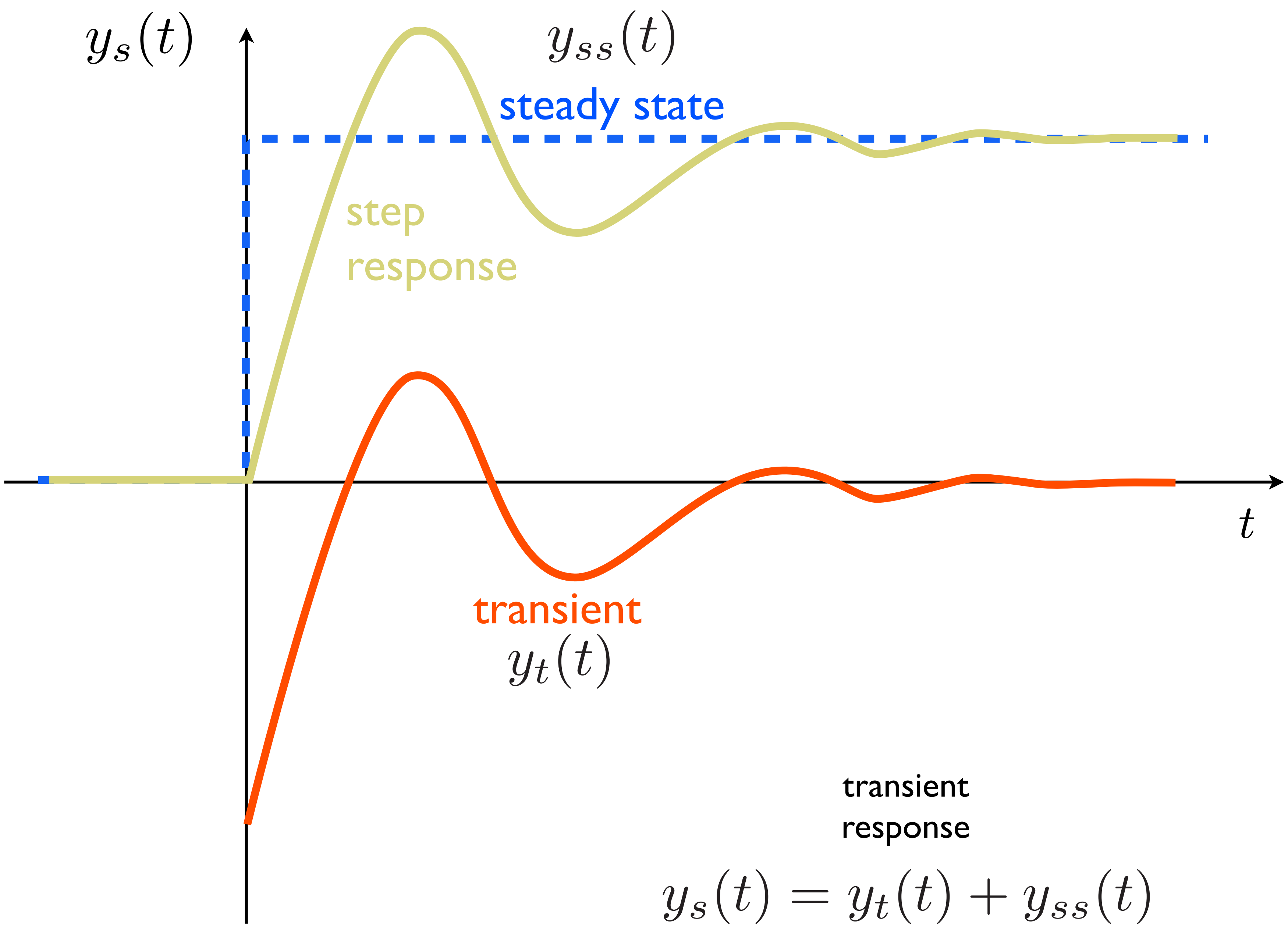
the response can be conceptually split in two parts: **transient** and **steady state**



do not confuse transient/steady state with free/forced response

step response: transient + steady state

let’s clarify the separation between the **transient** and **steady state** contributions



for the step response the steady state is the constant value at which the output asymptotically tends to, from the response expression this is

$$R_0\delta_{-1}(t)$$

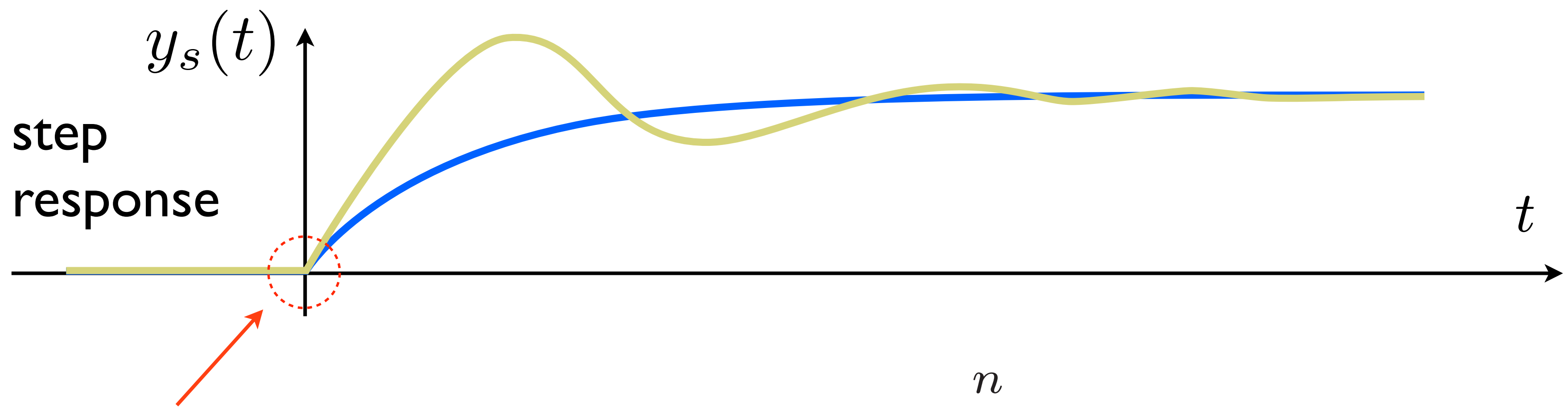
transient response

$$y_s(t) = y_t(t) + y_{ss}(t)$$

steady-state response

Laplace transform: initial value theorem

for an asymptotically stable system (distinct poles) $\quad y_s(t) = \left(R_0 + \sum_{i=1}^{n} R_i e^{p_i t} \right) \delta_{-1}(t)$



- what is the value at $t = 0$? $\quad y_s(0) = R_0 + \sum_{i=1}^{n} R_i = \ ?$ $\quad \xleftarrow{t\,=\,0}$

Initial value theorem

$$\boxed{\lim_{t \to 0^+} f(t) = \lim_{s \to +\infty} sF(s)}$$

true for any asymptotically stable system

therefore

$$y_s(0) = \lim_{s \to +\infty} sW(s)\frac{1}{s} = \lim_{s \to +\infty} W(s) = \begin{cases} D & \text{if } W(s) \text{ proper} \\ 0 & \text{if } W(s) \text{ strictly proper} \end{cases}$$

Laplace transform: final value theorem

- what is the behavior at steady state ($t = \infty$)?

Final value theorem

$$\boxed{\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)}$$

general result

provided the limit on the left exists $\longleftrightarrow$ if $\boxed{sF(s)}$ is analytic (all the roots of the denominator of $sF(s)$ in the open left half-plane)

example $\quad W(s) = \dfrac{1}{s+1} \quad$ asymptotically stable system

$$u_1(t) = \delta_{-1}(t) \qquad U_1(s) = \frac{1}{s} \qquad sY_1(s) = \cancel{s}\frac{1}{(s+1)}\frac{1}{\cancel{s}}$$

ok

$$u_2(t) = t\delta_{-1}(t) \qquad U_2(s) = \frac{1}{s^2} \qquad sY_2(s) = \cancel{s}\frac{1}{(s+1)}\frac{1}{\cancel{s^2}}$$

no

$$u_3(t) = \sin\omega t \qquad U_3(s) = \frac{\omega}{s^2+\omega^2} \qquad sY_3(s) = s\frac{1}{(s+1)}\frac{\omega}{(s^2+\omega^2)}$$

no

step response: steady-state

S asymptotically stable ($\mathrm{Re}[p_i] < 0$) $\qquad W(s) = \dfrac{N(s)}{\prod_{i=1}^{n}(s-p_i)}$ (distinct poles)

we already found

$$Y_s(s) = \frac{R_0}{s} + \sum_{i=1}^{n} \frac{R_i}{s-p_i} \quad \rightarrow \quad y_s(t) = \boxed{R_0 \delta_{-1}(t)} + \sum_{i=1}^{n} R_i e^{p_i t} \delta_{-1}(t)$$

these terms decay

by the definition of residue R_0

$$R_0 = [sY_s(s)]_{s=0} = \left[sW(s)\frac{1}{s} \right]_{s=0} = W(0)$$

steady-state

or the application of the final value theorem

steady-state output

$$\boxed{y_{ss}(t) = \lim_{s\to 0} sY_s(s) = \lim_{s\to 0} sW(s)\frac{1}{s} = W(0) = R_0}$$

dc-gain

note that $\qquad W(0) = D - CA^{-1}B$

step response: steady-state

Since the output (total) response to a step input is given by

$$y(t) = \underbrace{y_{zi}(t)}_{\text{zero-input response}} + \underbrace{y_s(t)}_{\text{step response}}$$

and the zero-input response tends to zero for an asymptotically stable system, we can state that:

> The output of a linear asymptotically stable system $W(s)$ to a unit step input tends to a constant value given by the system's gain $W(0)$

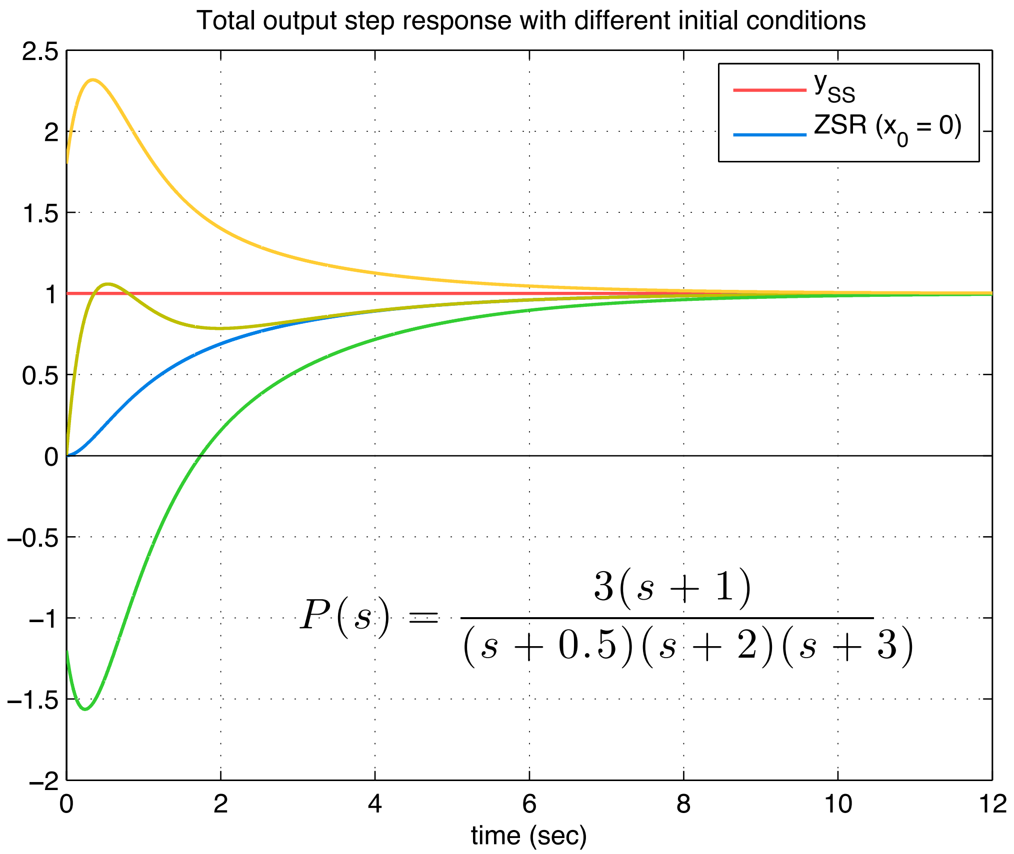
the reasoning is still true for repeated poles (provided the system is asymptotically stable)

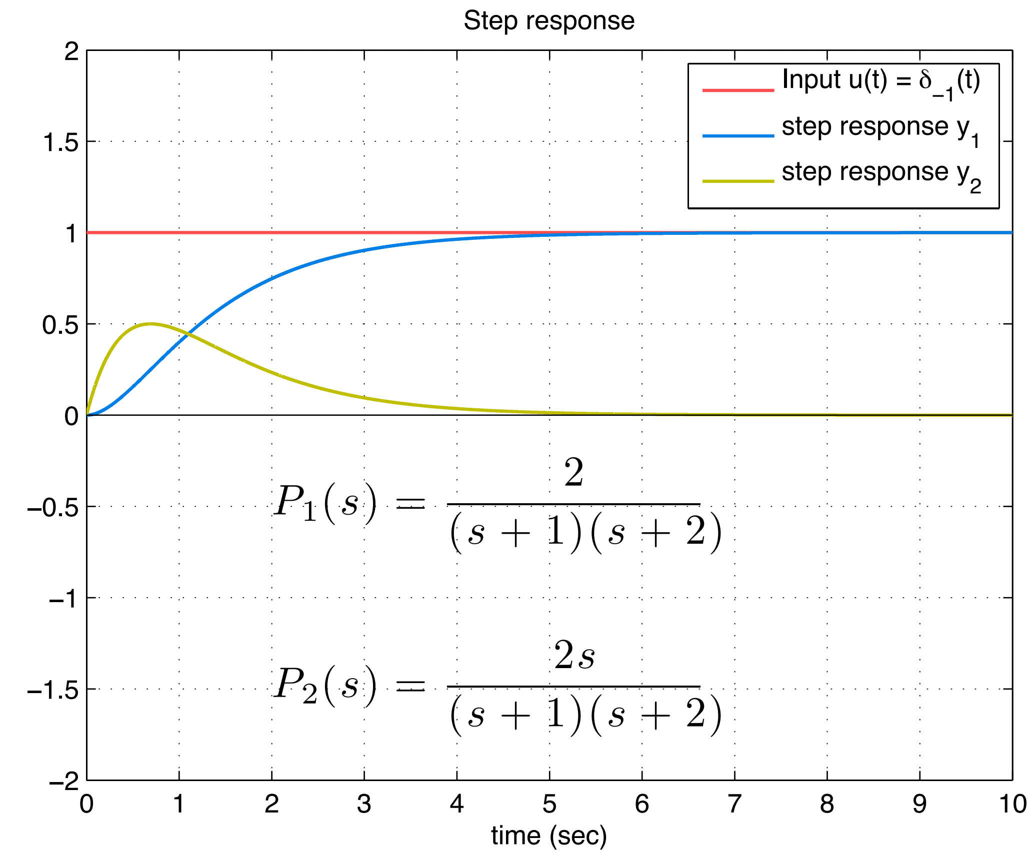
do not mix up $$y(t) = \overbrace{y_{zi}(t)}^{\text{zero-input response}} + \underbrace{\overbrace{y_s(t)}^{\text{zero-state response}}}_{\text{step response}}$$ with $$y_s(t) = \overbrace{y_t(t)}^{\text{transient response}} + \overbrace{y_{ss}(t)}^{\text{steady-state response}}$$

step response: steady-state

even with non-zero initial condition, due to the asymptotic stability of S all the responses tend to the same constant value

the final constant value coincides with the system gain which can also be zero (due to the presence of a zero in $s = 0$)





steady-state is independent from the initial conditions

$$P_2(0) = 0$$

step response: transient

transient is defined as the **forced response** minus the **steady-state** $y_t(t) = y_s(t) - y_{ss}(t)$

$$y_s(t) = R_0\delta_{-1}(t) + \sum_{i=1}^{n} R_i e^{p_i t}\delta_{-1}(t)$$

$$y_{\text{ss}}(t) = R_0$$

$\xrightarrow{\text{transient}}$

$$\boxed{y_t(t) = \sum_{i=1}^{n} R_i e^{p_i t}}$$

distinct poles case

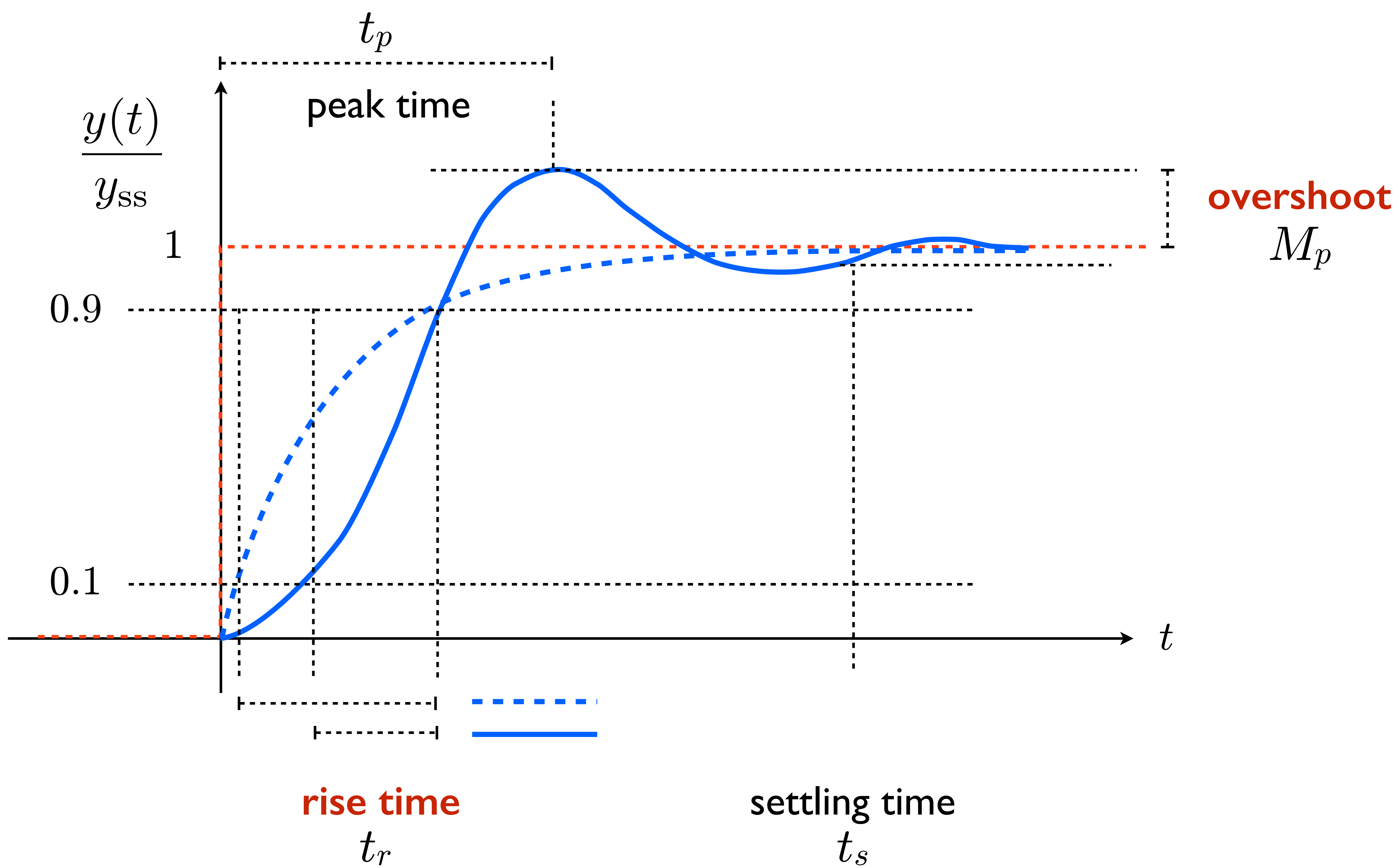
alternative

$$W(s) = \sum_{i=1}^{n} \frac{R_i'}{s - p_i} \qquad\qquad w(t) = \sum_{i=1}^{n} R_i' e^{p_i t}$$

$$y_s(t) = \int_0^t w(\tau)d\tau = \sum_{i=1}^{n} \frac{R_i'}{p_i}\left[e^{p_i t} - 1\right] = \sum_{i=1}^{n} \frac{R_i'}{p_i} e^{p_i t} \underbrace{- \sum_{i=1}^{n} \frac{R_i'}{p_i}}_{W(0)}$$

transient behavior of a system

defined on a particular time response: step response



transient

t_r **rise time**: amount of time required for the signal to go from 10% to 90% of its final value

y_{ss} **steady state value**: asymptotic output value

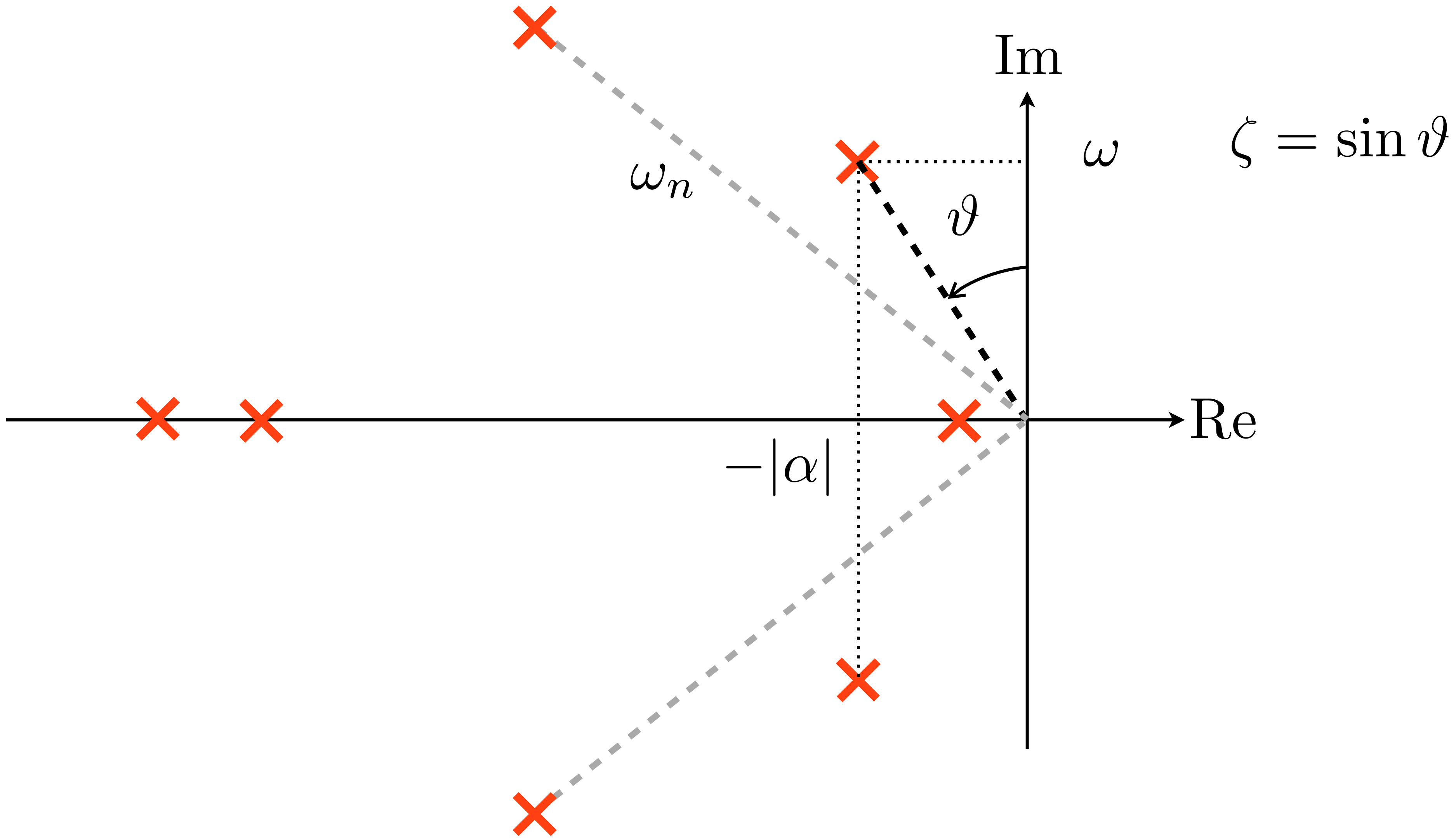
M_p **overshoot**: maximum excess of the output w.r.t. the final value (can be defined as a percentage of the final value). In a normalized $y_s(t)/y_{ss}$ plot the overshoot is given by the maximum of the normalized output minus one.

t_p **peak time**: time required for the step response to reach the overshoot

t_s **settling time**: amount of time required for the step response to stay within 2% of its final value for all future times

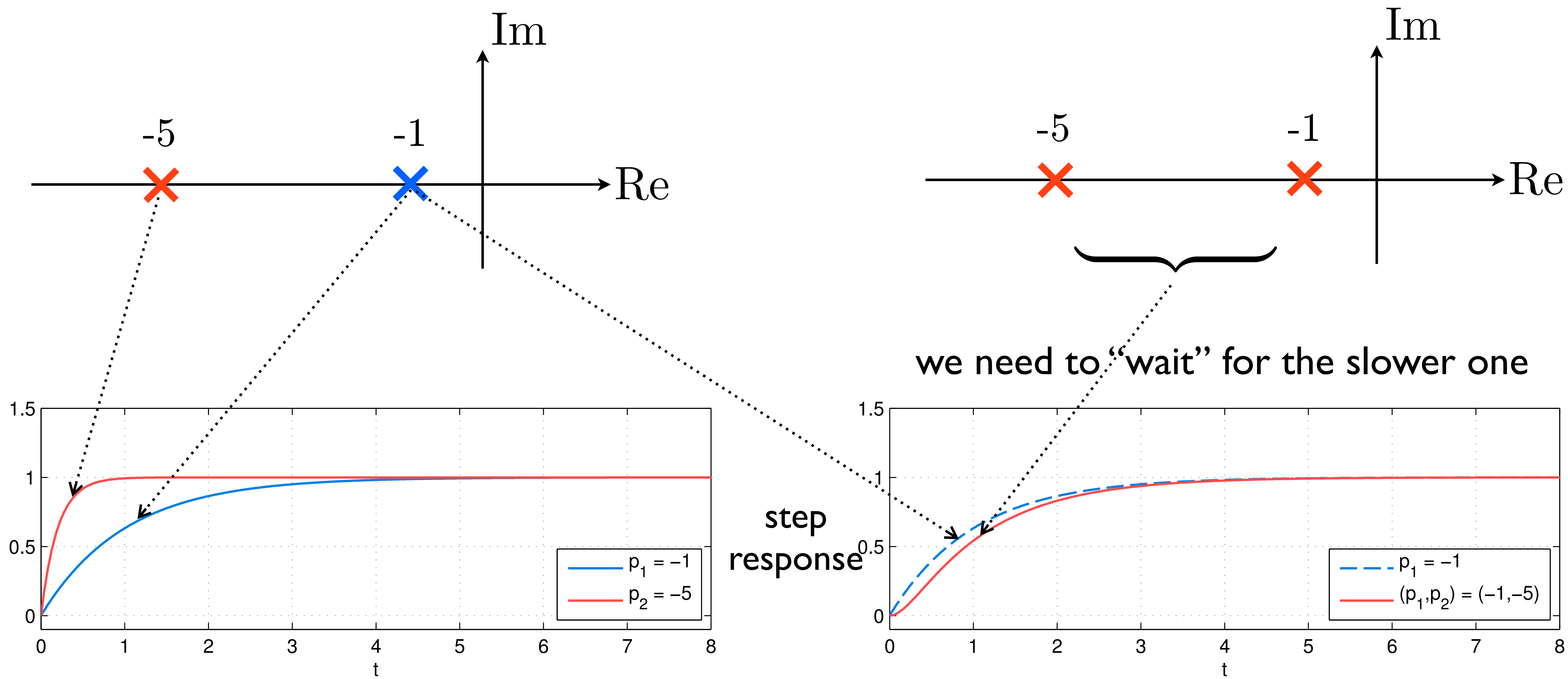
transient

Quantities related to complex plane position of the poles



transient

Quantities related to complex plane position of the poles (real poles)



comparison between the response of the two systems taken separately

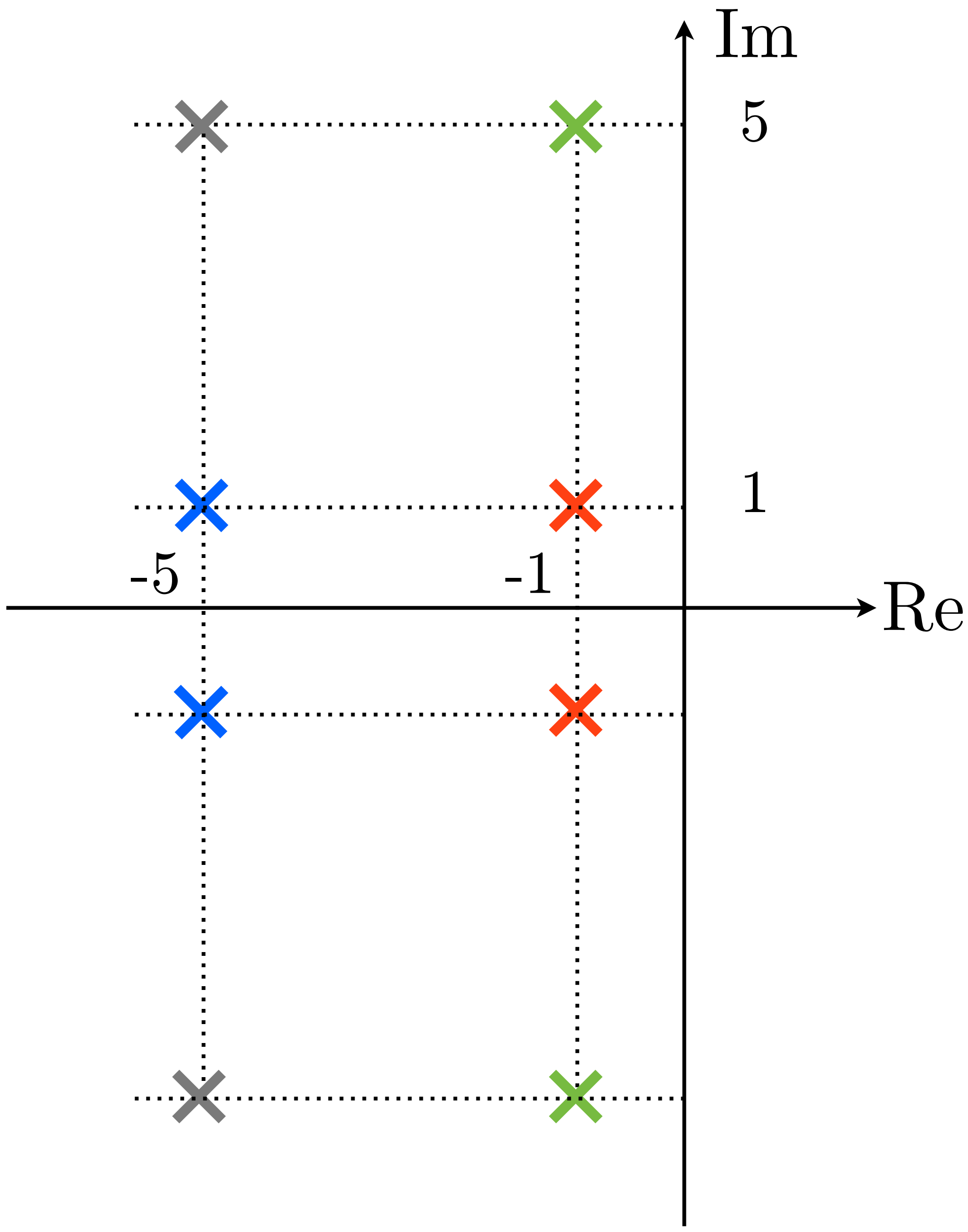
$$P_i(s) = \frac{-p_i}{s - p_i}$$

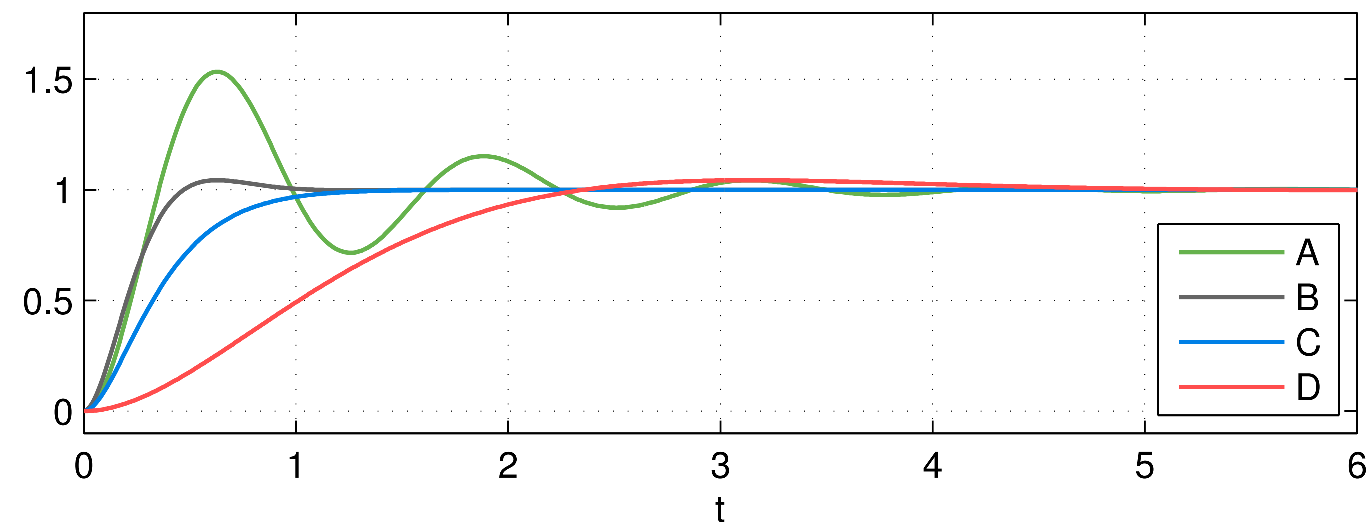
single system with both poles

$$P(s) = \frac{p_1 p_2}{(s - p_1)(s - p_2)}$$

transient

quantities related to complex plane position of the poles (complex poles)

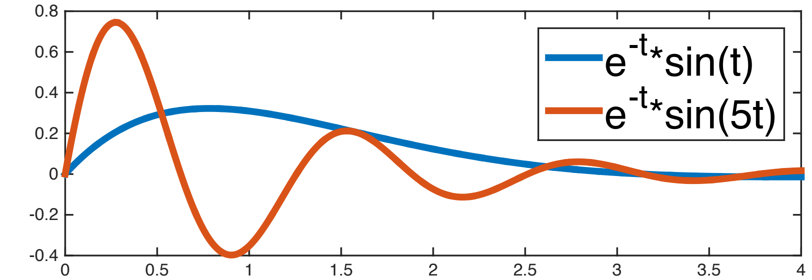




comparison between four systems with pair of complex poles and unit gain

$$P(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

explain why **x** is faster than **x** even if eigenvalues have same real part



steady state

the steady state will be also defined for other classes of inputs (not only step)

Existence

we require

1. the existence of the steady state also for the state (not only for the output)
2. and the independence, of the asymptotic behavior, from the initial conditions $x(0)$

being
$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

$x(t)$ will be independent for $t \to \infty$ from the initial condition iff all the modes are converging (all the eigenvalues have negative real part)

i.e. the system is **asymptotically stable**

steady state

- exists **only** for **asymptotically stable** systems
- is **independent** from the **initial state**
- **depends** on the particular **input** applied to the system

polynomial inputs $\longrightarrow$ polynomial steady state

sinusoidal inputs $\longrightarrow$ sinusoidal steady state

(canonical test signals)

polynomial input

let the canonical input (signal of order k) be $u(t) = \frac{t^k}{k!}\delta_{-1}(t)$ with transform $U(s) = \frac{1}{s^{k+1}}$

for an asymptotically stable system with distinct poles, the output is

$$Y(s) = P(s)U(s) = \frac{N(s)}{\prod_{i=1}^{n}(s-p_i)} \frac{1}{s^{k+1}}$$

and expanding

$$Y(s) = \frac{R_{11}}{s} + \frac{R_{12}}{s^2} + \cdots + \frac{R_{1,k+1}}{s^{k+1}} + \boxed{\sum_{i=1}^{n} \frac{R_i}{s-p_i}}$$

these give in t contributions that tend to 0 as t increases

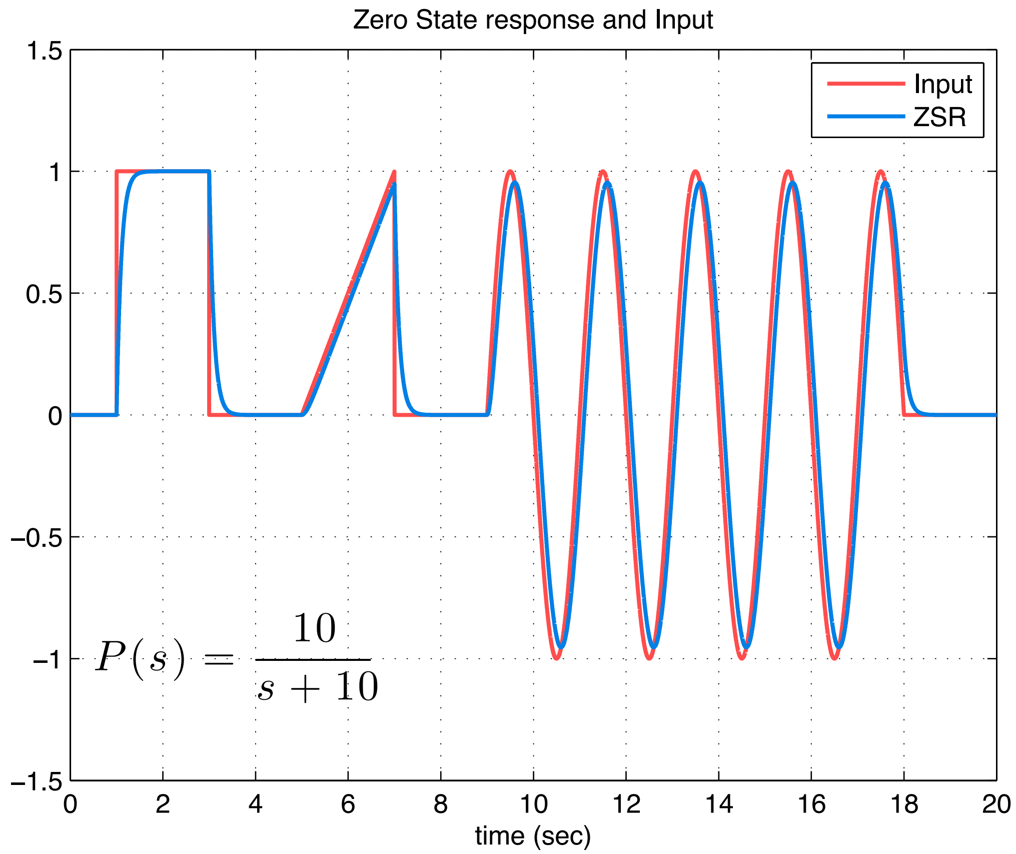
$$Y_{\mathrm{ss}}(s) = \frac{R_{11}}{s} + \frac{R_{12}}{s^2} + \cdots + \frac{R_{1,k+1}}{s^{k+1}}$$

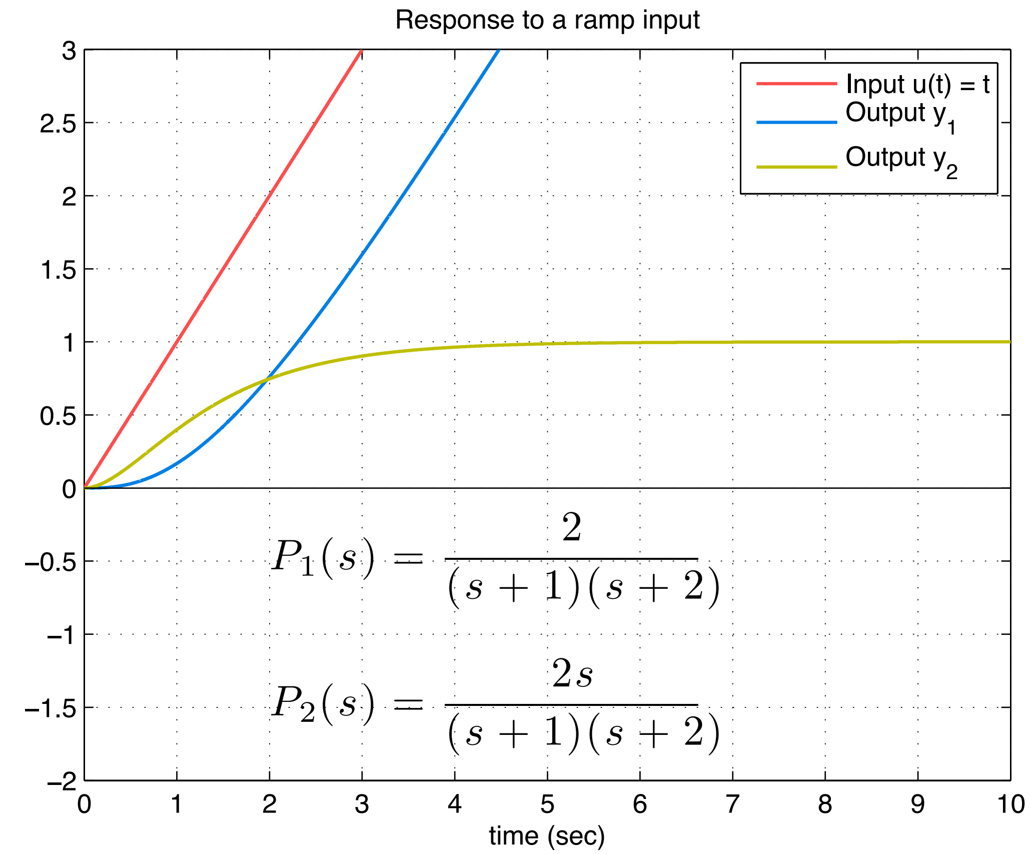
$$u(t) = \frac{t^k}{k!}\delta_{-1}(t) \xrightarrow{\text{steady state}} \boxed{y_{\mathrm{ss}}(t) = \left(R_{11} + R_{12}t + \cdots + R_{1,k+1}\frac{t^k}{k!}\right)\delta_{-1}(t)}$$

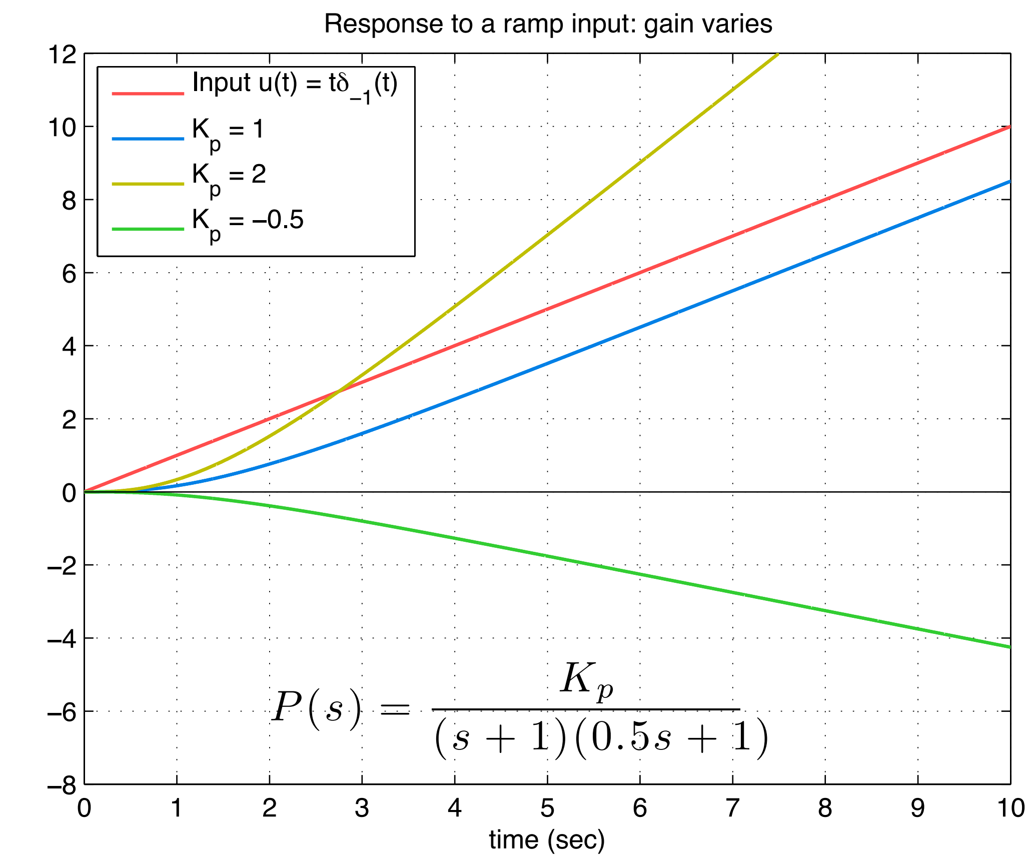
note that

$$R_{i,k+1} = \left[\frac{1}{((k+1)-(k+1))!} \frac{d^{((k+1)-(k+1))}}{s^{((k+1)-(k+1))}} s^{k+1} \frac{1}{s^{s+1}} P(s)\right]_{s=0} = P(0)$$

steady state







sinusoidal input

input $u(t) = \sin \bar{\omega} t = \dfrac{e^{j\bar{\omega}t} - e^{-j\bar{\omega}t}}{2j}$ $\longrightarrow$ $U(s) = \mathcal{L}\{\sin \bar{\omega} t\} = \dfrac{\bar{\omega}}{s^2 + \bar{\omega}^2} = \dfrac{\bar{\omega}}{(s + j\bar{\omega})(s - j\bar{\omega})}$

system (asymptotically stable) $P(s) = \dfrac{N(s)}{\prod(s - p_i)}$

output $Y(s) = P(s)U(s) = \dfrac{N'(s)}{\prod(s - p_i)} + \dfrac{R_1}{s - j\bar{\omega}} + \dfrac{R_2}{s + j\bar{\omega}}$

with

$$\begin{aligned} R_1 &= [(s - j\bar{\omega})Y(s)]_{s=j\bar{\omega}} = \left[P(s)\frac{\bar{\omega}}{s + j\bar{\omega}}\right]_{s=j\bar{\omega}} = \frac{1}{2j}P(j\bar{\omega}) \\ R_2 &= [(s + j\bar{\omega})Y(s)]_{s=-j\bar{\omega}} = \left[P(s)\frac{\bar{\omega}}{s - j\bar{\omega}}\right]_{s=-j\bar{\omega}} = -\frac{1}{2j}P(-j\bar{\omega}) = R_1^* \end{aligned}$$

rational function $P(-j\omega) = P^*(j\omega)$

sinusoidal input

asymptotic stability

$$\mathcal{L}^{-1}\left\{\frac{N'(s)}{\prod(s-p_i)}\right\} = \mathcal{L}^{-1}\left\{\sum\sum \frac{R_{ik}}{(s-p_i)^{m_i-k}}\right\} \quad \to 0 \qquad \text{when} \qquad t \to \infty$$

asymptotic behavior (steady-state) is

$$y_{\text{ss}}(t) = \mathcal{L}^{-1}\left\{\frac{R_1}{s-j\bar{\omega}} + \frac{R_2}{s+j\bar{\omega}}\right\}$$

but being $P(j\bar{\omega}) = |P(j\bar{\omega})|e^{j\angle P(j\bar{\omega})}$ with $|P(-j\bar{\omega})| = |P(j\bar{\omega})|$ and $\angle P(-j\bar{\omega}) = -\angle P(j\bar{\omega})$

$P(-j\omega) = P^*(j\omega)$

we have

$$\begin{aligned} y_{\text{ss}}(t) &= R_1 e^{j\bar{\omega}t} + R_2 e^{-j\bar{\omega}t} \\ &= \frac{1}{2j}\left(P(j\bar{\omega})e^{j\bar{\omega}t} - P(-j\bar{\omega})e^{-j\bar{\omega}t}\right) \\ &= \frac{1}{2j}\left(|P(j\bar{\omega})|e^{j\angle P(j\bar{\omega})}e^{j\bar{\omega}t} - |P(-j\bar{\omega})|e^{-j\angle P(j\bar{\omega})}e^{-j\bar{\omega}t}\right) \\ &= \frac{|P(j\bar{\omega})|}{2j}\left(e^{j(\bar{\omega}t+\angle P(j\bar{\omega}))} - e^{-j(\bar{\omega}t+\angle P(j\bar{\omega}))}\right) \\ &= |P(j\bar{\omega})|\sin(\bar{\omega}t + \angle P(j\bar{\omega})) \end{aligned}$$

sinusoidal input

the steady-state response of an asymptotically stable system $P(s)$
to a sinusoidal input $u(t) = \sin \bar{\omega} t$ is given by

$$\boxed{y_{\text{ss}}(t) = |P(j\bar{\omega})| \sin(\bar{\omega} t + \angle P(j\bar{\omega}))}$$

- steady-state has **same frequency** than input
- can be

  amplified $|P(j\bar{\omega})| > 1$

  attenuated $|P(j\bar{\omega})| < 1$
- also phase variation
- depends only on the frequency of the input and the system characteristics

$\vert P(j\omega)\vert$	$\angle P(j\omega)$
gain curve	**phase curve**

sinusoidal input