# Chapter 5

# Elastic joints

This chapter deals with modelling and control of robot manipulators with joint flexibility. The presence of such a flexibility is a common aspect in many current industrial robots. When motion transmission elements such as harmonic drives, transmission belts and long shafts are used, a dynamic time-varying displacement is introduced between the position of the driving actuator and that of the driven link.

Most of the times, this intrinsic small deflection is regarded as a source of problems, especially when accurate trajectory tracking or high sensitivity to end-effector forces is mandatory. In fact, an oscillatory behaviour is usually observed when moving the links of a robot manipulator with nonnegligible joint flexibility. These vibrations are of small magnitude and occur at relatively high frequencies, but still within the bandwidth of interest for control.

On the other hand, there are cases when compliant elements (in our case, at the joints) may become useful in a robotic structure, e.g., as a protection against unexpected "hard" contacts during assembly tasks. Moreover, when using harmonic drives, the negative side effect of flexibility is balanced by the benefit of working with a compact, in-line component, with high reduction ratio and large power transmission capability.

From the modelling viewpoint, the above deformation can be characterized as being *concentrated* at the joints of the manipulator, and thus we often refer to this situation by the term *elastic joints* in lieu of flexible joints. This is a main feature to be recognized, because it will limit the complexity both of the model derivation and of the control synthesis. In particular, we emphasize the difference with lightweight manipulator links, where flexibility involves bodies of larger mass (as opposed to an elastic transmission shaft) undergoing deformations distributed over longer segments. In that case, flexibility cannot be reduced to an effect concentrated at the joint. As we will see, this has relevant consequences in the control analysis and design; the case of flexible joints should then be treated separately from that of flexible links.

We also remark that the assumption of perfect rigidity is an ideal one for all robot manipulators. However, the primary concern in deriving a mathematical model including any kind of flexibility is to evaluate quantitatively its relative effects, as superimposed to the rigid body motion. The additional modelling effort allows verifying whether a control law derived on the rigidity assumption (valid for rigid manipulators) will still work in practice, or should be modified and if so up to what extent. If high performance cannot be reached in this way, new specific control laws should be investigated, explicitly based on the more complete manipulator model.

When compared to the rigid case, the dynamic model of robot manipulators with elastic joints (but rigid links) requires *twice* the number of generalized coordinates to completely characterize the configuration of all rigid bodies (motors and links) constituting the manipulator. On the other hand, since actual joint deformations are quite small, elastic forces are typically limited to a domain of linearity.

The case of elastic joint manipulators is a first example in which the number of control inputs does not reach the number of mechanical degrees of freedom. Therefore, control tasks are supposed to be more difficult than the equivalent ones for rigid manipulators. In particular, the implementation of a full state feedback control law will require twice the number of sensors, measuring quantities that are *before and after* (or across) the elastic deformation.

Conversely, the strong couplings imposed by the elastic joints are helpful in obtaining a convenient behaviour for all variables. Also, both the elastic and input torques act on the same joint axes (they are *physically co-located*) and this induces nice control characteristics to the system.

The control goal is then to properly handle the vibrations induced by elasticity at the joints, so as to achieve fast positioning and accurate tracking at the manipulator end-effector level. In the following, in view of the assumed link rigidity, the task space control problem is not considered, but attention will be focused only on the motion of the links, i.e., in the *joint space*. In fact, by a proper choice of coordinates, the direct kinematics of a manipulator with elastic joints is exactly the same as that of a rigid manipulator, and no further problems arise in this respect. Notice that the position of a link is a variable already *beyond* the point where elasticity is introduced, and thus the control objective is definitely not a restricted one.

This chapter is organized in two parts, covering modelling issues and control problems.

First, the dynamic equations of general elastic joint robot manipulators are derived, their internal structure is highlighted, and possible simplifications are discussed. There are two different, and both common, modelling assumptions that lead to two kinds of dynamic models: a *complete* and a *reduced* one. In what follows, it will be clear that these two models do not share the same structural properties from the control point of view. Furthermore, when the manipulator elastic joints are relatively stiff, it will be possible to suitably rewrite the dynamic equations in a singularly perturbed form.

A series of control strategies are then investigated for the problems of set-point *regulation* and trajectory *tracking control*.

For point-to-point motion, *linear controllers* usually provide satisfactory performance. A single link driven through an elastic joint and moving on the horizontal plane is introduced as a paradigmatic case study for showing properties and difficulties encountered even in a linear setting. This simple example shows immediately what can be achieved with different sets of state measurements. More in general, the inclusion of gravity will be handled by the addition of a constant compensation term to a PD controller, with feedback only from the motor variables.

For the *reduced model* of multilink elastic joint manipulators, the trajectory tracking problem is solved using two nonlinear control methods: *feedback linearization* and *singular perturbation*. The former is a global exact method, while the latter exploits the feasibility of an incremental design, moving from the rigid case up to the desired accuracy order.

When the *complete model* is considered, use of *dynamic state feedback* for obtaining exact linearization and decoupling will be illustrated by means of an example. We also show that the key assumption in this result holds for the whole class of manipulators with elastic joints, thus guaranteeing that full linearization can be achieved in general. Finally, a *nonlinear regulation* approach will be presented; its implementation is quite simple, since the computational burden is considerably reduced.

Only nonadaptive control schemes based on full or partial state feedback will be discussed. Results for the unknown parameter case and on the use of state observers are quite recent and not yet completely settled down. They can be found in the list of references at the end of the chapter for further reading.

## 5.1 Modelling

We refer to a robot manipulator with elastic joints as an open kinematic chain having $n + 1$ rigid bodies, the base (link 0) and the $n$ links, inter-

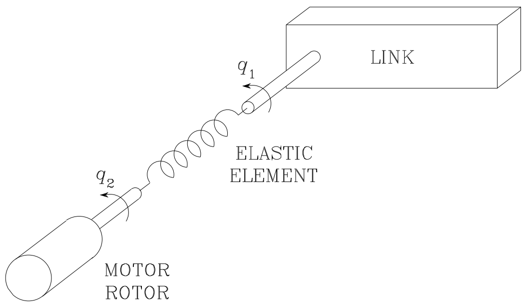


Figure 5.1: Schematic representation of an elastic joint.

connected by $n$ joints undergoing elastic deformation. The manipulator is actuated by electrical drives which are assumed to be located at the joints. More specifically, we consider the standard situation in which motor $i$ is mounted on link $i-1$ and moves link $i$. When reduction gears are present, they are modelled as being placed ***before*** the elastic element. All joints are considered to be elastic, though mixed situations may be encountered in practice due to the use of different transmission devices. The following quite general assumptions are made about the mechanical structure.

**Assumption 5.1** Joint deformations are small, so that elastic effects are restrained to the domain of linearity.

□

**Assumption 5.2** The elasticity in the joint is modelled as a spring, torsional for revolute joints and linear for prismatic joints; Fig. 5.1 shows an elastic revolute joint.

□

**Assumption 5.3** The rotors of the actuators are modelled as uniform bodies having their centers of mass on the rotation axes.

□

We emphasize the relevance of Assumption 5.3 on the geometry of the rotors: it implies that both the inertia matrix and the gravity term in the dynamic model are independent of the actual internal position of the motors.

Following the usual Lagrange formulation, a set of generalized coordinates has to be introduced to characterize uniquely the system configuration. Since the manipulator chain is composed of $2n$ rigid bodies, $2n$ coordinates are needed. Let $q_1$ be the $(n \times 1)$ vector of link positions, and

$q_2$ represents the $(n \times 1)$ vector of actuator (rotor) positions, as reflected through the gear ratios. With this choice, the difference $q_{1i} - q_{2i}$ is joint $i$ deformation. Moreover, the direct kinematics of the whole manipulator (and of each link end point) will be a function of the link variables $q_1$ only.

The *kinetic energy* of the manipulator structure is given as usual by

$$T = \frac{1}{2}\dot{q}^T H(q)\dot{q}, \tag{5.1}$$

where $q = (q_1^T \quad q_2^T)^T$ and $H(q)$ is the $(2n \times 2n)$ inertia matrix, which is symmetric and positive definite for all $q$. Moreover, for revolute joints all elements of $H(q)$ are bounded. According to the previous assumptions, $H(q)$ has the following internal structure:

$$H(q) = H(q_1) = \begin{pmatrix} H_1(q_1) & H_2(q_1) \\ H_2^T(q_1) & H_3 \end{pmatrix}. \tag{5.2}$$

All blocks in (5.2) are $(n \times n)$ matrices: $H_1$ contains the inertial properties of the rigid links, $H_2$ accounts for the inertial couplings between each spinning actuator and the previous links, while $H_3$ is the constant diagonal matrix depending on the rotor inertias of the motors and on the gear ratios.

The *potential energy* is given by the sum of two terms. The first one is the gravitational term for both actuators and links; on the symmetric mass assumption for the rotors, it takes on the form

$$U_g = U_g(q_1). \tag{5.3}$$

The second one, arising from joint elasticity, can be written as

$$U_e = \frac{1}{2}(q_1 - q_2)^T K(q_1 - q_2) \tag{5.4}$$

in which $K = \mathrm{diag}\{k_1, \ldots, k_n\}$ is the joint *stiffness matrix*, $k_i > 0$ being the elastic constant of joint $i$. By defining the matrix

$$K_e = \begin{pmatrix} K & -K \\ -K & K \end{pmatrix}, \tag{5.5}$$

the elastic energy (5.4) can be rewritten as

$$U_e = \frac{1}{2}q^T K_e q. \tag{5.6}$$

The dynamic equations of motion are obtained from the Lagrangian function $L(q, \dot{q}) = T(q, \dot{q}) - U(q)$ as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = e_i \qquad i = 1, \ldots, 2n \tag{5.7}$$

where $e_i$ is the generalized force performing work on $q_i$. Since only the motor coordinates $q_2$ are directly actuated, we collect all forces on the right-hand side of (5.7) in the $(2n \times 1)$ vector

$$e = \begin{pmatrix} 0 & \dots & 0 & u_1 & \dots & u_n \end{pmatrix}^T, \tag{5.8}$$

where $u_i$ denotes the external torque supplied by the motor at joint $i$. Link coordinates $q_1$ are indirectly actuated only through the elastic coupling.

Computing the derivatives needed in (5.7) leads to the set of $2n$ second-order nonlinear differential equations of the form

$$H(q_1)\ddot{q} + C(q, \dot{q})\dot{q} + K_e q + g(q_1) = e, \tag{5.9}$$

in which the Coriolis and centrifugal terms are

$$C(q, \dot{q})\dot{q} = \dot{H}(q_1)\dot{q} - \frac{1}{2}\left(\frac{\partial}{\partial q}(\dot{q}^T H(q_1)\dot{q})\right)^T, \tag{5.10}$$

and the gravity vector is

$$g(q_1) = \left(\frac{\partial U_g(q_1)}{\partial q}\right)^T = \begin{pmatrix} g_1(q_1) \\ 0 \end{pmatrix}, \tag{5.11}$$

with $g_1 = (\partial U_g / \partial q_1)^T$. Eq. (5.9) is also said to be the *full model* of an elastic joint manipulator.

Viscous friction terms acting both on the link and on the motor sides of the elastic joints could be easily included in the dynamic model.

### 5.1.1 Dynamic model properties

Referring to the general dynamic model (5.9), the following useful properties can be derived, some of which are already present for the rigid manipulator model.

**Property 5.1** The elements of $C(q, \dot{q})$ can always be defined so that the matrix $\dot{H} - 2C$ is *skew-symmetric*. In particular, one such feasible choice is provided by the Christoffel symbols, i.e.,

$$C_{ij}(q, \dot{q}) = \frac{1}{2}\left(\frac{\partial H_{ij}}{\partial q}\dot{q} + \sum_{k=1}^{2n}\left(\frac{\partial H_{ik}}{\partial q_j} - \frac{\partial H_{jk}}{\partial q_i}\right)\dot{q}_k\right), \tag{5.12}$$

for $i, j = 1, \dots, 2n$.

□

**Property 5.2** If $C(q,\dot{q})$ is defined by (5.12), then it can be decomposed as

$$C(q,\dot{q}) = C_A(q_1,\dot{q}_2) + C_B(q_1,\dot{q}_1), \tag{5.13}$$

with

$$C_A(q_1,\dot{q}_2) = \begin{pmatrix} C_{A1}(q_1,\dot{q}_2) & 0 \\ 0 & 0 \end{pmatrix} \tag{5.14}$$

$$C_B(q_1,\dot{q}_1) = \begin{pmatrix} C_{B1}(q_1,\dot{q}_1) & C_{B2}(q_1,\dot{q}_1) \\ C_{B3}(q_1,\dot{q}_1) & 0 \end{pmatrix} \tag{5.15}$$

where the elements of the $(n \times n)$ matrices $C_{A1}$, $C_{B1}$, $C_{B2}$, $C_{B3}$ are:

$$C_{A1ij}(q_1,\dot{q}_2) = \frac{1}{2}\left(\frac{\partial(H_2)_i}{\partial q_{1j}} - \frac{\partial(H_2)_j}{\partial q_{1i}}\right)\dot{q}_2 \tag{5.16}$$

$$C_{B1ij}(q_1,\dot{q}_1) = \frac{1}{2}\left(\frac{\partial H_{1ij}}{\partial q_1}\dot{q}_1 + \left(\frac{\partial(H_1)_i}{\partial q_{1j}} - \frac{\partial(H_1)_j}{\partial q_{1i}}\right)\dot{q}_1\right) \tag{5.17}$$

$$C_{B2ij}(q_1,\dot{q}_1) = \frac{1}{2}\left(\frac{\partial H_{2ij}}{\partial q_1}\dot{q}_1 - \frac{\partial(H_2^T)_j}{\partial q_{1i}}\dot{q}_1\right) \tag{5.18}$$

$$C_{B3ij}(q_1,\dot{q}_1) = \frac{1}{2}\left(\frac{\partial H_{2ji}}{\partial q_1}\dot{q}_1 + \frac{\partial(H_2^T)_i}{\partial q_{1j}}\dot{q}_1\right) \tag{5.19}$$

with $(H)_i$ denoting the $i$-th *row* of a matrix $H$. These expressions follow directly from the dependency of the inertia matrix (5.2) and from Property 5.1.

□

**Property 5.3** Matrix $H_2(q_1)$ has the upper triangular structure

$$\begin{pmatrix} 0 & H_{212}(q_{11}) & H_{213}(q_{11},q_{12}) & \cdots & H_{21n}(q_{11},\ldots,q_{1,n-1}) \\ 0 & 0 & H_{223}(q_{12}) & \cdots & H_{22n}(q_{12},\ldots,q_{1,n-1}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & H_{2,n-1,n}(q_{1,n-1}) \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{5.20}$$

where the most general cascade dependence is shown for each single term. Indeed, the elements of $H_2$ can be obtained as

$$H_{2ij} = \frac{\partial^2 T}{\partial \dot{q}_{1j}\partial \dot{q}_{2i}}, \tag{5.21}$$

where the kinetic energy $T$ is given by the sum of the kinetic energy of each link (including the stator of the successive motor) and of each motor rotor.

However, since the total kinetic energy of the links is a quadratic form of $\dot{q}_1$ only, by virtue of the chosen variable definition, contributions to $H_2$ may only come from that part of $T$ which is due to the rotors. For rotor $i$, the kinetic energy is given by

$$T_{ri} = \frac{1}{2} m_{ri} {}^{r_i}v_{2i}^{T} {}^{r_i}v_{2i} + \frac{1}{2} {}^{r_i}\omega_{2i}^{T} {}^{r_i}I_{ri} {}^{r_i}\omega_{2i} \tag{5.22}$$

where ${}^{r_i}v_{2i}$ and ${}^{r_i}\omega_{2i}$ are respectively the linear and angular velocity of the rotor expressed in the frame $r_i$ attached to the corresponding stator, while $m_{ri}$ and ${}^{r_i}I_{ri}$ are respectively the mass and the inertia tensor of the rotor. Since the rotor center of mass lies on its axis of rotation, only the second term on the right-hand side of (5.22) will contribute to $H_2$. The angular velocity ${}^{r_i}\omega_{2i}$ can be calculated recursively as (for revolute joints)

$$\begin{aligned} {}^{r_i}\omega_{2i} &= {}^{r_i}R_{i-1} {}^{i-1}\omega_{1,i-1} + \dot{q}_{2i} {}^{r_i}a_{2i} \\ {}^{i}\omega_{1i} &= {}^{i}R_{i-1}(q_{1i}) {}^{i-1}\omega_{1,i-1} + \dot{q}_{1i} {}^{i}a_{1i} \end{aligned} \tag{5.23}$$

where ${}^{i}\omega_{1i}$ is the angular velocity of link $i$ in the frame $i$ attached to the link itself, ${}^{r_i}R_{i-1}$ is the constant $(3 \times 3)$ rotation matrix from frame $r_i$ attached to the rotor to frame $i-1$, ${}^{r_i}a_{2i} = (0 \quad 0 \quad 1)^T$, ${}^{i}R_{i-1}$ is the $(3 \times 3)$ rotation matrix from frame $i$ to frame $i-1$, and ${}^{i}a_{1i} = (0 \quad 0 \quad 1)^T$. Eqs. (5.22) and (5.23) imply (5.20).

□

**Property 5.4** A positive constant $\alpha$ exists such that

$$\left\| \frac{\partial g_1(q_1)}{\partial q_1} \right\| \leq \alpha \qquad \forall q_1. \tag{5.24}$$

This property follows from the fact that $g_1(q_1)$ is formed by trigonometric functions of the link variables $q_{1i}$ in the case of revolute joints, and also by linear functions in $q_{1i}$ if some prismatic joint is present. The previous inequality implies, by the mean value theorem, that

$$\|g_1(q_1) - g_1(q_1')\| \leq \alpha \|q_1 - q_1'\| \qquad \forall q_1, q_1'. \tag{5.25}$$

□

### 5.1.2 Reduced models

In many common manipulator kinematic arrangements, the block $H_2$ in the inertia matrix of the elastic joint model (5.9) is *constant*. For instance, this occurs in the case of a two-revolute-joint planar arm or of a three-revolute joint anthropomorphic manipulator. This implies several simplifications in the dynamic model, due to vanishing of terms. In particular,

$$H_2 = \text{const} \quad \Longrightarrow \quad C_{A1} = C_{B2} = C_{B3} = 0 \tag{5.26}$$

so that Coriolis and centrifugal terms, which are always independent of $q_2$, become also independent of $\dot{q}_2$. As a result of (5.26), model (5.9) can be rewritten in partitioned form as

$$\begin{aligned} H_1(q_1)\ddot{q}_1 + H_2\ddot{q}_2 + C_1(q_1,\dot{q}_1)\dot{q}_1 + K(q_1 - q_2) + g_1(q_1) &= 0 \\ H_2^T\ddot{q}_1 + H_3\ddot{q}_2 + K(q_2 - q_1) &= u, \end{aligned} \tag{5.27}$$

where $C_1 = C_{B1}$ for compactness. Note that no velocity terms appear in the second set of $n$ equations, the one associated with the motor variables.

For some special kinematic structures it is found that $H_2 = 0$ and further simplifications are induced; this is the case of a single elastic joint, and of a 2-revolute-joint polar arm, i.e., with orthogonal joint axes. As a consequence, no inertial couplings are present between the link and motor dynamics, i.e.,

$$\begin{aligned} H_1(q_1)\ddot{q}_1 + C_1(q_1,\dot{q}_1)\dot{q}_1 + K(q_1 - q_2) + g_1(q_1) &= 0 \\ H_3\ddot{q}_2 + K(q_2 - q_1) &= u. \end{aligned} \tag{5.28}$$

For general elastic joint manipulators, a *reduced model* of the form (5.28) can also be obtained by neglecting some contributions in the energy of the system. In particular, $H_2$ will be *forced to zero* if the angular part of the kinetic energy of each rotor is considered to be due only to its own rotation, i.e., $\omega_{2i} = \dot{q}_{2i}{}^{r_i}a_{2i}$ —compare with (5.23)— or

$$T_{ri} = \frac{1}{2}m_{ri}{}^{r_i}v_{2i}^T{}^{r_i}v_{2i} + \frac{1}{2}I_{mi}\dot{q}_{2i}^2 \tag{5.29}$$

with the positive scalar $I_{mi} = {}^{r_i}I_{rizz}$. When the gear reduction ratios are very large, this approximation is quite reasonable since the fast spinning of each rotor dominates the angular velocity of the previous carrying links.

We note that the full model (5.9) (or (5.27)) and the reduced model (5.28) display different characteristics with respect to certain control problems. As will be discussed later, while the reduced model is always feedback linearizable by *static* state feedback, the full model needs in general *dynamic* state feedback for achieving the same result.

### 5.1.3 Singularly perturbed model

A different modelling approach can be pursued, which is convenient for designing simplified control laws. When the joint stiffness is large, the system naturally exhibits a *two-time scale* dynamic behaviour in terms of rigid and elastic variables. This can be made explicit by properly transforming the $2n$ differential equations of motion in terms of the new generalized coordinates

$$\begin{pmatrix} q_1 \\ z \end{pmatrix} = \begin{pmatrix} I & 0 \\ -K & K \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} q_1 \\ K(q_2 - q_1) \end{pmatrix}, \tag{5.30}$$

where the $i$-th component of $z$ is the *elastic force* transmitted through joint $i$ to the driven link.

For simplicity, only the model (5.27) is considered next. Solving the second equation in (5.27) for the motor acceleration and using (5.30) gives

$$\ddot{q}_2 = H_3^{-1}(u - z - H_2^T \ddot{q}_1). \tag{5.31}$$

Substituting (5.31) into the first equation in (5.27) yields

$$\Delta(q_1)\ddot{q}_1 + C_1(q_1, \dot{q}_1)\dot{q}_1 + g_1(q_1) - (I + H_2 H_3^{-1})z + H_2 H_3^{-1} u = 0, \tag{5.32}$$

where $\Delta(q_1) = H_1(q_1) - H_2 H_3^{-1} H_2^T$ is a block appearing on the diagonal of the inverse of inertia matrix, and thus is positive definite for all $q_1$. Combining (5.32) and (5.31) yields

$$\begin{aligned} \ddot{z} &= K(\ddot{q}_2 - \ddot{q}_1) \\ &= K(I + H_3^{-1} H_2^T)\Delta^{-1}(q_1)(C_1(q_1, \dot{q}_1)\dot{q}_1 + g_1(q_1)) \\ &\quad -K\big((I + H_3^{-1} H_2^T)\Delta^{-1}(q_1)(I + H_2 H_3^{-1}) + H_3^{-1}\big)z \\ &\quad +K\big((I + H_3^{-1} H_2^T)\Delta^{-1}(q_1)H_2 H_3^{-1} + H_3^{-1}\big)u. \end{aligned} \tag{5.33}$$

Notice that the matrix premultiplying $z$ in (5.33) is always invertible, being the sum of a positive definite and a positive semi-definite matrix.

Since it is assumed that the diagonal matrix $K$ has all large and similar elements, we can extract a large common scalar factor $1/\epsilon^2 \gg 1$ from $K$

$$K = \frac{1}{\epsilon^2}\hat{K} = \frac{1}{\epsilon^2}\,\mathrm{diag}\{\hat{k}_1, \ldots, \hat{k}_n\}. \tag{5.34}$$

Then, eq. (5.33) can be compactly rewritten as

$$\epsilon^2 \ddot{z} = \hat{K} a_1(q_1, \dot{q}_1) + \hat{K} A_2(q_1) z + \hat{K} A_3(q_1) u. \tag{5.35}$$

Eqs. (5.32) and (5.35) take on the usual form of a *singularly perturbed* dynamic system once the fast time variable $\tau = t/\epsilon$ is introduced in (5.35), i.e.,

$$\epsilon^2 \ddot{z} = \epsilon^2 \frac{d^2 z}{dt^2} = \frac{d^2 z}{d\tau^2}. \tag{5.36}$$

Eq. (5.32) characterizes the *slow* dynamics of the rigid robot manipulator, while (5.35) describes the *fast* dynamics associated with the elastic joints.

We note that (5.32) and (5.35) are considerably simplified when the reduced model with $H_2 = 0$ is used. In that case, they become:

$$\begin{aligned} H_1(q_1)\ddot{q}_1 + C_1(q_1, \dot{q}_1)\dot{q}_1 + g_1(q_1) &= z \\ \epsilon^2 \ddot{z} = \hat{K} H_1^{-1}(q_1)\big(C_1(q_1, \dot{q}_1)\dot{q}_1 &+ g_1(q_1)\big) \\ -\hat{K}\big(H_1^{-1}(q_1) + H_3^{-1}\big)z &+ \hat{K} H_3^{-1} u. \end{aligned} \tag{5.37}$$

## 5.2 Regulation

We analyze first the problem of controlling the position of the end effector of a robot manipulator with joint elasticity in simple point-to-point tasks. As shown in the modelling section, this corresponds to regulation of the link variables $q_1$ to a desired *constant* value $q_{1d}$, achieved using control inputs $u$ applied to the motor side of the elastic joints. A major aspect of the presence of joint elasticity is that the feedback part of the control law may depend in general on *four* variables for each joint; namely, the motor and link position, and the motor and link velocity. However, in the most common robot manipulator configurations only *two* sensors are available for joint measurements. We will study a single elastic joint with no gravity (leading to a linear model) to point out what are the control possibilities and the drawbacks in this situation. This provides some indications on how to handle the general multilink case in presence of gravity. In particular, it will be shown that a PD controller on the motor variables and a constant gravity compensation are sufficient to ensure global asymptotic stabilization of any manipulator configuration.

### 5.2.1 Single link

Consider a single link rotating on a horizontal plane and actuated with a motor through an elastic joint coupling. For the sake of simplicity, all friction or damping effects are neglected. Let $\vartheta_m$ and $\vartheta_\ell$ be the motor and link angular positions, respectively. Then, the dynamic equations are

$$\begin{aligned} I_\ell \ddot{\vartheta}_\ell + k(\vartheta_\ell - \vartheta_m) &= 0 \\ I_m \ddot{\vartheta}_m + k(\vartheta_m - \vartheta_\ell) &= u, \end{aligned} \tag{5.38}$$

where $I_m$ and $I_\ell$ are the motor and the link inertia about the rotation axis, and $k$ is the joint stiffness. Assuming $y = \vartheta_\ell$ as system output, the open-loop transfer function is

$$\frac{y(s)}{u(s)} = \frac{k}{I_m I_\ell s^2 + (I_m + I_\ell)k} \, \frac{1}{s^2}, \tag{5.39}$$

which has all poles on the imaginary axis. Note, however, that no zeros appear in (5.39).

In the following, one position variable and one velocity variable will be used for designing a linear stabilizing feedback. Since the desired position is given in terms of the link variable ($q_{1d} = \vartheta_{\ell d}$), the most natural choice is a feedback from the *link variables*

$$u = v_1 - (k_{P\ell}\vartheta_\ell + k_{D\ell}\dot{\vartheta}_\ell), \tag{5.40}$$

where $k_{P\ell}, k_{D\ell} > 0$ and $v_1$ is the external input used for defining the set point. In this case, the closed-loop transfer function is

$$\frac{y(s)}{v_1(s)} = \frac{k}{I_m I_\ell s^4 + (I_m + I_\ell)ks^2 + kk_{D\ell}s + kk_{P\ell}}. \tag{5.41}$$

No matter how the gain values are chosen, the system is still unstable due to the vanishing coefficient of $s^3$ in the denominator of (5.41). Indeed, if some viscous friction or spring damping were present, there would exist a small interval for the two positive gains $k_{P\ell}$ and $k_{D\ell}$ which guarantees closed-loop stability; however, in that case, the obtained performance would be very poor.

Another possibility is offered by a full feedback from the *motor variables*

$$u = v_2 - (k_{Pm}\vartheta_m + k_{Dm}\dot{\vartheta}_m), \tag{5.42}$$

leading to the transfer function

$$\frac{y(s)}{v_2(s)} = \frac{k}{I_m I_\ell s^4 + I_\ell k_{Dm} s^3 + \big(I_\ell(k + k_{Pm}) + I_m k\big)s^2 + kk_{Dm}s + kk_{Pm}}. \tag{5.43}$$

It is easy to see that strictly positive values for both $k_{Pm}$ and $k_{Dm}$ are necessary and sufficient for closed-loop stability. Notice that, in the absence of gravity, the equilibrium position $\vartheta_{md}$ for the motor variable coincides with the desired link position $\vartheta_{\ell d}$, and thus the reference value is $v_2 = k_{Pm}\vartheta_{\ell d}$. However, this is no longer true when gravity is present, deflecting the joint at steady state; in that case, the value of $v_2$ has to be computed using also the model parameters.

A third feedback strategy is to use the *motor velocity* and the *link position*

$$u = v_3 - (k_{P\ell}\vartheta_\ell + k_{Dm}\dot{\vartheta}_m). \tag{5.44}$$

This combination is rather convenient since it corresponds to what is actually measured in a robotic drive, when a tachometer is mounted on the DC motor and an optical encoder senses position on the load shaft, without any knowledge about the relevance of joint elasticity. Use of (5.44) leads to

$$\frac{y(s)}{v_3(s)} = \frac{k}{I_m I_\ell s^4 + I_\ell k_{Dm} s^3 + (I_\ell + I_m) k s^2 + k k_{Dm} s + k k_{P\ell}} \tag{5.45}$$

which differs in practice from (5.43) only for the coefficient of the quadratic term in the denominator. Using Routh's criterion, asymptotic stability occurs if and only if the feedback gains are chosen as

$$0 < k_{P\ell} < k \qquad 0 < k_{Dm}. \tag{5.46}$$

Hence, the proportional feedback on the link variable should not "override" the spring stiffness. Even for a set-point task with gravity, there is no need to transform the desired reference with this scheme, since the link position error is directly available and the steady-state motor velocity is zero anyway; hence, it is $v_3 = k_{P\ell}\vartheta_{\ell d}$.

Following the same lines, it is immediate to see that the combination of motor position and link velocity feedback is always unstable. Note also that other combinations would be possible, depending on the available sensing devices. For instance, mounting a strain gauge on the transmission shaft provides a direct measure of the elastic force $z = k(\vartheta_m - \vartheta_\ell)$ for control use.

To summarize, the use of alternate output measures may be a critical issue in the presence of joint elasticity. Conversely, a full state feedback may certainly guarantee asymptotic stability; however, this would be obtained at the cost of additional sensors and would require a proper tuning of the four gains. The previous developments were presented for set-point regulation, but similar considerations apply also to tracking control. These simple facts should be carefully kept in mind when moving from this canonical linear example to the more complex nonlinear dynamics of articulated manipulators.

### 5.2.2 PD control using only motor variables

Following the results of the previous section, we focus here our attention on general multilink robot manipulators with elastic joints modelled by (5.9), i.e., with $H_2(q_1) \neq 0$. It has been shown that feeding back the motor position and velocity guarantees asymptotic stability for a single link with elastic joint and no gravity. Moving to robot manipulators under the action of gravity imposes some caution in the selection of the control gains. However, it can be shown that a simple PD controller with constant gravity compensation globally stabilizes any desired link reference position $q_{1d}$.

**Theorem 5.1** *Consider the control law*

$$u = K_P(q_{2d} - q_2) - K_D\dot{q}_2 + g_1(q_{1d}) \tag{5.47}$$

*where $K_P$ and $K_D$ are $(n \times n)$ symmetric positive definite matrices and the motor reference position $q_{2d}$ is chosen as*

$$q_{2d} = q_{1d} + K^{-1}g_1(q_{1d}). \tag{5.48}$$

*If*

$$\lambda_{\min}(K_q) = \lambda_{\min}\begin{pmatrix} K & -K \\ -K & K + K_P \end{pmatrix} > \alpha, \tag{5.49}$$

*with $\alpha$ as defined by Property 5.4, then*

$$q_1 = q_{1d} \qquad q_2 = q_{2d} \qquad \dot{q} = 0$$

*is a globally asymptotically stable equilibrium point for the closed-loop system (5.9) and (5.47).*

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**Proof.** The equilibrium positions of (5.9) and (5.47) are the solutions to

$$K(q_1 - q_2) + g_1(q_1) = 0 \tag{5.50}$$

$$K(q_1 - q_2) - K_P(q_2 - q_{2d}) + g_1(q_{1d}) = 0. \tag{5.51}$$

By recalling (5.25), we can add the null term $K(q_{2d} - q_{1d}) - g_1(q_{1d})$ to (5.50) and (5.51) leading to

$$K(q_1 - q_{1d}) - K(q_2 - q_{2d}) + g_1(q_1) - g_1(q_{1d}) = 0 \tag{5.52}$$

$$K(q_1 - q_{1d}) - (K + K_P)(q_2 - q_{2d}) = 0, \tag{5.53}$$

which can be rewritten in matrix form as

$$K_q(q - q_d) = g(q_{1d}) - g(q_1), \tag{5.54}$$

where $q_d = (q_{1d}, q_{2d})$. The inequality (5.25) enables us to write, $\forall q \neq q_d$,

$$\|K_q(q - q_d)\| \geq \lambda_{\min}(K_q)\|q - q_d\| > \alpha\|q - q_d\| \geq \|g(q_{1d}) - g(q_1)\|. \tag{5.55}$$

Hence, (5.54) has the unique solution $q = q_d$.

Define the position-dependent function

$$P_1(q) = \frac{1}{2}(q - q_d)^T K_q (q - q_d) + U_g(q_1) - q^T g(q_{1d}). \tag{5.56}$$

The stationary points of $P_1(q)$ are given by the solutions to

$$\left(\frac{\partial P_1(q)}{\partial q}\right)^T = 0 \tag{5.57}$$

which coincides with (5.54). Therefore, $P_1(q)$ has the unique stationary point $q = q_d$. Moreover,

$$\frac{\partial^2 P_1(q)}{\partial q^2} = K_q + \frac{\partial g(q_1)}{\partial q}. \tag{5.58}$$

By virtue of Property 5.4 and of (5.49), (5.58) is positive definite and thus $q = q_d$ is an absolute minimum for $P_1(q)$.

Consider now the Lyapunov function candidate

$$V(q, \dot{q}) = \frac{1}{2}\dot{q}^T H(q_1)\dot{q} + P_1(q) - P_1(q_d) \tag{5.59}$$

that is positive definite with respect to $q = q_d$, $\dot{q} = 0$. The time derivative of (5.59) along a closed-loop trajectory is given by

$$\begin{aligned}\dot{V}(q, \dot{q}) = &\frac{1}{2}\dot{q}^T \dot{H}(q_1)\dot{q} - \dot{q}^T\big(C(q, \dot{q})\dot{q} + Kq + g(q_1)\big)\\ &-\dot{q}_2^T K_P (q_2 - q_{2d}) - \dot{q}_2^T K_D \dot{q}_2 + \dot{q}_2^T g_1(q_{1d})\\ &+\dot{q}^T K_q (q - q_d) + \left(\frac{\partial U_g(q_1)}{\partial q}\right)^T \dot{q} - \dot{q}^T g(q_{1d}).\end{aligned} \tag{5.60}$$

By recalling Property 5.1, (5.60) reduces to

$$\dot{V}(q, \dot{q}) = -\dot{q}_2^T K_D \dot{q}_2 + (\dot{q}_1 - \dot{q}_2)^T \big(K(q_{2d} - q_{1d}) - g_1(q_{1d})\big) \tag{5.61}$$

which, in turn, by virtue of (5.48) becomes

$$\dot{V}(q, \dot{q}) = -\dot{q}_2^T K_D \dot{q}_2. \tag{5.62}$$

Therefore, $\dot{V}$ is negative semi-definite and vanishes if and only if $\dot{q}_2 = 0$. Imposing this condition in (5.9) for all times and recalling the structure of terms from Property 5.2, we get

$$H_1(q_1)\ddot{q}_1 + C_{B1}(q_1, \dot{q}_1)\dot{q}_1 + Kq_1 + g_1(q_1) = Kq_2 = \text{const} \tag{5.63}$$

and

$$H_2^T(q_1)\ddot{q}_1 + C_{B3}(q_1,\dot{q}_1)\dot{q}_1 - Kq_1 = -Kq_2 - K_P(q_2 - q_{2d}) + g_1(q_{1d}) = \text{const.} \tag{5.64}$$

By taking (5.20) and (5.19) into account, the first scalar equation in (5.64) becomes

$$q_{11} = \text{const.} \tag{5.65}$$

Substitution of (5.65) into the second scalar equation in (5.64) yields $q_{12} =$ const. Proceeding in the same way, we finally obtain

$$q_1 = \text{const.} \tag{5.66}$$

This, substituted into (5.63) and (5.64), leads to

$$\begin{aligned} K(q_1 - q_2) + g_1(q_1) &= 0 \\ K(q_2 - q_1) + K_P(q_2 - q_{2d}) - g_1(q_{1d}) &= 0. \end{aligned} \tag{5.67}$$

Since, as previously shown, (5.67) has the unique solution $q = q_d$, then $q = q_d$, $\dot{q} = 0$ is the largest invariant subset in the set $\dot{V} = 0$. The thesis is proved by applying La Salle's theorem.

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**Remarks**

- The assumption $\lambda_{\min}(K_q) > \alpha$ in the above theorem is not restrictive; in fact, joint stiffness $K$ dominates gravity so that, by increasing the smallest eigenvalue of $K_P$, inequality (5.49) can always be satisfied.

- The PD control law (5.47) is robust with respect to some model uncertainty. In particular, asymptotic stability is guaranteed even though the inertial parameters of the manipulator are not known. Conversely, uncertainty on the gravitational and elastic parameters may affect the performance of the controller since these terms appear explicitly in the control law (see also (5.48)). However, it can be shown that the PD controller is still stable subject to uncertainty on these parameters, but the equilibrium point of the closed-loop system is, in general, different from the desired one. If $\widehat{g}_1(q_1)$ and $\widehat{K}$ are the available estimates of the gravity vector and of the stiffness matrix, then the control law

$$u = K_P(\widehat{q}_{2d} - q_2) - K_D\dot{q}_2 + \widehat{g}_1(q_{1d}) \tag{5.68}$$

with

$$\widehat{q}_{2d} = q_{1d} + \widehat{K}^{-1}\widehat{g}_1(q_{1d}) \tag{5.69}$$

asymptotically stabilizes the equilibrium point $q = \bar{q}_d$, $\dot{q} = 0$, where $\bar{q}_d$ is the solution to the steady-state equation

$$K_e(\bar{q}_d - q_d) = \begin{pmatrix} g_1(q_{1d}) - g_1(\bar{q}_{1d}) \\ K_P(\widehat{q}_{2d} - q_{2d}) + \widehat{g}_1(q_{1d}) - g_1(q_{1d}) \end{pmatrix}. \tag{5.70}$$

This solution is unique provided that $\lambda_{\min}(K_q) > \alpha$, as before. It is apparent from (5.70) that the better is the model estimate, the closer $\bar{q}_d$ will be to the desired $q_d$.

- Since uncertainty on gravity and elastic terms affect directly the reference value for the motor variables, inclusion of an integral term in the control law (5.68) is not useful for recovering regulation at the desired set point $q_d$. In order for the integral term to be effective, the proportional as well as the integral parts of the PID controller should be driven by the link error $q_{1d} - q_1$. However, as shown in the simple one-joint linear case, velocity should be still fed back at the motor level in order to prevent unstable behaviour, leading to

$$u = K_P(q_{1d} - q_1) - K_D\dot{q}_2 + K_I\int_0^t (q_{1d} - q_1)d\tau + \widehat{g}_1(q_{1d}). \tag{5.71}$$

The asymptotic stability of such a controller has not yet been proved. In particular, the choice of the integral matrix gain $K_I$ is a critical one.

## 5.3 Tracking control

As for rigid robot manipulators, also for manipulators with elastic joints the problem of tracking link (end-effector) trajectories is harder than achieving constant regulation. Nonlinear state feedback control may be useful in order to transform the closed-loop system into an equivalent linear and decoupled one for which the tracking task is easily accomplished. However, the application of this inverse dynamics control strategy is not straightforward in the presence of joint elasticity. Furthermore, it can be shown that the general dynamic model (5.9) may satisfy neither the necessary conditions for feedback linearization nor those for input–output decoupling.

On the other hand, the reduced model (5.28) (with $H_2 = 0$) is more tractable from this point of view, and it always allows exact linearization via *static state feedback*. Therefore, we will use the reduced model to illustrate this nonlinear control approach to the trajectory tracking problem. The same reduced model, in its format (5.37), will also be used to present a *two-time scale control* approach. In particular, the design of this approximate nonlinear control is fully carried out for a one-link arm with joint elasticity.

Two further control strategies will be presented with reference to the complete model. The use of a larger class of control laws, based on *dynamic state feedback*, allows us to recover exact linearization and decoupling results in general. As a preliminary step, it will be shown that the robot manipulator system (5.9), with the link position taken as output, displays *no zero dynamics*. In a nonlinear setting, this is equivalent to state that no internal motion is possible when the input is chosen so as to constrain the output to be constantly zero.

The second control approach is a simpler one, making use only of a feedforward command plus linear feedback from the full state. In this case, convergence to the desired trajectory is only locally guaranteed, i.e., the initial error should be small enough. This technique is referred to as *nonlinear regulation*.

### 5.3.1 Static state feedback

Consider the reduced model (5.28) and define as system output the link position vector

$$y = q_1, \tag{5.72}$$

i.e., the variables to be controlled for tracking purposes.

We will show next that the robot manipulator system with output (5.72) can be input–output decoupled with the use of a nonlinear static state feedback. Moreover, the same decoupling control law will automatically linearize the closed-loop system equations. Also, the coordinate transformation needed to display this linearity is provided as a byproduct of the same approach. The decoupling algorithm requires us to differentiate each output component $y_i$ until the input torque $u$ appears explicitly. The control law is then computed from the last set of obtained differential equations, under proper conditions. This procedure does not require transforming the manipulator dynamic model into the usual state space form, although it is completely equivalent.

By taking the first time derivative of the output

$$\dot{y} = \dot{q}_1 \tag{5.73}$$

and the second one

$$\ddot{y} = \ddot{q}_1 = -H_1^{-1}(C_1\dot{q}_1 + K(q_1 - q_2) + g_1), \tag{5.74}$$

it is immediate to see that the link acceleration is not instantaneously dependent of the applied motor torque $u$. In (5.74), model dependence has been dropped for compactness. Proceeding further, we have

$$y^{(3)} = \frac{d^3 q_1}{dt^3} = -(\dot{H_1^{-1}})\big(C_1\dot{q}_1 + K(q_1 - q_2) + g_1\big) \\ -H_1^{-1}\big(\dot{C}_1\dot{q}_1 + C_1\ddot{q}_1 + K(\dot{q}_1 - \dot{q}_2) + \dot{g}_1\big). \tag{5.75}$$

The right-hand side depends twice on the link acceleration, once directly and once through $\dot{C}_1$. By using (5.74), the link jerk can be rewritten as

$$y^{(3)} = a_3(q_1, \dot{q}_1, q_2) + H_1^{-1}K\dot{q}_2, \tag{5.76}$$

where

$$a_3 = H_1^{-1}\left(3C_1 - \sum_{i=1}^{n}\frac{\partial c_{1i}}{\partial \dot{q}_1}\dot{q}_{1i}\right)H_1^{-1}\big(C_1\dot{q}_1 + K(q_1 - q_2) + g_1\big) \\ -H_1^{-1}\left(\sum_{i=1}^{n}\frac{\partial c_{1i}}{\partial q_1}\dot{q}_{1i}\dot{q}_1 + K\dot{q}_1 + \dot{g}_1\right), \tag{5.77}$$

with $c_{1i}$ denoting the $i$-th column of matrix $C_1$. Next, the fourth derivative of the output gives

$$y^{(4)} = \frac{d^4 q_1}{dt^4} = \dot{a}_3 + (\dot{H_1^{-1}})K\dot{q}_2 + H_1^{-1}K\ddot{q}_2. \tag{5.78}$$

Substituting $\ddot{q}_2$ from the model, differentiating (5.77) with respect to time, and using again (5.74), yields finally

$$y^{(4)} = a_4(q_1, q_2, \dot{q}_1, \dot{q}_2) + H_1^{-1}KH_3^{-1}u \tag{5.79}$$

with

$$a_4 = \dot{a}_3 - H_1^{-1}\big(\dot{H}_1 H_1^{-1}K\dot{q}_2 - KH_3^{-1}K(q_1 - q_2)\big). \tag{5.80}$$

Since the matrix premultiplying $u$ is always *nonsingular*, we can set $y^{(4)} = u_0$ (the external control input) in (5.79) and solve for the feedback control $u$ as

$$u = H_3 K^{-1} H_1(q_1)\big(u_0 - a_4(q_1, q_2, \dot{q}_1, \dot{q}_2)\big). \tag{5.81}$$

The matrix $H_1^{-1}KH_3^{-1}$ premultiplying $u$ in (5.79) is the so-called *decoupling matrix* of the system. Moreover, the *relative degree* $r_i$ of output $y_i$ is equal to 4, uniformly for all outputs. Thus, the sum of all relative degrees equals the state space dimension, i.e., $\sum_{i=1}^{n} r_i = 4n$, which is a sufficient condition for obtaining full linearization, both for the input–output and the state equations. This is obtained by using the same static state feedback decoupling control (5.81). The coordinate transformation which, after the application of (5.81), displays linearity is defined by (5.72) through (5.75). This global diffeomorphism has the inverse transformation given by

$$\begin{aligned} q_1 &= y \\ \dot{q}_1 &= \dot{y} \\ q_2 &= y + K^{-1}\big(H_1(y)\ddot{y} + C_1(y,\dot{y})\dot{y} + g_1(y)\big) \\ \dot{q}_2 &= \dot{y} + K^{-1}\big(H_1(y)y^{(3)} + \dot{H}_1(y)\ddot{y} + C_1(y,\dot{y})\ddot{y} \\ &\qquad + \dot{C}_1(y,\dot{y})\dot{y} + \dot{g}_1(y)\big). \end{aligned} \tag{5.82}$$

Notice that the linearizing coordinates are the link position, velocity, acceleration and jerk. However, in order to perform feedback linearization it is not needed to measure link acceleration and jerk since the control law (5.81) is completely defined in terms of the original states (including motor position and velocity).

By defining

$$z_1 = y \quad z_2 = \dot{y} \quad z_3 = \ddot{y} \quad z_4 = y^{(3)}, \tag{5.83}$$

the transformed system is described by

$$\begin{aligned} \dot{z}_{1i} &= z_{2i} \\ \dot{z}_{2i} &= z_{3i} \\ \dot{z}_{3i} &= z_{4i} \qquad i = 1,\ldots,n, \\ \dot{z}_{4i} &= u_{0i} \\ y_i &= z_{1i} \end{aligned} \tag{5.84}$$

that corresponds to $n$ independent chains of 4 integrators. To complete a tracking controller for a desired trajectory $y_{di}(t)$ of joint $i$, we should design the new input $u_{0i}$ as

$$u_{0i} = y_{di}^{(4)} + \sum_{j=0}^{3} \alpha_{ji}(y_{di}^{(j)} - z_{j+1,i}) \tag{5.85}$$

where the scalar constants $\alpha_{ji}$, $j = 0,\ldots,3$ are coefficients of a Hurwitz polynomial. Note that the control law (5.85) implicitly assumes that the reference trajectory is differentiable up to order four. If the model parameters are known and full state feedback is available, the control (5.81) and (5.85) guarantees trajectory tracking with exponentially decaying error. If the initial state $q_1(0)$, $q_2(0)$, $\dot{q}_1(0)$, $\dot{q}_2(0)$ is matched with the reference trajectory and its derivatives at time $t = 0$ —in this respect, equations (5.82) are to be used— exact reproduction of the reference trajectory is achieved.

### 5.3.2 Two-time scale control

A simpler strategy for trajectory tracking exploits the two-time scale nature of the flexible part and the rigid part of the dynamic equations. The use of this approach allows us to develop a *composite controller*, just by adding terms accounting for joint elasticity to any original control law designed for the rigid manipulator.

In the following, we consider only the reduced model in its singularly perturbed form (5.37). For simplicity, the control approach will be illustrated on a one-link elastic joint manipulator under gravity. All steps followed for this single-input case can be easily adapted to the general multi-input case, by replacing scalar terms with matrix expressions.

The dynamic equations of a robot manipulator having one revolute elastic joint and one link moving in the vertical plane are

$$\begin{aligned} I_\ell \ddot{q}_1 + mg\ell \sin q_1 + k(q_1 - q_2) &= 0 \\ I_m \ddot{q}_2 - k(q_1 - q_2) &= u, \end{aligned} \tag{5.86}$$

where $I_\ell$ and $I_m$ are the link and motor inertia, respectively, $k$ is the joint stiffness, $m$ is the link mass and $\ell$ is the distance of the link center of mass from the joint axis. By setting

$$z = k(q_2 - q_1) \qquad \epsilon^2 = \frac{1}{k}, \tag{5.87}$$

the singularly perturbed model is written as

$$I_\ell \ddot{q}_1 + mg\ell \sin q_1 = z \tag{5.88}$$

$$\epsilon^2 \ddot{z} = -\left(\frac{1}{I_\ell} + \frac{1}{I_m}\right) z + \frac{1}{I_\ell} mg\ell \sin q_1 + \frac{1}{I_m} u, \tag{5.89}$$

with the link position $q_1$ as the slow variable and the joint elastic force $z$ as the fast variable. Since the joint stiffness $k$ is usually quite large, in the limit we can set $\epsilon = 0$ and obtain the approximate dynamic representation

$$I_\ell \ddot{q}_1 + mg\ell \sin q_1 = z \tag{5.90}$$

$$0 = -\left(\frac{1}{I_\ell} + \frac{1}{I_m}\right) z + \frac{1}{I_\ell} mg\ell \sin q_1 + \frac{1}{I_m} u_s \tag{5.91}$$

where $u_s = u|_{\epsilon=0}$. The first step in a singular perturbation approach requires solving (5.91) for $z$ and substitute it into (5.90), so as to obtain a dynamic equation in terms of the slow variable only. Note that this can always be done when $u_s = 0$. When a nonzero control input is present in (5.91), its structure should still allow expressing (5.91) with respect to $z$.

To this purpose, it is sufficient to choose the dependence of the overall control input as

$$u = u_s(q_1, \dot{q}_1, t) + \epsilon u_f(z, \dot{z}, q_1, \dot{q}_1, t), \tag{5.92}$$

where $u_f$ does *not* contain terms of order $1/\epsilon$ or higher. Thus, a two-time scale control law is obtained which is composed of the *slow* part $u_s$, designed using only slow variables, and of the *fast* part $\epsilon u_f$ (vanishing for $\epsilon = 0$) which counteracts the effects of joint elasticity.

Plugging (5.92) into (5.91), and solving for $z$ gives

$$z = \frac{1}{I_\ell + I_m}(I_m mg\ell \sin q_1 + I_\ell u_s). \tag{5.93}$$

This algebraic relation defines a control dependent manifold in the four-dimensional state space of the system. Substituting $z$ in (5.90) yields the so-called *slow reduced system*

$$I_\ell \ddot{q}_1 + mg\ell \sin q_1 = \frac{1}{I_\ell + I_m}(I_m mg\ell \sin q_1 + I_\ell u_s) \tag{5.94}$$

or else

$$(I_\ell + I_m)\ddot{q}_1 + mg\ell \sin q_1 = u_s, \tag{5.95}$$

which is the equivalent rigid manipulator model. The synthesis of the slow control part is based only on this representation of the system. Given a desired trajectory $q_{1d}(t)$ for the link (the system output), a convenient choice could be an inverse dynamics control law

$$u_s = (I_\ell + I_m)u_{s0} + mg\ell \sin q_1, \tag{5.96}$$

with the linear tracking part

$$u_{s0} = \ddot{q}_{1d} + k_D(\dot{q}_{1d} - \dot{q}_1) + k_P(q_{1d} - q_1). \tag{5.97}$$

This is an exact feedback linearizing control law, performed only on the rigid equivalent model. Note that any other control strategy could be used for defining $u_s = u_s(q_1, \dot{q}_1, t)$ (time dependence is introduced through the reference trajectory), without affecting the subsequent steps.

Substituting the control structure (5.92) in the fast dynamics (5.89) yields

$$\begin{aligned}\epsilon^2 \ddot{z} = &-\left(\frac{1}{I_\ell} + \frac{1}{I_m}\right) z + \frac{1}{I_\ell} mg\ell \sin q_1 + \frac{1}{I_m} u_s(q_1, \dot{q}_1, t) \\ &+ \frac{1}{I_m}\epsilon u_f(q_1, \dot{q}_1, z, \dot{z}, t).\end{aligned} \tag{5.98}$$

Due to the time scale separation, we can assume that slow variables are at steady state with respect to variations of the fast variable $z$ and rewrite (5.98) as

$$\epsilon^2 \ddot{z} = -\left(\frac{1}{I_\ell} + \frac{1}{I_m}\right) z + \frac{1}{I_m}\epsilon u_f(q_1, \dot{q}_1, z, \dot{z}, t) + w_s(\hat{q}_1, \hat{\dot{q}}_1, \hat{t}), \tag{5.99}$$

where

$$w_s(\hat{q}_1, \hat{\dot{q}}_1, \hat{t}) = \frac{1}{I_\ell} mg\ell \sin \hat{q}_1 + \frac{1}{I_m} u_s(\hat{q}_1, \hat{\dot{q}}_1, \hat{t}) \tag{5.100}$$

and a hat characterizes steady-state values. Note that $\hat{t}$ stands for the slow nature of the reference trajectory for the link variable.

By comparing (5.100) with the expression of the manifold (5.93), we have

$$w_s(\hat{q}_1, \hat{\dot{q}}_1, \hat{t}) = \left(\frac{1}{I_\ell} + \frac{1}{I_m}\right) \hat{z}, \tag{5.101}$$

with $\hat{z}$ as a parameter in the fast time scale. Defining $\zeta = z - \hat{z}$, the *fast error* dynamics becomes

$$\epsilon^2 \ddot{\zeta} = \left(\frac{1}{I_\ell} + \frac{1}{I_m}\right) \zeta + \frac{1}{I_m}\epsilon u_f. \tag{5.102}$$

The fast control $u_f$ should stabilize this linear error dynamics, which means that the fast variable $z$ asymptotically converges to its *boundary layer* behaviour $\hat{z}$. A possible choice is

$$u_f = -k_f \dot{\zeta} = -k_f \dot{z} \qquad k_f > 0. \tag{5.103}$$

This yields

$$\epsilon^2 \ddot{\zeta} + \frac{k_f}{I_m}\epsilon \zeta + \left(\frac{1}{I_\ell} + \frac{1}{I_m}\right) \zeta = 0 \tag{5.104}$$

or, by setting $\tau = t/\epsilon$ as the fast time scale,

$$\frac{d^2\zeta}{d\tau^2} + a\frac{d\zeta}{d\tau} + b\zeta = 0 \qquad a, b > 0, \tag{5.105}$$

which is exponentially stable.

The final composite *two-time scale* control law is

$$u = u_s(q_1, \dot{q}_1, t) - \epsilon k_f \dot{z}. \tag{5.106}$$

For example, using the inverse dynamics control law (5.96) and (5.97) as the slow controller, (5.106) becomes in the original link and motor variables

$$\begin{aligned} u = {} & (I_m + I_\ell)\big(\ddot{q}_{1d} + k_D(\dot{q}_{1d} - \dot{q}_1) + k_P(q_{1d} - q_1)\big) \\ & + mg\ell \sin q_1 - k_f\sqrt{k}(\dot{q}_2 - \dot{q}_1), \end{aligned} \tag{5.107}$$

since $\epsilon = 1/\sqrt{k}$. The fast control part is just a damping action on the relative motion of the motor and the link. In order to keep the time scale separation between the rigid and elastic dynamics, the gain $k_f$ should be chosen so that $k_f \ll 1/\epsilon = \sqrt{k}$.

In the above analysis, the slow control part has been designed so as to suitably work for the case $\epsilon = 0$. Its action around the manifold (5.93) is only an approximate one. At the expense of a greater complexity, this approach can be improved by adding correcting terms in $\epsilon$ which expand the validity of the slow control also beyond $\epsilon = 0$, i.e.,

$$u_s = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \dots \tag{5.108}$$

where $u_0$ is the previously designed slow control term. For large values of $k$ the correcting terms are small with respect to $u_0$. Associated with $u_s$ in (5.108), a modified control dependent manifold can be defined which characterizes the slow behaviour, similarly to (5.93). It can be shown that a second-order expansion in (5.108) is enough to guarantee that this manifold becomes an *invariant* one; if the initial state is on this manifold, the control $u_s$ will keep the system evolution within this manifold. In particular, this means that the robot manipulator will exactly track the desired link trajectory if the initial state is properly set. The fast control is then needed to counteract mismatched initial conditions and/or disturbances.

### 5.3.3 Dynamic state feedback

We turn now our attention to the general case of robot manipulators with elastic joints, described by the complete dynamic model (5.9). It will be shown next that input–output decoupling in this case is generically impossible using only static state feedback. We remind that the necessary and sufficient condition for this is that the system decoupling matrix is nonsingular. It is then convenient to rewrite the model in the following partitioned form, where dependence is dropped for compactness:

$$\begin{pmatrix} H_1 & H_2 \\ H_2^T & H_3 \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} C_1\dot{q} \\ C_2\dot{q} \end{pmatrix} + \begin{pmatrix} g_1 \\ 0 \end{pmatrix} + \begin{pmatrix} K(q_1 - q_2) \\ -K(q_1 - q_2) \end{pmatrix} = \begin{pmatrix} 0 \\ u \end{pmatrix}. \tag{5.109}$$

Solving the second set of equations for $\ddot{q}_2$ and substituting it into the first set yields

$$\begin{aligned} &(H_1 - H_2H_3^{-1}H_2^T)\ddot{q}_1 + (C_1 - H_2H_3^{-1}C_2)\dot{q} \\ &+(I + H_2H_3^{-1})K(q_1 - q_2) + g_1 + H_2H_3^{-1}u = 0. \end{aligned} \tag{5.110}$$

If the link position is chosen as output

$$y = q_1, \tag{5.111}$$

application of the decoupling algorithm requires, as before, differentiation of each component of (5.111) as many times as until the input explicitly appears. Using (5.110), it is immediate to show that after *two steps*, we obtain

$$\ddot{y} = \ddot{q}_1 = a_2(q, \dot{q}) - (H_1 - H_2 H_3^{-1} H_2^T)^{-1} H_2 H_3^{-1} u. \tag{5.112}$$

Provided that the matrix

$$A(q_1) = -(H_1 - H_2 H_3^{-1} H_2^T)^{-1} H_2 H_3^{-1} \tag{5.113}$$

has *at least one nonzero element for each row*, this will be exactly the decoupling matrix of the system. The first and last matrices on the right-hand side of (5.113) are $(n \times n)$ nonsingular matrices, being respectively the first diagonal block of the inverse of inertia matrix and the inverse of the second diagonal block of the inertia matrix. Thus, nonsingularity of the decoupling matrix depends only on $H_2$. However, this matrix is always singular since its structure is given by (5.20). As a consequence, input–output decoupling via static state feedback is impossible on the above assumption. Indeed, if one row of (5.113) is identically zero, the associated output component should be differentiated further in order to obtain an explicit dependence from the input $u$. Notice that for the reduced dynamic model, $H_2 \equiv 0$ implies that no input appears in the second time derivative of the output (see also (5.74)), and so the decoupling matrix will be completely different from (5.113).

Unfortunately, no general conclusion can be inferred on the rank of the decoupling matrix for the full dynamic model, because its structure strongly depends on the kinematic arrangement of the manipulator with elastic joints. For instance, the single elastic joint case and the 2-revolute-joint polar arm have a nonsingular decoupling matrix (both in fact have $H_2 \equiv 0$). The same considerations apply also to the case of prismatic elastic joints: the cylindric manipulator (prismatic-revolute-prismatic joints), with all joints being elastic, has a nonsingular decoupling matrix. On the other hand, common structures such as the two-revolute-joint planar arm, the 3-revolute-joint anthropomorphic manipulator, as well as manipulators with *mixed* rigid and elastic joints have a structurally singular decoupling matrix.

Similar arguments can be used for the analysis of the feedback linearization property (i.e., the existence of a static state feedback that transforms the closed-loop system into a linear one, not taking into account the output functions), which is also found to depend on the specific kinematic arrangement of the robot manipulator.

For both the input–output decoupling and the exact state linearization problems, a more general class of control laws can be considered. As a matter of fact, we may try to design a ***dynamic state feedback*** law of the form

$$\begin{aligned} u &= \alpha(q, \dot{q}, \xi) + \beta(q, \dot{q}, \xi)u_0 \\ \dot{\xi} &= \gamma(q, \dot{q}, \xi) + \delta(q, \dot{q}, \xi)u_0 \end{aligned} \tag{5.114}$$

where the $(\nu \times 1)$ vector $\xi$ is the state of the dynamic compensator, $\alpha$, $\beta$, $\gamma$, $\delta$ are suitable nonlinear vector functions, and the $(n \times 1)$ vector $u_0$ ($n$ is the dimension of the joint space) is the new external input used for trajectory tracking purposes. In the multi-input case, the conditions for obtaining noninteraction and/or exact linearization in the closed-loop system using (5.114) are indeed weaker than those based on static state feedback. Note that the latter is a special case of (5.114) for $\nu = 0$.

In particular, a sufficient condition for the existence of a linearizing and input–output decoupling dynamic controller is that the given system has *no zero dynamics*, so that no internal motion is compatible with the output being kept at a fixed (zero) value. Such an interpretation of zero dynamics allows us to generalize the concept of transfer function zeros of a linear system. In the following, we will show that the complete dynamic model (5.109) with output chosen as

$$y = q_1 - q_{10}, \tag{5.115}$$

for any constant $q_{10}$, is a nonlinear system with no zero dynamics. As just said, this will be the nonlinear analogue of the fact that the transfer function (5.39) from motor torque to link position has no zeros. Imposing $y \equiv 0$ implies

$$q_1 = q_{10} \qquad \dot{q}_1 = 0 \qquad \ddot{q}_1 = 0 \tag{5.116}$$

which, substituted into the first set of equations in (5.109), gives

$$H_2(q_{10})\ddot{q}_2 + K(q_{10} - q_2) + g_1(q_{10}) = 0, \tag{5.117}$$

where the expressions (5.14) and (5.15) of the velocity terms have been used. Due to the strict upper triangular structure of matrix $H_2$, the set of $n$ equations (5.117) can be analyzed starting from the last one which is

$$k_n(q_{10n} - q_{2n}) + g_{1n}(q_{10}) = 0, \tag{5.118}$$

or

$$q_{2n} = q_{10n} + \frac{1}{k_n} g_{1n}(q_{10}) = \text{const.} \tag{5.119}$$

Proceeding backward, all components of $q_2$ are found to be constant and equal to

$$q_2 = q_{10} + K^{-1} g_1(q_{10}) = q_{20} \qquad \dot{q}_2 = 0. \tag{5.120}$$

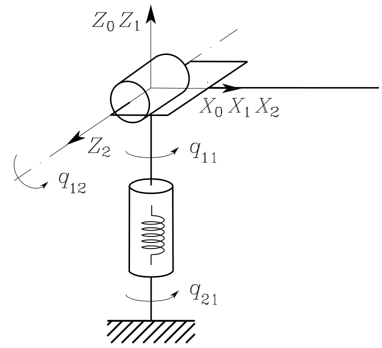


Figure 5.2: A 2-revolute-joint polar robot arm with the first joint being elastic.

Therefore, no internal motion is possible when the output is constantly zero. Notice also that the input needed for keeping this equilibrium condition is computed from the second set of equations in (5.109) as

$$u = K(q_{20} - q_{10}) = g_1(q_{10}). \tag{5.121}$$

As a result, all robot manipulators with elastic joints can be fully linearized and input–output decoupled provided that dynamic state feedback control is allowed.

### Two-revolute-joint polar arm

In order to illustrate the synthesis of such a controller, we will use, as an example, one of the simplest structures where dynamic feedback is needed. Consider a 2-revolute-joint polar arm as in Fig. 5.2, displaying relevant elasticity only in the first joint, whose axis is vertical; the second joint, whose axis is horizontal, is assumed to be perfectly rigid. For simplicity, links with uniform mass distribution are considered.

Let $q_{11}$ and $q_{12}$ be the two link variables, defined through the Denavit-Hartenberg notation, and $q_{21}$ be the first joint motor variable (beyond the reduction gearbox).

The total kinetic energy is the sum of the four contributions associated with the two motors and the two links (no additional coordinate is needed to describe the second motor position, due to the rigidity assumption for the second joint):

$$T_{m1} = \frac{1}{2} I_{m1zz} \dot{q}_{21}^2$$

$$
\begin{aligned}
T_{\ell 1} &= \frac{1}{2} I_{\ell 1zz} \dot{q}_{11}^2 \\
T_{m2} &= \frac{1}{2} I_{m2zz} \dot{q}_{12}^2 + \frac{1}{2} I_{m2xx} \dot{q}_{11}^2 \\
T_{\ell 2} &= \frac{1}{2} I_{\ell 2zz} \dot{q}_{12}^2 + \frac{1}{2} (I_{\ell 2xx} \sin^2 q_{12} + I_{\ell 2yy} \cos^2 q_{12}) \dot{q}_{11}^2 \\
&\quad + \frac{1}{2} m_2 r_{2x}^2 (\dot{q}_{12}^2 + \cos^2 q_{12} \dot{q}_{11}^2),
\end{aligned} \tag{5.122}
$$

where $m_2$ is the mass of the second link, $r_2$ is the position vector of the center of mass of the second link expressed in its frame, and $I_{mi}$ and $I_{\ell i}$ are respectively the constant rigid body inertia matrices of motor $i$ and link $i$ (diagonal in the associated frames). The inertia matrix is readily computed from the above expressions, resulting in a diagonal form; Coriolis and centrifugal terms are then obtained by proper differentiation. The total potential energy is the sum of the elastic energy of the first joint and of the gravitational energy of the second link:

$$
\begin{aligned}
U_{e1} &= \frac{1}{2} k_1 (q_{21} - q_{11})^2 \\
U_{g2} &= g m_2 r_{2x} \sin q_{12},
\end{aligned} \tag{5.123}
$$

where $k_1$ is the elastic constant of the first joint. By introducing the following parameters:

$$
\begin{aligned}
\pi_1 &= I_{\ell 1yy} + I_{m2xx} + I_{\ell 2zz} \\
\pi_2 &= I_{\ell 2yy} - I_{\ell 2xx} + m_2 r_{2x}^2 \\
\pi_3 &= I_{m2zz} + I_{\ell 2zz} + m_2 r_{2x}^2 \\
\pi_4 &= I_{m1zz} \\
\pi_5 &= g m_2 r_{2x},
\end{aligned} \tag{5.124}
$$

the dynamic equations can be finally written as

$$
\begin{aligned}
&(\pi_1 + \pi_2 \cos^2 q_{12}) \ddot{q}_{11} - 2\pi_2 \sin q_{12} \cos q_{12} \dot{q}_{11} \dot{q}_{12} + k_1 (q_{11} - q_{21}) = 0 \\
&\pi_3 \ddot{q}_{12} + \pi_2 \sin q_{12} \cos q_{12} \dot{q}_{11}^2 + \pi_5 \cos q_{12} = u_2 \\
&\pi_4 \ddot{q}_{21} + k_1 (q_{21} - q_{11}) = u_1.
\end{aligned} \tag{5.125}
$$

Note that *three* second-order differential equations result, due to the mixed nature of rigid and elastic joints. Also, it should be mentioned that for the same reason the general model structure investigated in Section 5.1 cannot be directly applied to (5.125).

Choosing as output the link positions

$$
\begin{aligned}
y_1 &= q_{11} \\
y_2 &= q_{12},
\end{aligned} \tag{5.126}
$$

the application of the input–output decoupling algorithm requires, in this case, to differentiate three times $y_1$ and two times $y_2$ in order to have the input explicitly appearing:

$$
\begin{aligned}
y_1^{(3)} = \frac{d^3 q_{11}}{dt^3} &= \frac{d}{dt}\left(\frac{k_1(q_{21}-q_{11}) + 2\pi_2 \sin q_{12} \cos q_{12} \dot{q}_{11} \dot{q}_{12}}{\pi_1 + \pi_2 \cos^2 q_{12}}\right) \\
&= a_3(q_{11}, q_{12}, q_{21}, \dot{q}_{11}, \dot{q}_{12}, \dot{q}_{21}) + \frac{2\pi_2 \sin q_{12} \cos q_{12} \dot{q}_{11}}{\pi_3(\pi_1 + \pi_2 \cos^2 q_{12})} u_2 \\
\ddot{y}_2 = \ddot{q}_{12} &= -\frac{\pi_2 \sin q_{12} \cos q_{12} \dot{q}_{11}^2 + \pi_5 \cos q_{12}}{\pi_3} + \frac{1}{\pi_3} u_2,
\end{aligned}
\tag{5.127}
$$

where the accelerations are obtained from the dynamic model (5.125). As a result, the decoupling matrix has the form

$$
A(q_{12}, \dot{q}_{11}) = \begin{pmatrix} 0 & \dfrac{1}{\pi_3} \dfrac{2\pi_2 \sin q_{12} \cos q_{12} \dot{q}_{11}}{\pi_1 + \pi_2 \cos^2 q_{12}} \\ 0 & \dfrac{1}{\pi_3} \end{pmatrix}
\tag{5.128}
$$

which is *always* singular. This means that the second input appears "too soon" in both outputs, before the action of the first input torque is felt through the natural path of joint elasticity. Therefore, decoupling can never be achieved without the use of dynamic components which slow down the action of the second input. In fact, consider the addition of *two integrators* on the second input channel. Denote by $\xi_1$, $\xi_2$ the corresponding states, by $u_2'$ the input to the second integrator, and by $u_1' = u_1$ the other input which does not change. The system equations are rewritten as:

$$
\begin{aligned}
(\pi_1 + \pi_2 \cos^2 q_{12})\ddot{q}_{11} - 2\pi_2 \sin q_{12} \cos q_{12}\ \dot{q}_{11}\dot{q}_{12} + k_1(q_{11} - q_{21}) &= 0 \\
\pi_3 \ddot{q}_{12} + \pi_2 \sin q_{12} \cos q_{12}\ \dot{q}_{11}^2 + \pi_5 \cos q_{12} - \xi_1 &= 0 \\
\ddot{\xi}_1 &= u_2' \\
\pi_4 \ddot{q}_{21} + k_1(q_{21} - q_{11}) &= u_1'.
\end{aligned}
\tag{5.129}
$$

The problem is now turned to control the link positions by acting on the torque of the first motor and on the second derivative of the torque of the second motor. By applying the decoupling algorithm to the *extended system* (5.129), it is immediate to check that neither the second nor the third derivatives of both outputs depend on the new inputs $u_1'$ and $u_2'$, which appear instead in $y^{(4)}$. To see this dependence, it is convenient to take the second derivative of the first set of two equations in (5.129)

$$
(\pi_1 + \pi_2 \cos^2 q_{12}) q_{11}^{(4)} + 2\pi_2 \sin q_{12} \cos q_{12}\ \dot{q}_{12} q_{11}^{(3)}
$$

$$
\begin{aligned}
&+2\pi_2(\cos^2 q_{12} - \sin^2 q_{12})\,\ddot{q}_{11}\ddot{q}_{12} \\
&-2\pi_2 \frac{d^2}{dt^2}(\sin q_{12} \cos q_{12}\, \dot{q}_{11}\dot{q}_{12}) + k_1(\ddot{q}_{11} - \ddot{q}_{21}) = 0 \\
&\pi_3 q_{12}^{(4)} + \frac{d^2}{dt^2}(\pi_2 \sin q_{12} \cos q_{12}\, \dot{q}_{11}^2 + \pi_5 \cos q_{12}) - \ddot{\xi}_1 = 0
\end{aligned}
\tag{5.130}
$$

and substitute therein $\ddot{\xi}_1$ and $\ddot{q}_{21}$, as obtained from the second set of two equations:

$$
\begin{aligned}
\ddot{\xi}_1 &= u_2' \\
\ddot{q}_{21} &= \frac{1}{\pi_4} u_1' + \frac{1}{\pi_4} k_1 (q_{11} - q_{21}).
\end{aligned}
\tag{5.131}
$$

This yields

$$
\begin{aligned}
\begin{pmatrix} q_{11}^{(4)} \\ q_{12}^{(4)} \end{pmatrix} &= \begin{pmatrix} a_{41}(q_{11}, q_{12}, q_{21}, \dot{q}_{11}, \dot{q}_{12}, \dot{q}_{21}, \xi_1, \xi_2) \\ a_{42}(q_{11}, q_{12}, q_{21}, \dot{q}_{11}, \dot{q}_{12}, \dot{q}_{21}, \xi_1, \xi_2) \end{pmatrix} \\
&+ \begin{pmatrix} \dfrac{k_1}{\pi_4(\pi_1 + \pi_2 \cos^2 q_{12})} & 0 \\ 0 & \dfrac{1}{\pi_3} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix},
\end{aligned}
\tag{5.132}
$$

from which it is apparent that the decoupling matrix for the extended system is *always nonsingular*. Therefore, a *static* state feedback decoupling control law for the *extended* system is obtained as

$$
\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} \dfrac{\pi_4(\pi_1 + \pi_2 \cos^2 q_{12})}{k_1} & 0 \\ 0 & \pi_3 \end{pmatrix} \begin{pmatrix} u_{01} - a_{41} \\ u_{02} - a_{42} \end{pmatrix}.
\tag{5.133}
$$

Moreover, the relative degrees for the two outputs are $r_1 = r_2 = 4$, so that their sum is equal to the dimension of the extended state space (the six states of the robot arm plus the two states of the compensator). Thus, the same control law (5.133) will also fully linearize the closed-loop system. In terms of the original system, the combination of the control law (5.133) and of the dynamic extension performed with the addition of the two integrators is equivalent to the following *dynamic* state feedback controller

$$
\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \pi_3(u_{02} - a_{42}) \\
u_1 &= \frac{\pi_4(\pi_1 + \pi_2 \cos^2 q_{12})}{k_1}(u_{01} - a_{41}) \\
u_2 &= \xi_1.
\end{aligned}
\tag{5.134}
$$

To summarize, the polar robot arm with the first joint being elastic has been transformed under the action of control (5.134) into two decoupled chains of four integrators. The tracking control problem can then be solved using standard linear techniques for the synthesis of $u_{01}$ and $u_{02}$. Notice that it is sufficient to have a four times differentiable reference link trajectory in order to obtain its exact reproduction.

### 5.3.4 Nonlinear regulation

All previous control approaches for trajectory tracking are based on the use of nonlinear state feedback, which is in general rather complex. This is true both in the static case (e.g., when using the reduced model) and in the dynamic case (i.e., when the decoupling matrix of the system is singular). Roughly speaking, the purpose of these controllers is to set up a way to predict the evolution of the robot manipulator state by enforcing a linear behaviour through model-based feedback. On the other hand, given a desired link trajectory $q_1 = q_{1d}(t)$, it is always possible to compute the nominal trajectory of the state variables which is associated with the given output behaviour. Similarly, the nominal torque producing this robot manipulator motion can also be computed in closed form. This allows us to design a simpler tracking controller based on feedforward plus linear feedback of the computed state error. Such a control scheme is called *nonlinear regulation* since the error linear feedback stabilizes the system around the desired trajectory while computation of the feedforward term and of the reference state trajectory is based on the full nonlinear robot manipulator dynamics.

The approach can be developed directly for the complete dynamic model (5.9), using conveniently its partitioned form (5.109), and applies to the static linearizable as well as to the dynamic linearizable case. In the following, we will refer to the model (5.27), i.e., to the case $H_2 = \text{const}$, for ease of exposition.

Once the output is assigned, we have immediately a specified behaviour for the $2n$ state variables

$$q_1 = q_{1d}(t) \qquad \dot{q}_1 = \dot{q}_{1d}(t). \tag{5.135}$$

Our objective is to compute also the nominal trajectory for the remaining $2n$ state variables. To this purpose, notice that we cannot use the simple coordinate transformation (5.82) when the model is not linearizable by static state feedback. Using (5.135) in the first set of equations (5.27) yields

$$H_1(q_{1d})\ddot{q}_{1d} + H_2\ddot{q}_2 + C_1(q_{1d}, \dot{q}_{1d})\dot{q}_{1d} + K(q_{1d} - q_2) + g_1(q_{1d}) = 0, \tag{5.136}$$

which can be compactly rewritten as

$$H_2\ddot{q}_2 + K(q_{1d} - q_2) + w_d(t) = 0. \tag{5.137}$$

The term $w_d(t)$ depends only on the reference trajectory and its derivatives, collecting all known quantities. By exploiting the structure (5.20) of matrix $H_2$, we can solve for $q_{2n}$ the $n$-th equation in (5.137) as

$$q_{2dn} = q_{1dn} + \frac{1}{k_n} w_{dn}. \tag{5.138}$$

Differentiating twice $q_{2dn}$

$$\ddot{q}_{2dn} = \ddot{q}_{1dn} + \frac{1}{k_n} \ddot{w}_{dn}, \tag{5.139}$$

and substituting it into the $(n-1)$-th equation in (5.137) provides the evolution for $q_{2,n-1}$ as

$$q_{2d,n-1} = q_{1d,n-1} + \frac{1}{k_{n-1}}(w_{d,n-1} + H_{2,n-1,n}\ddot{q}_{2d,n}). \tag{5.140}$$

Proceeding backward recursively, it is then possible to define the nominal evolution of all motor variables

$$q_2 = q_{2d}(t) \qquad \dot{q}_2 = \dot{q}_{2d}(t). \tag{5.141}$$

At this point, the nominal torque for the given trajectory is computed in closed form using the second set of equations in (5.27), i.e.,

$$u_d(t) = H_2^T \ddot{q}_{1d} + H_3 \ddot{q}_{2d} + K(q_{2d} - q_{1d}). \tag{5.142}$$

We notice that the above algebraic computation is allowed by the absence of zero dynamics in the system. Otherwise, the derivation of a state reference trajectory from an output trajectory would require the integration of some differential equations.

To complete the design of a nonlinear regulator we need to find a stabilizing matrix gain $F$ for a linear approximation of the robot manipulator system. This approximation may be derived around a fixed equilibrium point or around the nominal reference trajectory, leading respectively to a linear time-invariant or to a linear time-varying system. In any case, the existence of a (possibly time-varying) stabilizing feedback matrix is guaranteed by the controllability of the linear approximation. The resulting controller becomes

$$u = u_d + F \begin{pmatrix} q_{1d} - q_1 \\ q_{2d} - q_2 \\ \dot{q}_{1d} - \dot{q}_1 \\ \dot{q}_{2d} - \dot{q}_2 \end{pmatrix}. \tag{5.143}$$

**Remarks**

- The validity of this approach is only *local* in nature and the region of convergence depends both on the given trajectory and on the robustness of the designed linear feedback. On the other hand, the final control structure (5.143) is quite simple.

- A special pattern may be selected for the feedback matrix $F$, so as to avoid the measurement of the full robot manipulator state. Following the previous set-point regulation result for elastic joint manipulators, we can attempt using

$$F = \begin{pmatrix} 0 & K_P & 0 & K_D \end{pmatrix}, \tag{5.144}$$

  where only motor variables are fed back. However, there is no proof of global validity for this choice in the tracking case.

- Similar arguments can be used to derive a *dynamic* nonlinear regulator, based only on the measure of the link positions. This controller includes a model-based state observer and therefore requires the assumption that the linear approximation is observable. This is certainly true for robot manipulators with elastic joints, when the output is the link position.

## 5.4 Further reading

An early study on the inclusion of joint elasticity in the modelling of robot manipulators is due to [32]. The general dynamic model of manipulators with elastic joints can be generated automatically using symbolic manipulation programs [4]. Subsequent investigations include, e.g., [45], while the detailed analysis of the model structure presented in Section 5.1 comes from [47]. In [31], the special case of motors mounted on the driven links is treated. Modelling and control analysis of robot manipulators having some joint rigid and some other elastic is presented in [8]. The relevant mechanical considerations involved in the design of robot manipulators as well as in the evaluation of their compliant elements are collected in [39].

A large interest for the control problem of manipulators with elastic joints was excited by the experimental findings of [46, 15] on the GE P-50 robot manipulator. Since then, a number of conventional linear controllers have been proposed, see, e.g., [24], [17], and [26]. However, schemes with proved convergence characteristics have appeared only recently: the linear PD controller with constant compensation of Section 5.2 is a contribution of [48]. An iterative scheme that learns the desired gravity compensation at the set point has been developed in [11].

The reduced model was first introduced in [41], where its exact linearizability via static state feedback is shown. Results on feedback linearization and decoupling for special classes of manipulators had already been found in [13] and [14]; indeed, all these robot manipulators display the reduced model format. The robustness of feedback linearization (or inverse dynamics) control was studied in [16]. A comparative study on the errors induced by inverse dynamics control used on robotic systems which are not linearizable by static state feedback has been carried out in [33]. Practical implementation of inverse dynamics control in discrete time can be found in [20].

The observation that joints with limited elasticity lead to a singularly perturbed dynamic model dates back to the work in [29]. Nonlinear controllers based on the two-time scale separation property were then proposed by [23] and [44]. The corrective control is an outcome of these singular perturbation methods.

When considering the complete dynamic model, feedback linearizability and input–output decoupling were investigated by [28], reporting many negative results (most significantly, on the 3-revolute-joint articulated manipulator). On the other hand, it was found that robot manipulators with elastic joints possess nice structural properties such as nonlinear controllability [3]. Other interesting differential geometric results and a classification of the control characteristics of robot manipulators with different kinematic arrangements can be found in [6].

The use of dynamic state feedback was first proposed in [9] for the 3-revolute-joint robot manipulator, while the general approach to dynamic linearization and decoupling is described in [5]. The proof that absence of zero dynamics in an invertible nonlinear system is a sufficient condition for full linearization via dynamic feedback can be found in [18]. Checking of these sufficient conditions for the general models of robot manipulators with elastic joints is given in [10]. The analysis of manipulator zero dynamics in the presence of damping at the joints can be found in [7]. On the other hand, the reduced model of robot manipulators with mixed elastic/rigid joints may require either static or dynamic feedback for linearization and decoupling [8].

The nonlinear regulation approach for robot manipulators with joint elasticity has been introduced in [7]. The same strategy of state trajectory and feedforward computation can be found in [25] and [36].

Although not treated in this chapter, different state observers have been presented, starting from an approximate one in [38] up to the exact ones in [34] and [47], where a tracking controller based on the estimated state is also tested. In all the above cases, link position and velocity measurements are assumed. When only link positions are measured, instead, regulation schemes have been proposed in [1, 21] while the tracking problem has been considered in [37].

Adaptive control results for robot manipulators with elastic joints include approximate schemes based either on high-gain [42] or on singular perturbations [22], as well as the global solution obtained in [27] and extended in [2]; all analyzed using the reduced dynamic model. Another approach valid for the scalar case can be found in [35]. Moreover, robust control schemes have been proposed in [40] and later in [49], while iterative learning has been used in [12].

Finally, an interesting problem concerns force control of manipulators with elastic joints in constrained tasks, which is discussed in [43] and [30], following a singular perturbation technique, and in [19], using the inverse dynamics approach.

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