Modeling of Robots in Contact with a Dynamic Environment

Alessandro De Luca and Costanzo Manes

Abstract—**A control-oriented modeling approach for describing kinematics and dynamics of robots in contact with a dynamic environment is presented. In many robotic tasks the manipulator in contact cannot be simply modeled as a kinematically constrained system. Conversely, modeling of robot-environment interactions through dynamic impedance may not fit the task layout. A suitable model structure is proposed in this note that handles the more general case in which purely kinematic constraints on the robot end-effector live together with dynamic interactions. Feasible end-effector configurations are parameterized from the environment point of view, using a minimal set of coordinates. Accordingly, a description is obtained also for admissible velocities and contact forces. In particular, a force parameterization is chosen so as to separate static reaction forces from active forces responsible for energy transfer between robot and environment. The overall dynamics of the coupled robot-environment system is obtained in a single framework. The introduced modeling technique naturally leads to the design of new hybrid control laws.**

I. Introduction

Most robotic tasks involve intentional contact between the manipulator and the environment. In addition to a path specification for the robot end-effector, a proper definition of forces to be exerted is necessary to complete tasks such as polishing, deburring, or assembly operations.

In the usual analysis, interaction between robot and environment may or may not imply energy exchange. If the environment imposes purely kinematic constraints on end-effector motion, only a static balance of forces and torques occurs at the contact, when friction effects are neglected. These modeling assumptions, which imply no energy transfer or dissipation, underlie the constrained approach of Yoshikawa [1] and McClamroch and Wang [2], where an algebraic vector equation restricts the feasible end-effector poses. On the other hand, an energy exchange between robot and environment is commonly treated using a full-dimensional linear impedance model for the dynamic interaction, as done by Hogan [3] and Kazerooni *et al.* [4]. This approach is limited by the assumption of small deformation of the workpiece, with no relative motion allowed in the coupling.

As opposed to "completely static" or "pure dynamic" interaction, there are cases in which the robot, while being subject to kinematic constraints, may exert also dynamic forces at the tip, i.e., forces not compensated by a constraint reaction and producing active work on the environment. A paradigmatic example is the task of a robot turning a crank, when crank dynamics is relevant. Therefore, an effective modeling technique should be able to handle these mixed situations as well.

In this note, a general modeling approach is proposed that allows to deal with all those cases in which the end-effector is dynamically coupled and/or kinematically constrained to the external world. The kinematic description of robot-environment interaction is revisited, expressing the end-effector pose in terms of a proper set of parameters and determining admissible directions of robot end-effector motion. Using energy transfer arguments, we introduce *active contact forces*, due to the presence of environment dynamics, beside the usual reaction forces. Both types of forces will be conveniently expressed using another set of independent parameters. This description is coupled with the dynamics of the robot arm to give the equations of motion of the overall system.

Different formats can be worked out for the dynamic description, each suitable for simulation or control design purposes. The dynamics of constrained rigid multibody systems is extensively treated in the literature (see e.g., [5]), not focusing on the use of models for motion and force control design. Instead, the introduced parameterization of forces and displacements is naturally oriented to the synthesis of *hybrid control laws* for constrained robotic tasks. In particular, inverse dynamics computations can be organized in a straightforward way so to accomplish a desired hybrid task. Interestingly enough, the presence of environment dynamics introduces two alternatives in the execution of a contact task, since *either* forces *or* accelerations can be assigned along properly defined *dynamic directions*. In this respect, our modeling approach leads also to new control results. The basic ideas of *task-space* and of hybrid control itself are generalized within the present framework: in fact, the introduced *parameter space* may include also dynamic variables, differently from classical hybrid control task models [6]–[8].

Simple examples are used to illustrate the modeling technique. For further details, the reader is referred to [9].

II. Kinematic and Static Modeling

Consider a robot with n degrees of freedom, constituted by an open kinematic chain of rigid bodies. The robot arm configuration is identified by the joint variables vector $\mathbf{q} \in R^n$ (the *robot parameterization*). Let $\mathbf{r}$ be the position vector of the arm tip, while a minimal representation is used for its orientation, e.g., Euler angles $\mathbf{o} = (\varphi, \vartheta, \psi)$. Position and orientation can be organized in a single 6-dimensional pose $\mathbf{p} = (\mathbf{r}, \mathbf{o})$. As a consequence, end-effector direct and differential kinematics are defined *from the robot side* by

$$\mathbf{p} = \mathbf{k}(\mathbf{q}), \qquad \dot{\mathbf{p}} = \frac{\partial \mathbf{k}(\mathbf{q})}{\partial \mathbf{q}}\dot{\mathbf{q}} = \mathbf{J}_k(\mathbf{q})\dot{\mathbf{q}}. \tag{1}$$

The generalized end-effector velocity $\mathbf{v} = (\dot{\mathbf{r}}, \omega)$ is composed of linear velocity $\dot{\mathbf{r}}$ and angular velocity ω. Its expression is related to $\dot{\mathbf{p}}$ by means of a matrix $\mathbf{G}$ depending on the set of orientation angles

$$\mathbf{v} = \mathbf{G}(\mathbf{p})\dot{\mathbf{p}} \quad \text{with} \quad \mathbf{G}(\mathbf{p}) = \begin{bmatrix} \mathbf{I}_{3\times 3} & \mathbf{O} \\ \mathbf{O} & \widehat{\mathbf{G}}(\varphi, \vartheta, \psi) \end{bmatrix}, \tag{2}$$

where $\widehat{\mathbf{G}}$ is the 3×3 matrix mapping the time derivatives of the chosen representation of orientation into the angular velocity (see e.g., [10, Appendix B]). As a result, one has

$$\mathbf{v} = \mathbf{J}(\mathbf{q})\dot{\mathbf{q}} \quad \text{with} \quad \mathbf{J}(\mathbf{q}) = \mathbf{G}(\mathbf{k}(\mathbf{q}))\mathbf{J}_{\mathbf{k}}(\mathbf{q}), \tag{3}$$

where $\mathbf{J}(\mathbf{q})$ is the basic robot Jacobian.

We assume that the environment is a mechanical system with $d \leq 6$ degrees of freedom, so that it can be described by a second-order dynamic model in terms of a set of environment configuration variables $\mathbf{s}_D \in R^d$. If the robot end-effector exerts a power grasp on the environment,then the set of end-effector poses is restricted to a d-dimensional manifold. However, weaker types of contacts are allowed in general (point, edge, surface, etc.) and an additional set of purely kinematic variables $\mathbf{s}_K \in R^k$ may be required to specify uniquely the end-effector pose $\mathbf{p}$, as seen *from the environment side*. This aspect is not considered in the common literature (see e.g., [11],

Manuscript received July 29, 1991; revised April 14, 1993. This paper is based on work partly supported by the *Consiglio Nazionale delle Ricerche*, contract no. 92.01115.PF67 (*Progetto Finalizzato Robotica*).

The authors are with the Dipartimento di Informatica e Sistemistica, Universitá degli Studi di Roma "La Sapienza", 00184 Roma, Italy.

IEEE Log Number 9402095.

[12]). The two vectors are then merged to form the *environment parameterization* $\mathbf{s} = (\mathbf{s}_K, \mathbf{s}_D) \in R^e$, with $e = k + d \leq 6$. Thus,

$$\mathbf{p} = \Gamma(\mathbf{s}), \qquad \dot{\mathbf{p}} = \frac{\partial\Gamma(\mathbf{s})}{\partial\mathbf{s}}\dot{\mathbf{s}}. \tag{4}$$

where we assume that $\mathbf{\Gamma}$ is a smooth and full rank mapping. This description implies that kinematic constraints on the end-effector, when present, are holonomic [10].

Remark 1: Equations (1) and (4) are used to couple robot and environment kinematics. If $e = 6$, this does not impose any kinematic constraint on the robot end-effector motion, resulting just in a mapping between robot coordinates and environment parameters. Moreover, if $k = 0$ and $d = 6$ the case of a generalized (nonlinear) contact impedance is recovered. Vice versa, if $e < 6$ and $k = e$ (and so $d = 0$) there are effectively $6 - k$ kinematic constraints imposed on the end-effector.

We note that, as with all modeling processes, there are many possible choices for the parameterization $\mathbf{s}$, leading to different but *equivalent* descriptions of the same physical system.

Using (2) and (4), one has

$$\mathbf{v} = \mathbf{T}(\mathbf{s})\mathbf{s}, \qquad \text{with} \quad \mathbf{T}(\mathbf{s}) = \mathbf{G}\big(\mathbf{\Gamma}(\mathbf{s})\big)\frac{\partial\mathbf{\Gamma}(\mathbf{s})}{\partial\mathbf{s}}. \tag{5}$$

where matrix $\mathbf{T}$ is assumed full rank in the operating region. It is possible to group columns of $\mathbf{T}$ and separate velocity contributions due to *kinematic* and to *dynamic* degrees of freedom of the environment as

$$\mathbf{v} = \mathbf{T}_K(\mathbf{s})\dot{\mathbf{s}}_K + \mathbf{T}_D(\mathbf{s})\dot{\mathbf{s}}_D. \tag{6}$$

The generalized *reaction forces* $\mathbf{F}_R$ are defined as those which do not deliver power on admissible velocities at the contact, i.e., such that

$$\mathbf{v}^T\mathbf{F}_R = [\dot{\mathbf{r}}^T \quad \omega^T]\begin{bmatrix}\mathbf{f}_R \\ \mathbf{m}_R\end{bmatrix} = 0, \tag{7}$$

where $\mathbf{f}_R$ are reaction forces and $\mathbf{m}_R$ are reaction torques acting at the robot tip on the environment. Dual to the parameterization of velocity, a full column rank matrix $\mathbf{Y}_R$ can be determined so that reaction forces are expressed as

$$\mathbf{F}_R = \mathbf{Y}_R(\mathbf{s})\lambda_R. \tag{8}$$

Since *all* reaction forces should belong to *span* $[\mathbf{Y}_R]$, the number of columns defining this matrix is maximal, and from the reciprocity[1] condition (7) it follows that

$$\mathbf{T}^T(\mathbf{s})\mathbf{Y}_R(\mathbf{s}) = 0_{e\times(6-e)}. \tag{9}$$

Thus, vector $\lambda_R \in R^{6-e}$ will parameterize reaction forces in the same way as $\dot{\mathbf{s}}$ parameterizes admissible velocities.

Remark 2: The definition of $\mathbf{Y}_R$ is not unique, although *span* $[\mathbf{Y}_R]$ is uniquely identified from (9) as the kernel of $\mathbf{T}^T$ and interpreted as all those directions yielding zero energy transfer between the robot and the environment. The arbitrariness resides in the choice of a basis for *span* $[\mathbf{Y}_R]$. Each feasible choice will lead to a different physical interpretation for the directions associated to the columns of the matrix $\mathbf{Y}_R$.

Remark 3: The columns of $\mathbf{T}$ and $\mathbf{Y}_R$ specify 6-dimensional directions respectively for velocities and forces, intended in a generalized sense. Therefore, a column of $\mathbf{T}$ may represent an angular velocity *together with* a related linear one. Similarly, a column of $\mathbf{Y}_R$ may represent a combination of a force and of a momentum.

[1] Force and velocity belong to dual spaces and the term *reciprocity* should be preferred to *orthogonality*, defined only among vectors of the same linear space.

Remark 4: The constrained kinematic approach of [1], [2] can be recovered within the present formalism, considering as a special case an environment with *no dynamics* ($d = 0$ and $\mathbf{s} \equiv \mathbf{s}_K$).

III. Dynamic Modeling

A robot interacts with a dynamic environment not only through the balance of reaction forces associated with purely kinematic constraints. Instead, an energy exchange between robot and environment is allowed when *active contact forces* come into play, defined along dynamic directions that are not reciprocal to those of admissible motions. These forces will appear as input to the dynamic model both of the environment and of the robot. A Lagrangian approach will be followed for deriving the equations of motion, using the set of generalized coordinates $\mathbf{s}_D$ for the environment in the same way as $\mathbf{q}$ are used for the manipulator.

Let the kinetic energy, potential energy, and Lagrangian of the robot be

$$K = \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{B}(\mathbf{q})\dot{\mathbf{q}}, \qquad P = P(\mathbf{q}), \qquad L = K - P. \tag{10a}$$

while the corresponding quantities for the environment are

$$K_E = \frac{1}{2}\dot{\mathbf{s}}_D^T\mathbf{B}_E(\mathbf{s}_D)\dot{\mathbf{s}}_D, \qquad P_E = P_E(\mathbf{s}_D), \qquad L_E = K_E - P_E, \tag{10b}$$

so that the total Lagrangian of the interacting system is $L_T = L + L_E$. A symmetric form is taken for the positive definite inertia matrices $\mathbf{B}$ and $\mathbf{B}_E$. Non-conservative forces performing work on $\mathbf{q}$ are the torques $\mathbf{u}$ supplied by the motors, and the viscous friction modeled by a dissipative term $-\mathbf{D}\dot{\mathbf{q}}$ (with $\mathbf{D} > 0$). Similarly, the only non-conservative force performing work on $\mathbf{s}_D$ is $-\mathbf{D}_E\dot{\mathbf{s}}_D$ (with $\mathbf{D}_E > 0$) since the environment is assumed without external actuation. The dynamic variables $\mathbf{q}$ and $\mathbf{s}_D$ are related by (1) and (4):

$$\mathbf{p} = \mathbf{k}(\mathbf{q}) = \mathbf{\Gamma}(\mathbf{s}) \quad \Rightarrow \quad \mathbf{\Gamma}(\mathbf{s}) - \mathbf{k}(\mathbf{q}) = 0. \tag{11}$$

Note that *also* the kinematic variables $\mathbf{s}_K$ appear in this coupling equation. In the presence of (11), the composite Lagrangian becomes

$$L_C = L_T + \eta^T[\mathbf{\Gamma}(\mathbf{s}) - \mathbf{k}(\mathbf{q})]. \tag{12}$$

where $\eta \in R^6$ is a multiplier vector.

The Euler-Lagrange equations of motion [10] are derived as

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\mathbf{q}}}^T - \frac{\partial L}{\partial\mathbf{q}}^T + \frac{\partial\mathbf{k}}{\partial\mathbf{q}}^T\eta = \mathbf{u} - \mathbf{D}\dot{\mathbf{q}}, \tag{13a}$$

$$\frac{d}{dt}\frac{\partial L_E}{\partial\dot{\mathbf{s}}_D}^T - \frac{\partial L_E}{\partial\mathbf{s}_D}^T - \frac{\partial\mathbf{\Gamma}}{\partial\dot{\mathbf{s}}_D}^T\eta = -\mathbf{D}_E\mathbf{s}_D, \tag{13b}$$

$$-\frac{\partial\mathbf{\Gamma}}{\partial\mathbf{s}_K}^T\eta = 0, \tag{13c}$$

$$\frac{\partial L_C}{\partial\eta}^T = \mathbf{\Gamma}(\mathbf{s}) - \mathbf{k}(\mathbf{q}) = 0. \tag{13d}$$

Using virtual work arguments, the multiplier η can be interpreted as the generalized force performing work on $\mathbf{p}$, i.e., delivering power on $\dot{\mathbf{p}}$. Thus, the force $\mathbf{F} = \mathbf{G}^{-T}\eta$ delivers power on $\mathbf{v}$.

Developing computations and defining $\mathbf{n} = \mathbf{c} + (\partial P/\partial\mathbf{q})^T + \mathbf{D}\dot{\mathbf{q}}$ and $\mathbf{n}_E = \mathbf{c}_E + (\partial P_E/\partial\mathbf{s}_D)^T + \mathbf{D}_E\dot{\mathbf{s}}_D$, where $\mathbf{c}$ and $\mathbf{c}_E$ are Coriolis and centrifugal contributions, the final dynamic model consists of $n + d$ second-order differential equations

$$\mathbf{B}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{n}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{u} - \mathbf{J}^T(\mathbf{q})\mathbf{F}, \qquad \text{robot} \tag{14a}$$

$$\mathbf{B}_E(\mathbf{s}_D)\ddot{\mathbf{s}}_D + \mathbf{n}_E(\mathbf{s}_D, \dot{\mathbf{s}}_D) = \mathbf{T}_D^T(\mathbf{s})\mathbf{F}, \qquad \text{environment} \tag{14b}$$

together with

$$\mathbf{T}_K^T(\mathbf{s})\mathbf{F} = 0 \tag{14c}$$

and coupled with the algebraic relation (13d) or, in its differential form,

$$\mathbf{v} = \mathbf{T}_K(\mathbf{s})\dot{\mathbf{s}}_K + \mathbf{T}_D(\mathbf{s})\dot{\mathbf{s}}_D = \mathbf{J}(\mathbf{q})\dot{\mathbf{q}}, \tag{15}$$

where all terms have been multiplied by the transformation matrix $\mathbf{G}$. It follows from (15) that the end-effector is restricted to have zero cartesian velocity if and only if *span* $[\mathbf{J}(\mathbf{q})] \cap$ *span* $[\mathbf{T}(\mathbf{s})] = \{0\}$.

Condition (14c) states that admissible contact forces do not perform work along kinematic directions. Moreover, equation (14b) shows that only contact forces $\mathbf{F}$ not reciprocal to the columns of $\mathbf{T}_D$ will affect environment dynamics. Since $\mathbf{T}_D$ is part of $\mathbf{T}$, from (9) these active forces will not belong to *span* $[\mathbf{Y}_R]$. Therefore, they can be generated as combinations of the columns of another matrix $\mathbf{Y}_A$ such that *span* $[\mathbf{Y}_R] \cap$ *span* $[\mathbf{Y}_A] = \{0\}$. Any contact force, being composed of reaction and active terms, will be parameterized as

$$\mathbf{F} = \mathbf{F}_R + \mathbf{F}_A = \mathbf{Y}_R(\mathbf{s})\lambda_R + \mathbf{Y}_A(\mathbf{s})\lambda_A = \mathbf{Y}(\mathbf{s})\lambda, \tag{16}$$

where $\mathbf{Y}_R$ and λ_R are the same as in (8). We will refer to λ as the *force parameterization*. The actual dimension of λ_A is obtained by expressing the power transfer from robot to environment:

$$\mathbf{v}^T\mathbf{F} = \left(\mathbf{T}_K(\mathbf{s})\dot{\mathbf{s}}_K + \mathbf{T}_D(\mathbf{s})\dot{\mathbf{s}}_D\right)^T\left(\mathbf{Y}_R(\mathbf{s})\lambda_R + \mathbf{Y}_A(\mathbf{s})\lambda_A\right). \tag{17}$$

By definition of reaction forces (see also (9))

$$[\mathbf{T}_K \quad \mathbf{T}_D]^T\mathbf{Y}_R = 0, \tag{18}$$

and from (14c) and (16),

$$\mathbf{T}_K^T[\mathbf{Y}_R \quad \mathbf{Y}_A] = 0, \tag{19}$$

stating that kinematic displacements are orthogonal to all forces because there cannot be work performed on a kinematic variable. Being $\mathbf{T}$ (and thus $\mathbf{T}_K$) full column rank by assumption, $\mathbf{Y}$ and $\mathbf{Y}_R$ have $6-k$ and $6-e$ columns, so that $\mathbf{Y}_A$ has d independent columns and $\lambda_A \in R^d$. Accordingly, the power transfer at the contact simplifies to

$$\mathbf{v}^T\mathbf{F} = \dot{\mathbf{s}}_D^T\mathbf{T}_D^T(\mathbf{s})\mathbf{Y}_A(\mathbf{s})\lambda_A, \tag{20}$$

which is not definite in sign. The matrix $\mathbf{T}_D^T\mathbf{Y}_A$ is always nonsingular, since for any $\lambda_A \neq 0$ there exists a $\dot{\mathbf{s}}_D$ such that $\mathbf{v}^T\mathbf{F} \neq 0$; if not, $\mathbf{Y}_A\lambda_A$ would be a reaction force, contrary to its definition.

Remark 5: As for $\mathbf{Y}_R$, the definition of a matrix $\mathbf{Y}_A$ is not unique although *span* $[\mathbf{Y}]$ is uniquely identified as the kernel of $\mathbf{T}_K^T$ and contains *span* $[\mathbf{Y}_R]$. Any completion of a basis in *span* $[\mathbf{Y}]$, given the columns of $\mathbf{Y}_R$ as a sub-basis, will be feasible. In any case, a convenient selection of $\mathbf{Y}_A$ is possible such that each component of λ_A will have the physical dimension of a force or of a torque.

Remark 6: One possible choice for $\mathbf{Y}_A$ is given by the weighted pseudoinverse of $\mathbf{T}_D^T$

$$\mathbf{Y}_A = (\mathbf{T}_D^T)^{\#}_{\mathbf{W}} = \mathbf{W}\mathbf{T}_D(\mathbf{T}_D^T\mathbf{W}\mathbf{T}_D)^{-1}, \tag{21}$$

where the positive definite matrix $\mathbf{W}$ can always be chosen so to retain consistency of physical dimensions of the columns of the resulting $\mathbf{Y}_A$ [13].

Having obtained the dynamic model of the robot plus environment, and with the given force parameterization at hand, we are ready to rewrite the equations of motion of the overall system eliminating the explicit appearance of the contact force $\mathbf{F}$ from (14). For, differentiate (15) to obtain

$$\begin{aligned}\mathbf{T}_K(\mathbf{s})\ddot{\mathbf{s}}_K + \dot{\mathbf{T}}_K(\mathbf{s},\dot{\mathbf{s}})\dot{\mathbf{s}}_K + \mathbf{T}_D(\mathbf{s})\ddot{\mathbf{s}}_D + \dot{\mathbf{T}}_D(\mathbf{s},\dot{\mathbf{s}})\dot{\mathbf{s}}_D \\ = \mathbf{J}(\mathbf{q})\ddot{\mathbf{q}} + \dot{\mathbf{J}}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}}.\end{aligned} \tag{22}$$

Solving equations (14a, b) in terms of the accelerations $\dot{\mathbf{q}}$ and $\ddot{\mathbf{s}}_D$, and substituting into (22) gives

$$\begin{aligned}\mathbf{T}_K\ddot{\mathbf{s}}_K + \dot{\mathbf{T}}_K\dot{\mathbf{s}}_K + \mathbf{T}_D\mathbf{B}_E^{-1}\mathbf{T}_D^T\mathbf{F} - \mathbf{T}_D\mathbf{B}_E^{-1}\mathbf{n}_E + \dot{\mathbf{T}}_D\dot{\mathbf{s}}_D \\ = \mathbf{J}\mathbf{B}^{-1}\mathbf{u} - \mathbf{J}\mathbf{B}^{-1}\mathbf{J}^T\mathbf{F} - \mathbf{J}\mathbf{B}^{-1}\mathbf{n} + \dot{\mathbf{J}}\dot{\mathbf{q}}.\end{aligned} \tag{23}$$

Introducing the force parameterization (16) into (23), using $\mathbf{T}_D^T\mathbf{Y}_R = 0$ and defining

$$\mathbf{m} = -\dot{\mathbf{T}}_K\dot{\mathbf{s}}_K - \dot{\mathbf{T}}_D\dot{\mathbf{s}}_D + \dot{\mathbf{J}}\dot{\mathbf{q}} + \mathbf{T}_D\mathbf{B}_E^{-1}\mathbf{n}_E - \mathbf{J}\mathbf{B}^{-1}\mathbf{n}, \tag{24a}$$

$$\mathbf{Q} = \left[(\mathbf{T}_D\mathbf{B}_E^{-1}\mathbf{T}_D^T + \mathbf{J}\mathbf{B}^{-1}\mathbf{J}^T)\mathbf{Y}_A \quad \mathbf{J}\mathbf{B}^{-1}\mathbf{J}^T\mathbf{Y}_R \quad \mathbf{T}_K\right], \tag{24b}$$

provides finally

$$\mathbf{Q}(\mathbf{q},\mathbf{s})\begin{bmatrix}\lambda_A \\ \lambda_R \\ \ddot{\mathbf{s}}_K\end{bmatrix} = \mathbf{m}(\mathbf{q},\dot{\mathbf{q}},\mathbf{s},\dot{\mathbf{s}}) + \mathbf{J}(\mathbf{q})\mathbf{B}^{-1}(\mathbf{q})\mathbf{u}. \tag{25}$$

This is the main equation that is needed for the synthesis of simultaneous motion and force control in the presence of environment dynamics. It relates robot actuator inputs $\mathbf{u}$ to contact forces and contact kinematics, expressed in terms of the introduced parameterization. Its use for control is detailed in the next section.

Remark 7: Given an input torque $\mathbf{u}^*$, a contact force $\mathbf{F}^* = \mathbf{Y}_A\lambda_A^* + \mathbf{Y}_R\lambda_R^*$, and a kinematic acceleration $\ddot{\mathbf{s}}_K^*$ satisfying (25), the same equation will be satisfied also by the triple $(\mathbf{u}^*, \mathbf{F}^* + \mathbf{F}_N, \ddot{\mathbf{s}}_K^*)$, with $\mathbf{F}_N \in$ *span* $[\mathbf{Y}_R] \cap$ *ker* $[\mathbf{J}^T]$. The presence of a multiplicity of solutions may result in the physical jamming of the robot with the constraining environment. The solution will be unique if and only if *span* $[\mathbf{Y}_R] \cap$ *ker* $[\mathbf{J}^T] = \{0\}$. This is equivalent to have a full rank matrix $\mathbf{J}^T\mathbf{Y}_R$. Moreover, this condition is necessary and sufficient for the nonsingularity of the 6×6 matrix $\mathbf{Q}(\mathbf{q},\mathbf{s})$ in (25) (see [9]). □

In the following, we show how (25) can be used to obtain dynamic equations for the overall system in a format useful for simulation and further analysis. Once the invertibility of $\mathbf{Q}$ is assured, one has

$$\begin{bmatrix}\lambda_A \\ \lambda_R \\ \ddot{\mathbf{s}}_K\end{bmatrix} = \mathbf{Q}^{-1}\mathbf{m} + \mathbf{Q}^{-1}\mathbf{J}\mathbf{B}^{-1}\mathbf{u}. \tag{26}$$

By defining the blocks of the inverse of $\mathbf{Q}$, vector $\overline{m}(\mathbf{q},\dot{\mathbf{q}},\mathbf{s},\dot{\mathbf{s}})$, and matrix $\overline{N}(\mathbf{q},\mathbf{s})$ as

$$\mathbf{Q}^{-1} = \begin{bmatrix}\mathbf{P}_A \\ \mathbf{P}_R \\ \mathbf{P}_K\end{bmatrix}, \qquad \begin{aligned}\overline{m} &= \left(\mathbf{Y}_R\mathbf{P}_R + \mathbf{Y}_A\mathbf{P}_A\right)\mathbf{m}, \\ \overline{N} &= \left(\mathbf{Y}_R\mathbf{P}_R + \mathbf{Y}_A\mathbf{P}_A\right)\mathbf{J}\mathbf{B}^{-1},\end{aligned} \tag{27}$$

the contact force can be rewritten as a rather complex but explicit function of the robot-environment state $(\mathbf{q},\dot{\mathbf{q}},\mathbf{s},\dot{\mathbf{s}})$ and of the torque input $\mathbf{u}$:

$$\mathbf{F} = \overline{m}(\mathbf{q},\dot{\mathbf{q}},\mathbf{s},\dot{\mathbf{s}}) + \overline{N}(\mathbf{q},\mathbf{s})\mathbf{u}. \tag{28}$$

Remark 8: It should be stressed that this *physical* force depends only on the system state and input, *not* on the internal representation used, viz. on the force parameterization λ and, in particular, on the chosen $\mathbf{Y}_A$. Given any two different choices $\mathbf{Y}_{A,1}$ and $\mathbf{Y}_{A,2}$ both satisfying (19), it can be shown that the contact force $\mathbf{F}$ resulting from (28) is the same. A formal proof of this can be found in [9].

Replacing (28) into (14) yields finally

$$\begin{aligned}\mathbf{B}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{n}(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{J}^T(\mathbf{q})\overline{m}(\mathbf{q},\dot{\mathbf{q}},\mathbf{s},\dot{\mathbf{s}}) \\ = [\mathbf{I} - \mathbf{J}^T(\mathbf{q})\overline{N}(\mathbf{q},\mathbf{s})]\mathbf{u},\end{aligned} \tag{29a}$$

$$\begin{aligned}\mathbf{B}_E(\mathbf{s}_D)\ddot{\mathbf{s}}_D + \mathbf{n}_E(\mathbf{s}_D,\dot{\mathbf{s}}_D) - \mathbf{T}_D^T(\mathbf{s})\overline{m}(\mathbf{q},\dot{\mathbf{q}},\mathbf{s},\dot{\mathbf{s}}) \\ = \mathbf{T}_D^T(\mathbf{s})\overline{N}(\mathbf{q},\mathbf{s})\mathbf{u},\end{aligned} \tag{29b}$$

which should be completed with the third block equation in (26)

$$\ddot{\mathbf{s}}_K = \mathbf{P}_K(\mathbf{q},\mathbf{s})\mathbf{m}(\mathbf{q},\dot{\mathbf{q}},\mathbf{s},\dot{\mathbf{s}}) + \mathbf{P}_K(\mathbf{q},\mathbf{s})\mathbf{J}(\mathbf{q})\mathbf{B}^{-1}(\mathbf{q})\mathbf{u}. \quad (29c)$$

System (29) is in a convenient format for the simulation of the robot-environment system as well as for numerical analysis (e.g., linearization). The kinematic quantities $\mathbf{q},\dot{\mathbf{q}},\mathbf{s},\dot{\mathbf{s}}$ computed by integrating (29) automatically satisfy the coupling relations (11) and (15), and hence possible kinematic constraints for the robot. However, a numerically robust integration should be performed to avoid that accumulation of round-off errors may result in constraint violation.

We conclude this section by addressing how to *reduce* the number of dynamic equations, proceeding similarly to [2], [14]. The system (29) of $n+e$ second-order differential equations is indeed a redundant one, due to the presence of a kinematic coupling. Supposing non-degeneracy of the orientation representation, and letting

$$h = \text{rank}\begin{bmatrix}\dfrac{\partial \mathbf{k}(\mathbf{q})}{\partial \mathbf{q}} & \dfrac{\partial \mathbf{\Gamma}(\mathbf{s})}{\partial \mathbf{s}}\end{bmatrix} = \text{rank}\,[\mathbf{J}(\mathbf{q}) \quad \mathbf{T}(\mathbf{s})], \quad (30)$$

it is possible to locally express h of the $n+e$ variables $(\mathbf{q},\mathbf{s})$ in terms of the remaining ones. The number $n+e-h$ of obtained independent second-order dynamic equations coincides with the dimension of the space of admissible end-effector velocities when the robot Jacobian is full rank. Being $\mathbf{T}$ of full column rank e, then $h \geq e$ from (30). As a result, elimination can be performed so that *all* the components of $\mathbf{s}$ are expressed in terms of the robot variables $\mathbf{q}$, thus canceling completely the environment equations. With a slight abuse of notation, the dynamics of the robot in contact with the environment is then described by

$$\mathbf{B}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{n}(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{J}^T(\mathbf{q})\overline{m}(\mathbf{q},\dot{\mathbf{q}}) = [\mathbf{I} - \mathbf{J}^T(\mathbf{q})\overline{N}(\mathbf{q})]\mathbf{u}, \quad (31)$$

obtained by formally replacing $\mathbf{s} = \mathbf{s}(\mathbf{q})$ and $\dot{\mathbf{s}} = \dot{\mathbf{s}}(\mathbf{q},\dot{\mathbf{q}})$ in (29a). Equation (31) shows how the behavior of the robot is modified by the contact with the environment, both in the dynamic and non-dynamic case.

IV. Inverse Dynamics for Hybrid Control

The dynamic equation (25) can be used to compute nominal input torques which realize a specified hybrid task. However, as a result of the presence of environment dynamics, it is now possible to impose different modes of operation.

At every state $(\mathbf{q},\dot{\mathbf{q}},\mathbf{s},\dot{\mathbf{s}})$ of the robot-environment system, the columns of $\mathbf{T}_K$ give directions along which *only* accelerations can be imposed at will, while in the reciprocal directions characterized by the columns of $\mathbf{Y}_R$ *only* generalized forces can be exerted. This is not different from what occurs in the conventional hybrid control approach originated from the work of Mason [6] and refined in [1], [8,] [15]. A more subtle situation occurs in the directions characterized by $\mathbf{T}_D$ and $\mathbf{Y}_A$. In fact, one can *either* apply forces in the directions spanned by $\mathbf{Y}_A$, resulting in a unique acceleration along $\mathbf{T}_D$, *or* impose desired accelerations along $\mathbf{T}_D$, causing a unique active force in $\mathbf{Y}_A$.

To formalize these statements, one should look at acontact task as a succession of modifications of the environment to be performed by the robot. The desired motion involved in these modifications can be directly specified by the environment parameterization $\mathbf{s}_{\text{des}}(t)$, while the needed contact forces are obtained in a similar way from $\lambda_{\text{des}}(t)$. Thus, the environment and force parameterizations naturally define a *task-space*, where the formulation of the task is easily performed. This is well illustrated with a series of examples in [6], but no dynamic issues are considered there and the task parameterization is given directly in terms of cartesian coordinates associated with a particular reference frame attached to the task. Instead, following our approach leads to the more general expressions

$$\begin{aligned}\mathbf{p}_{\text{des}}(t) &= \mathbf{\Gamma}(\mathbf{s}_{\text{des}}(t)),\\ \mathbf{F}_{\text{des}}(t) &= \mathbf{Y}(\mathbf{s}_{\text{des}}(t))\lambda_{\text{des}}(t).\end{aligned} \quad (32)$$

When the environment is purely kinematic, the compatibility of these hybrid control specifications with the kinematic model of the contact is automatically verified. In the presence of environment dynamics, the simultaneous specification of *all* parameters $\mathbf{s}$ and λ is not free, but must satisfy the environment dynamic description if compatible end-effector poses and contact forces are to be obtained. However, desired time evolutions can be freely specified either for the subset of parameters$(\lambda_A, \lambda_R, \mathbf{s}_K)$, or for $(\mathbf{s}_D, \lambda_R, \mathbf{s}_K)$, in alternative. As a consequence, two types of hybrid control laws can be pursued, depending on whether forces or accelerations are imposed along dynamic directions. The computation of nominal input torques realizing the desired task follows slightly different steps in the two cases, as a result of the associated *inverse dynamics*.

Let first the task be specified by the triple

$$(\lambda_{A,\text{des}}(t), \lambda_{R,\text{des}}(t), \mathbf{s}_{K,\mathbf{des}}(t))$$

with a desired force assigned along the dynamic directions. The nominal input torque for this task formulation is computed directly from (25) as

$$\mathbf{u}_{\text{des}} = (\mathbf{J}\mathbf{B}^{-1})^*\left(\mathbf{Q}\begin{bmatrix}\lambda_{A,\text{des}}\\ \lambda_{R,\text{des}}\\ \ddot{\mathbf{s}}_{K,\text{des}}\end{bmatrix} - \mathbf{m}\right) \quad (33)$$

where $(\mathbf{J}\mathbf{B}^{-1})^*$ is any right generalized inverse of $\mathbf{J}\mathbf{B}^{-1}$ [16], and $\mathbf{J}$ is assumed to have full rank. One possible choice is $\mathbf{B}\mathbf{J}^{\#}$, where $\mathbf{J}^{\#}$ is the pseudoinverse of $\mathbf{J}$. The closed-loop system is then described by

$$\begin{bmatrix}\lambda_A\\ \lambda_R\\ \ddot{\mathbf{s}}_K\end{bmatrix} = \begin{bmatrix}\lambda_{A,\text{des}}\\ \lambda_{R,\text{des}}\\ \ddot{\mathbf{s}}_{K,\text{des}}\end{bmatrix}, \quad (34)$$

while the resulting (parametric) accelerations in the dynamic directions will be

$$\ddot{\mathbf{s}}_D = \mathbf{B}_E^{-1}\mathbf{n}_E + \mathbf{B}_E^{-1}\mathbf{T}_D^T\mathbf{Y}_A\lambda_{A,\text{des}}. \quad (35)$$

If instead the task is specified by the triple

$$(\mathbf{s}_{D,\text{des}}(t), \lambda_{R,\text{des}}(t), \mathbf{s}_{K,\text{des}}(t))$$

priority is given to the assignment of motion along the dynamic directions. Solving the environment dynamic model (14b) for the active force parameterization

$$\lambda_A = (\mathbf{T}_D^T\mathbf{Y}_A)^{-1}\mathbf{n}_E + (\mathbf{T}_D^T\mathbf{Y}_A)^{-1}\mathbf{B}_E\ddot{\mathbf{s}}_D, \quad (36)$$

using (23), and defining

$$\begin{aligned}\widehat{m} &= \dot{\mathbf{J}}\dot{\mathbf{q}} - \dot{\mathbf{T}}_K\dot{\mathbf{s}}_K - \dot{\mathbf{T}}_D\dot{\mathbf{s}}_D\\ &\quad - \mathbf{J}\mathbf{B}^{-1}\mathbf{n} - \mathbf{J}\mathbf{B}^{-1}\mathbf{J}^T\mathbf{Y}_A(\mathbf{T}_D^T\mathbf{Y}_A)^{-1}\mathbf{n}_E, \quad (37a)\\ \widehat{\mathbf{Q}} &= \Big[\mathbf{T}_D + \mathbf{J}\mathbf{B}^{-1}\mathbf{J}^T\mathbf{Y}_A(\mathbf{T}_D^T\mathbf{Y}_A)^{-1}\mathbf{B}_E\\ &\qquad \mathbf{J}\mathbf{B}^{-1}\mathbf{J}^T\mathbf{Y}_R \qquad \mathbf{T}_K\Big] \quad (37b)\end{aligned}$$

the system dynamics can be rewritten as

$$\widehat{\mathbf{Q}}(\mathbf{q},\mathbf{s})\begin{bmatrix}\ddot{\mathbf{s}}_D\\ \lambda_R\\ \ddot{\mathbf{s}}_K\end{bmatrix} = \widehat{m}(\mathbf{q},\dot{\mathbf{q}},\mathbf{s},\dot{\mathbf{s}}) + \mathbf{J}(\mathbf{q})\mathbf{B}^{-1}(\mathbf{q})\mathbf{u}. \quad (38)$$

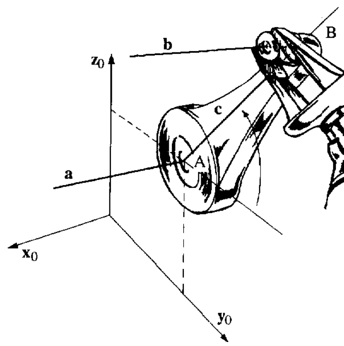

Fig. 1. Robot turning a crank with a free knob.

The nominal input for the task is computed in this case as

$$\mathbf{u}_{\rm des} = (\mathbf{J}\mathbf{B}^{-1})^*\left(\widehat{\mathbf{Q}}\begin{bmatrix}\ddot{\mathbf{s}}_{D,\rm des}\\ \lambda_{R,\rm des}\\ \ddot{\mathbf{s}}_{K,\rm des}\end{bmatrix} - \widehat{m}\right). \tag{39}$$

Similarly to (34), the closed-loop system will be described by

$$\begin{bmatrix}\ddot{\mathbf{s}}_D\\ \lambda_R\\ \ddot{\mathbf{s}}_K\end{bmatrix} = \begin{bmatrix}\ddot{\mathbf{s}}_{D,\rm des}\\ \lambda_{R,\rm des}\\ \ddot{\mathbf{s}}_{K,\rm des}\end{bmatrix}, \tag{40}$$

while the resulting (parametric) active forces in the dynamic directions are obtained by just replacing the desired value $\ddot{\mathbf{s}}_{D,\rm des}$ in (36), or

$$\lambda_A = (\mathbf{T}_D^T\mathbf{Y}_A)^{-1}\mathbf{n}_E + (\mathbf{T}_D^T\mathbf{Y}_A)^{-1}\mathbf{B}_E\ddot{\mathbf{s}}_{D,\rm des}. \tag{41}$$

Note that for the dynamic parameters $\mathbf{s}_D$ and λ_A, this second choice mimics the behavior obtained in impedance control schemes: active contact forces are implicitly determined by the selection of a desired end-effector pose, and compatibility with the environment is not programmed a priori but obtained through dynamic balance [3].

Equations (33) and (39) give nominal torques that have to be applied in the absence of any uncertainty. Indeed, an effective hybrid controller should include a strategy for dealing with model imperfections and other types of disturbances. Also, suitable transformations have to be implemented for mapping desired and measured end-effector motions and contact forces into their parametric descriptions. These and related control issues are discussed in more detail in [13], [17].

V. Examples

The following examples illustrate the modeling approach for simple tasks in the presence of environment dynamics. This analysis leads also to a task specification procedure to be used with generalized hybrid control laws.

A. Robot Turning a Crank with a Free Knob

Consider the robot task of turning a crank with non-negligible mass and inertia in the vertical plane. With reference to Fig. 1, we suppose first that the knob is free to rotate at the crank pin. All vectors will be expressed in an inertial frame ${}^0\mathbf{S}$ fixed with the robot base. Let $\mathbf{a}$ be the crankshaft axis, $\mathbf{b}$ the rotation axis of the knob, both parallel to $\mathbf{x}_0$, and $\mathbf{c}$ the axis of the crank web, normal to both $\mathbf{a}$ and $\mathbf{b}$ and intersecting $\mathbf{b}$ at the point B. The crankweb has length $\mathbf{r}$. A frame ${}^nS$ is attached at the robot end-effector with origin in B and the hand grasp is such that the $\mathbf{x}_n$ axis is always kept parallel to $\mathbf{b}$. This leaves one degree of freedom to the end-effector orientation.

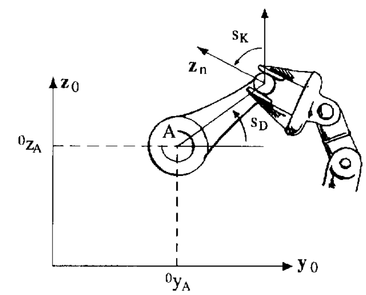

Fig. 2. Environmental parameterization for the case of a free knob.

Since the knob inertia is negligible (or, if not, it can be included in the robot end-effector inertia), the dynamic model of the environment can be written using the angle between the crank web $\mathbf{c}$ and the absolute axis y_0 as the only dynamic variable s_D. To determine uniquely the contact configuration between robot end-effector and environment, we need also one kinematic variable that can be chosen as the absolute angle s_K between $\mathbf{z}_n$ and $\mathbf{z}_0$ (see Fig. 2).

Representing the end-effector orientation with zxz-Euler angles, the absolute pose $\mathbf{p}$ is written as a function of s

$$\mathbf{p} = \begin{bmatrix}x\\ y\\ z\\ \varphi\\ \vartheta\\ \psi\end{bmatrix} = \begin{bmatrix}0\\ y_A + r\cos s_D\\ z_A + r\sin s_D\\ 0\\ s_K\\ 0\end{bmatrix} = \mathbf{\Gamma}(\mathbf{s}). \tag{42}$$

Differentiating (42), and multiplying it by $\mathbf{G}$ as in (2)[2], one obtains the parameterization of the admissible end-effector velocity $\mathbf{v}$:

$$\mathbf{v} = \begin{bmatrix}\dot x\\ \dot y\\ \dot z\\ \omega_x\\ \omega_y\\ \omega_z\end{bmatrix} = \begin{bmatrix}0\\0\\0\\1\\0\\0\end{bmatrix}\dot s_K + \begin{bmatrix}0\\ -r\sin s_D\\ r\cos s_D\\ 0\\ 0\\ 0\end{bmatrix}\dot s_D = \mathbf{T}_K\dot s_K + \mathbf{T}_D\dot s_D. \tag{43}$$

Using the reciprocity condition (9) to determine reaction forces, a parameterization of all contact forces is given by

$$\begin{aligned}\mathbf{F} &= \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos s_D & 0 & 0\\ 0 & \sin s_D & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\lambda_R + \begin{bmatrix}0\\ -\sin s_D\\ \cos s_D\\ 0\\ 0\\ 0\end{bmatrix}\lambda_A\\ &= \mathbf{Y}_R\lambda_R + \mathbf{Y}_A\lambda_A,\end{aligned} \tag{44}$$

where the independent column $\mathbf{Y}_A$ satisfies (19) and λ_A is a force.

Although other definitions of $\mathbf{Y}_R$ in (44) are possible, the chosen one has a 'natural' interpretation: the first two columns are associated to forces along axes $\mathbf{b}$ and $\mathbf{c}$ of the crank; the second two are associated to moments around Y_0 and $\mathbf{z}_0$. Indeed, the contact force component that does not lie in *span* $[\mathbf{Y}_R]$, i.e., $\mathbf{Y}_A\lambda_A$, appears only when explicitly considering the environment dynamics.

The overall dynamic model of the system can be written symbolically as

$$\begin{aligned}\mathbf{B}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{n}(\mathbf{q},\dot{\mathbf{q}}) &= \mathbf{u} - \mathbf{J}^T(\mathbf{q})\mathbf{F},\\ I_c\ddot s_D + D_c\dot s_D + gl_c m_c\cos s_D &= \mathbf{T}_D^T(\mathbf{s})\mathbf{F}.\end{aligned} \tag{45}$$

[2] In this case, for $\varphi = \psi = 0$ one has $\omega_x = \dot\vartheta$.

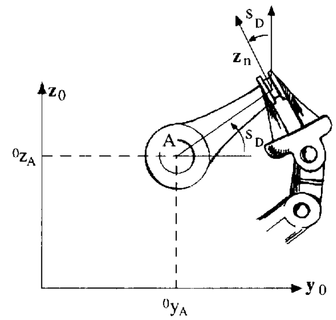

Fig. 3. Robot turning a crank with a fixed knob.

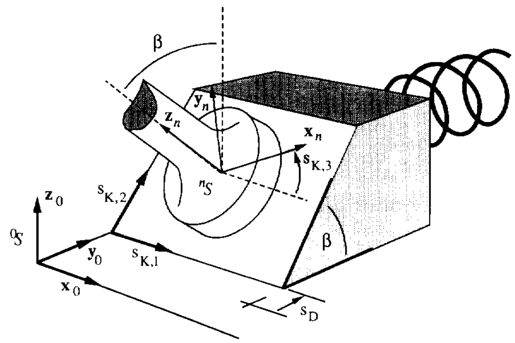

Fig. 4. Robot pushing a mass on a rail.

where m_c is the crank mass, ℓ_c is the distance from center of mass to axis $\mathbf{a}$, I_c is the moment of inertia of the crank w.r.t. $\mathbf{a}$ and D_c is the viscous friction coefficient at the same axis. From (43) and (44), the product $\mathbf{T}_D^T\mathbf{F} = r\lambda_A$. The elimination procedure of $\mathbf{F}$ from (45) leads to equations of the form (29), where in this case (29b) and (29c) would be scalar. Using the notation in (27), (29b) can be particularized as

$$I_c\ddot{s}_D + D_c\dot{s}_D + gl_cm_c\cos s_D = r\mathbf{P}_A(\mathbf{q},\mathbf{s})\big(\mathbf{m}(\mathbf{q},\dot{\mathbf{q}},\mathbf{s},\dot{\mathbf{s}}) + \mathbf{J}(\mathbf{q})\mathbf{B}(\mathbf{q})^{-1}\mathbf{u}\big). \tag{46}$$

In the case of a 6-dof robot with full rank Jacobian ($h = n = 6$), the coupled system will have a two-dimensional dynamics, function only of the environment variables s_D and s_K (see [9] for details). Two hybrid control laws can be proposed in this case: if the crank has to rotate at a constant speed ω, then $\dot{s}_{D,\mathrm{des}} = \omega$ and (39) should be used; instead, if a desired torque M is needed (mimicking the operation of bolt fastening), then $\lambda_{A,\mathrm{des}} = M/r$ and (33) has to be used. In both cases, the hybrid controller should keep track also of the four reaction forces/torques λ_R and of the single kinematic quantity $\dot{s}_K$, typically setting their desired values to zero.

B. Robot Turning a Crank with a Fixed Knob

In the case of a crank with a fixed knob at the web extremity (Fig. 3), only the first dynamic parameter is needed for specifying the end-effector velocity:

$$\mathbf{v} = \begin{bmatrix} 0 \\ -r\sin s_D \\ r\cos s_D \\ 1 \\ 0 \\ 0 \end{bmatrix}\dot{s}_D = \mathbf{T}_D(s_D)\dot{s}_D. \tag{47}$$

The column $\mathbf{T}_D$ represents the only generalized direction of admissible motion and implies that there cannot be a translational velocity without an angular one, and vice versa.

A straightforward choice for the 5-dimensional reaction force parameterization leads now to

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -\sin s_D & \cos s_D & 0 & 0 \\ 0 & \cos s_D & \sin s_D & 0 & 0 \\ 0 & -r & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\lambda_R + \mathbf{Y}_A\lambda_A = \mathbf{Y}_R\lambda_R + \mathbf{Y}_A\lambda_A. \tag{48}$$

Note that the second column of $\mathbf{Y}_R$ identifies a generalized direction in which the environment reaction balances a force along $\mathbf{b}\times\mathbf{c}$ *together with* a torque about $\mathbf{b}$. Three feasible choices for $\mathbf{Y}_A$ in (48) are

$$\mathbf{Y}_{A,1} = \begin{bmatrix} 0 \\ -\sin s_D \\ \cos s_D \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{Y}_{A,2} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

$$\mathbf{Y}_A(\alpha) = \begin{bmatrix} 0 \\ -\frac{r}{r^2+\alpha^2}\sin s_D \\ \frac{r}{r^2+\alpha^2}\cos s_D \\ \frac{\alpha^2}{r^2+\alpha^2} \\ 0 \\ 0 \end{bmatrix}, \tag{49}$$

where the first two are both special cases of the third one, which is obtained from (21) for an appropriate diagonal weighting matrix $\mathbf{W}$ (see also [13]). The scalar α has the dimension of a length. For any value of α, the force parameter $\lambda_A(\alpha)$ associated to $\mathbf{Y}_A(\alpha)$ has the dimension of a torque.

As for the system dynamics, when the choice $\mathbf{Y}_{A,1}$ is made, (29b) takes the same formal expression (46). The robot-environment dynamics is now described by $n+1$ second-order differential equations, out of which only $n+1-h$ are independent. Again, if $h = n$, the overall system will have a one-dimensional dynamic behavior.

C. Robot Pushing a Mass on a Rail

Consider a dynamic task in which a robot is pushing on a heavy body of mass m that slides within an horizontal rail as in Fig. 4, where the relevant task frames are also shown. The manipulator has a flat end-effector—a disk of radius r—which enables to exert torques around two independent directions on the frictionless skewed face of the body. The body itself is anchored to a wall through an elastic spring of stiffness k_E. We suppose that full planar contact is kept at all times, so that a bilateral constraint can be assumed for the end-effector.

Two kinematic variables are needed to define the position of the end-effector center on the skewed face, while a third one specifies the orientation angle around the face normal. Instead, one dynamic variable is needed to define the position of the body along the rail with its zero value chosen at the rest position of the spring. Representing the end-effector orientation with zxz-Euler angles, its pose $\mathbf{p}$ is written as

$$\mathbf{p} = \begin{bmatrix} x \\ y \\ z \\ \varphi \\ \vartheta \\ \psi \end{bmatrix} = \begin{bmatrix} s_{K,1} \\ y_A + s_{K,2}\cos\beta + s_D \\ s_{K,2}\sin\beta \\ 0 \\ \beta \\ s_{K,3} \end{bmatrix} = \mathbf{\Gamma}(\mathbf{s}) \quad (50)$$

where β is the angle of the skewed face with respect to the horizontal plane. Differentiating (50), and multiplying it by $\mathbf{G}$, one obtains

$$\mathbf{v} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & 0 \\ 0 & \sin\beta & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sin\beta \\ 0 & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} \dot{s}_{K,1} \\ \dot{s}_{K,2} \\ \dot{s}_{K,3} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \dot{s}_D = \mathbf{T}_K \dot{\mathbf{s}}_K + \mathbf{T}_D \dot{s}_D. \quad (51)$$

Based on the reciprocity condition (9) and on (19), reaction and active forces can be parameterized as

$$\mathbf{F} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & \cos\beta \\ 0 & \sin\beta \end{bmatrix} \begin{bmatrix} \lambda_{R,1} \\ \lambda_{R,2} \end{bmatrix} + \begin{bmatrix} 0 \\ \sin\beta \\ -\cos\beta \\ 0 \\ 0 \\ 0 \end{bmatrix} \lambda_A = \mathbf{Y}_R \lambda_R + \mathbf{Y}_A \lambda_A \quad (52)$$

where $\lambda_{R,1}$ and $\lambda_{R,2}$ have dimensions of torques, and λ_A is the parameterization of the linear force in the active direction. The two columns of $\mathbf{Y}_R$ correspond to two orthogonal directions lying on the skewed face of the body, while $\mathbf{Y}_A$ is associated to the surface normal.

The assumption of surface contact imposes certain restrictions on the force parameters. In particular, since the robot is "pushing," it is

$$\lambda_A \geq 0. \quad (53)$$

Accordingly, the parameterization for the reaction moments will have to satisfy

$$\sqrt{\lambda_{R,1}^2 + \lambda_{R,2}^2} \leq \lambda_A r \quad (54)$$

since the application point of the active force should be inside the tool disk. As a consequence, no torques can be applied without also pushing on the body. With this task specification, an hybrid controller will typically regulate λ_R to zero and either impose a positive value for the active force parameter λ_A, or specify a motion within a positive domain for s_D. Finally, the dynamic model of the environment is

$$m\ddot{s}_D + k_E s_D = \mathbf{T}_D^T \mathbf{F} = \lambda_A \sin\beta, \quad (55)$$

which has to be completed with the robot dynamics.

VI. Conclusion

A control-oriented framework for describing kinematics and dynamics of robots in contact with a dynamic environment has been presented. The main idea is to introduce a minimal parameterization of the basic quantities that arise in connection with hybrid control tasks. The environment and contact interaction with the robot end-effector are characterized through *both* dynamic and kinematic variables. From this description, generalized directions of admissible end-effector motion are derived, together with active force directions in which an energy transfer is realized from robot to environment and directions along which contact forces are balanced by the environment reaction. The overall dynamics of the robot-environment system has been derived in this framework.

Most peculiarly to the proposed modeling approach, dynamic directions can be determined along which active forces *and* end-effector motion exist at the same time. Thus, the classical vision of hybrid tasks where two kinds of directions exist, those where only force control is possible and those where only motion control is feasible, is enriched by the addition of a third set of directions along which force or motion control may be performed in alternative.

A suitable task-based dynamic modeling approach is available, and its flexible usage has been illustrated here on representative cases. The simplicity of this modeling allows an useful formulation of the control problem in a series of practical hybrid tasks, e.g., pushing a wheeled vehicle, opening doors, deburring with pneumatically suspended tool and in general in all those cases where a robot arm has to manipulate massive objects in constrained motion or payloads that possess relative degrees of freedom in the grasping.

References

[1] T. Yoshikawa, "Dynamic hybrid position/force control of robot manipulators—Description of hand constraints and calculation of joint driving force," *IEEE J. Trans. Robotics Automat.*, vol. 3, no. 5, pp. 386–392, 1987.

[2] N. H. McClamroch and D. Wang, "Feedback stabilization and tracking in constrained robots," *IEEE Trans. Automat. Contr.*, vol. 33, no. 5, pp. 419–426, 1988.

[3] N. Hogan, "Impedance control: An approach to manipulation: Part I—Theory," "Part II—Implementation," "Part III—Applications," *Trans. ASME J. of Dynamic Systems, Measurement, and Control*, vol. 107, no. 3, pp. 1–24, 1985.

[4] H. Kazerooni, T. B. Sheridan, and P. K. Houpt, "Robust compliant motion for manipulators, Part I: The fundamental concepts of compliant motion," *IEEE Trans. Robotics Automat.*, vol. 2, no. 2, pp. 83–92, 1986.

[5] R. E. Roberson and R. Schwertassek, *Dynamics of Multibody Systems*. New York: Springer-Verlag, 1988.

[6] M. T. Mason, "Compliance and force control for computer controlled manipulators," *IEEE Trans. Syst., Man Cyber.*, vol. 11, no. 6, pp. 418–432, 1981.

[7] M. H. Raibert and J. J. Craig, "Hybrid position/force control of manipulators," *Trans. ASME J. Dynamic Systems, Measurement, and Control*, vol. 102, no. 3, pp. 126–133, 1981.

[8] O. Khatib, "A unified approach to motion and force control of robot manipulators: the operational space formulation," *IEEE Trans. Robotics Automat.*, vol. 3, no. 1, pp. 43–53, 1987.

[9] C. Manes, "Robot-environment interaction models and hybrid control of position and force," Ph.D. Thesis, (in Italian), Università di Roma "La Sapienza", Feb. 1992.

[10] H. Goldstein, *Classical Mechanics*, 2nd Edition. Reading, MA: Addison-Wesley, 1980.

[11] J. Wen and S. Murphy, "Stability analysis of position and force control for robot arms," *IEEE Trans. Automat. Contr.*, vol. 36, no. 3, pp. 365–371, 1991.

[12] N. Hogan, "On the stability of manipulators performing contact tasks," *IEEE J. Robotics Automat.*, vol. 4, no. 6, pp. 677–686, 1988.

[13] C. Manes, "Recovering model consistence for force and velocity measures in robot hybrid control," *1992 IEEE Int. Conf. on Robotics and Automation*, Nice, France, pp. 1276–1281, 1992.

[14] M. A. Unseren and A. J. Koivo, "Reduced order model and decoupled control architecture for two manipulators holding an object," *1989 IEEE Int. Conf. on Robotics and Automation*, Scottsdale, AZ, pp. 1240–1245, 1989.

[15] A. De Luca, C. Manes and F. Nicolò, "A task space decoupling approach to hybrid control of manipulators," *2nd IFAC Symp. on Robot Control (SYROCO'88)*, Karlsruhe, Germany, pp. 157–162, 1988.

[16] T. L. Boullion and P. L. Odell, *Generalized Inverse Matrices*. New York: Wiley, 1971.

[17] A. De Luca and C. Manes, "Hybrid force-position control for robots in contact with dynamic environments," *3rd IFAC Symp. on Robot Control (SYROCO'91)*, Vienna, Austria, pp. 377–382, 1991.