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Regulation of Flexible Arms Under Gravity

Alessandro De Luca and Bruno Siciliano

***Abstract*—A simple controller is presented for the regulation problem of robot arms with flexible links under gravity. It consists of a joint PD feedback plus a constant feedforward. Global asymptotic stability of the reference equilibrium state is shown under a structural assumption about link elasticity and a mild condition on the proportional gain. The result holds also in the absence of internal damping of the flexible arm. A numerical case study is presented.**

I. Motivation

The regulation problem for articulated mechanical structures is often solved by designing simple control laws which strongly exploit the physical properties of the system. It is well known that a rigid robot arm can be globally asymptotically stabilized around a given joint configuration via a PD controller on the joint errors, provided that gravity is exactly cancelled by feedback [1]. Under a mild condition on the proportional gain, this scheme can be simplified by performing only a constant gravity compensation at the desired configuration [2]. This result was extended in [3] to the case of robots with elastic joints, under the further assumption that joint stiffness overcomes the gradient of the gravitational term. More recently, the design of simple controllers with guaranteed convergence has been addressed for robots where flexibility is distributed along the links and not concentrated at the joints. Asymptotic stability of a joint PD controller for planar (i.e. without gravity) robot arms with flexible links has been shown in [4], while the case of no internal damping has been considered for a single-link arm in [5]. Full state asymptotic stabilization of flexible arms using strain measures at the link bases was also presented in [6], still without gravity.

In this work, we consider the case of flexible manipulators under gravity, with or without internal damping of link vibration. Inspired by the approach of [3], we prove *global asymptotic stability* of a *joint PD controller,* i.e. avoiding feedback from the elastic coordinates, with *constant feedforward gravity compensation* (PD+ controller) for a full nonlinear model of *multilink flexible robots.* A structural assumption about link elasticity is required and a mild condition on the proportional gain is derived. The proof goes through a classical Lyapunov argument. A particular expression of the inertia matrix is then exploited to show asymptotic stability also in the absence of internal damping. A numerical case study is developed for a two-link flexible arm.

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A. de Luca is with the Dipartamento di Informatica e Sistemistica, Università di Roma "La Sapienza", Via Endossiana 18, 00185 Rome, Italy.

B. Siciliano is with the Dipartamento di Informatica e Sistemistica, Università di Napoli Federico II, Via Claudio 21, 80125 Napoli, Italy.

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II. Dynamic Model of Flexible Arms

Consider a robot arm composed of a serial chain of links, some of which are flexible. The Lagrangian technique can be used to derive the dynamic model, through the computation of global kinetic and potential energy of the system [7]. Due to link flexibility, the dynamic model is of distributed nature. Slender links can be modeled as Euler–Bernoulli beams satisfying proper boundary conditions at the actuated joint and at the link tip. While a linear model is in general sufficient to capture the dynamics of each flexible link, the interplay of rigid body motion and flexible deflections in the multilink case gives rise to fully nonlinear dynamic equations. However, the usual dynamic models are valid under the assumption of small link deformation [7]–[9].

In order to obtain a finite-dimensional model for convenient analysis and synthesis of control laws, basis functions for describing link deformation shapes are to be chosen with an associated set of generalized coordinates. Let θ denote the $n \times 1$ vector of joint coordinates, and δ the $m \times 1$ vector of link coordinates of an assumed modes description of link deflections; then, the $(n+m) \times 1$ vector $q = (\theta^T \delta^T)^T$ characterizes the arm configuration.

We suppose to include only bending deformations limited for each link to the plane of rigid motion. This can be enforced by proper structural design of the links so as to avoid torsional effects. The closed-form dynamic equations of the arm can be written as $n+m$ second-order nonlinear differential equations in the general form [7], [8]:

$$B(q)\ddot{q} + h(q,\dot{q}) + g(q) + \begin{pmatrix} 0 \\ K\delta + D\dot{\delta} \end{pmatrix} = \begin{pmatrix} u \\ 0 \end{pmatrix}. \qquad (1)$$

In (1), the $(n+m) \times (n+m)$ positive definite symmetric inertia matrix B may depend in general on both joint (rigid) and link (flexible) coordinates. The $(n+m) \times 1$ vector h contains Coriolis and centrifugal forces, and can be computed via the Christoffel symbols, i.e. via differentiation of the inertia matrix elements; it can be shown that a factorization of h exists

$$h(q,\dot{q}) = S(q,\dot{q})\dot{q} \qquad (2)$$

such that the matrix $\dot{B} - 2S$ is skew-symmetric. This is similar to the rigid case [1] and follows from energy arguments holding for all mechanical systems with positive definite inertia matrix. The positive *semi-definite* (diagonal) matrix D in (1) describes internal modal damping of the links, i.e. the case of no damping ($D = 0$) will also be considered. Notice that we have implicitly considered clamped boundary conditions at the joint actuators' side; this assumption, which is typically enforced under a joint PD feedback [9], implies that the control does not enter directly in the equations of motion for the flexible part.

Some further considerations are in order regarding the terms in (1) deriving from the potential energy U, composed of the gravity contribution U_g and of the elastic contribution U_δ. In view of the small deformation hypothesis we have, in the range of validity Δ of the model, that

$$U_\delta = \frac{1}{2}\delta^T K \delta \leq U_{\delta,\max} < \infty, \quad \delta \in \Delta \subset \mathbb{R}^m \qquad (3)$$

where K is the positive definite symmetric (diagonal) stiffness matrix associated with link elasticity. A direct consequence of (3) is that

$$\|\delta\| \leq \sqrt{\frac{2U_{\delta,\max}}{\lambda_{\max}(K)}} \tag{4}$$

where $\|v\|$ denotes the usual Euclidean norm of a vector v; also, we denote by $\lambda_{\max}(A)(\lambda_{\min}(A))$ the largest (smallest) eigenvalue of a symmetrix matrix A.

Concerning the gravity contribution, the $(n+m)\times 1$ vector of gravity forces $g = (\partial U_g/\partial q)^T$ can be partitioned as

$$g(q) = \begin{pmatrix} g_\theta(\theta,\delta) \\ g_\delta(\theta) \end{pmatrix} \tag{5}$$

where the dependence of the lower term is justified by the small deformation hypothesis. Further, the vector g satisfies the inequality

$$\left\| \frac{\partial g}{\partial q} \right\| \leq \alpha_0 + \alpha_1\|\delta\| \leq \alpha_0 + \alpha_1\sqrt{\frac{2U_{\delta,\max}}{\lambda_{\max}(K)}} =: \alpha \tag{6}$$

where $\alpha_0, \alpha_1, \alpha > 0$. This can be easily proven by observing that the gravity term contains only trigonometric functions of θ and linear/trigonometric functions of δ. Also, inequality (4) has been used in (6). As a direct consequence of (6), we have

$$\|g(q_1) - g(q_2)\| \leq \alpha\|q_1 - q_2\| \quad \forall q_i \in \mathbb{R}^n \times \Delta \subset \mathbb{R}^{n+m} \quad i = 1,2. \tag{7}$$

III. A Stable Joint PD+ Controller

Consider the joint PD+ control law

$$u = K_P(\theta_{\text{des}} - \theta) - K_D\dot{\theta} + g_\theta(\theta_{\text{des}}, \delta_{\text{des}}), \tag{8}$$

with $K_P > O$ (at least), $K_D > O$, and being δ_{des} defined by

$$\delta_{\text{des}} = -K^{-1}g_\delta(\theta_{\text{des}}). \tag{9}$$

The equilibrium states of the closed-loop system (1), (8) satisfy the equations

$$g_\theta(\theta,\delta) = K_P(\theta_{\text{des}} - \theta) + g_\theta(\theta_{\text{des}}, \delta_{\text{des}}) \tag{10a}$$
$$g_\delta(\theta) = -K\delta. \tag{10b}$$

It is easy to recognize that (10b) has a unique solution δ for any value of $\theta \in \mathbb{R}^n$. Adding $K\delta_{\text{des}} + g_\delta(\theta_{\text{des}}) = 0$ to the right-hand side of (10b) yields

$$\begin{aligned} K_q(q_{\text{des}} - q) :&= \begin{pmatrix} K_P & O \\ O & K \end{pmatrix}\begin{pmatrix} \theta_{\text{des}} - \theta \\ \delta_{\text{des}} - \delta \end{pmatrix} \\ &= \begin{pmatrix} g_\theta(\theta,\delta) - g_\theta(\theta_{\text{des}},\delta_{\text{des}}) \\ g_\delta(\theta) - g_\delta(\theta_{\text{des}}) \end{pmatrix} \\ &= g(q) - g(q_{\text{des}}). \end{aligned} \tag{11}$$

Under the assumption that

$$\lambda_{\min}(K_q) > \alpha, \tag{12}$$

we have, for $q \neq q_{\text{des}}$:

$$\begin{aligned} \|K_q(q_{\text{des}} - q)\| &\geq \lambda_{\min}(K_q)\|q_{\text{des}} - q\| \\ &> \alpha\|q_{\text{des}} - q\| \geq \|g(q) - g(q_{\text{des}})\| \end{aligned} \tag{13}$$

where the last inequality follows from (7). This implies that $q = q_{\text{des}}, \dot{q} = 0$ is the *unique* equilibrium state of the closed-loop system (1), (8).

Condition (12) will automatically be satisfied, provided that the assumption on the structural link flexibility

$$\lambda_{\min}(K) > \alpha \tag{14}$$

holds, and that the proportional control gain is chosen so that the condition

$$\lambda_{\min}(K_P) > \alpha \tag{15}$$

is verified.

First, we consider the case of $D > 0$. The following result can be established.

Theorem: The equilibrium state $q = q_{\text{des}}, \dot{q} = 0$ of system (1) is asymptotically stable under the control (8), provided that (12) holds.

Proof: Consider the energy-based Lyapunov function candidate

$$\begin{aligned} V = \frac{1}{2}\dot{q}^TB\dot{q} + \frac{1}{2}(q_{\text{des}} - q)^TK_q(q_{\text{des}} - q) \\ + U_g(q) - U_g(q_{\text{des}}) + (q_{\text{des}} - q)^Tg(q_{\text{des}}) \geq 0 \end{aligned} \tag{16}$$

which vanishes only at the desired equilibrium state, due to (10)–(13). The time derivative of (16) along the trajectories of the closed-loop system (1), (8) is

$$\begin{aligned} \dot{V} &= \dot{q}^T\left(B\ddot{q} + \frac{1}{2}\dot{B}\dot{q}\right) - \dot{q}^TK_q(q_{\text{des}} - q) \\ &\quad + \dot{q}^T(g(q) - g(q_{\text{des}})) \\ &= \dot{q}^T\left(\begin{pmatrix} K_P(\theta_{\text{des}} - \theta) - K_D\dot{\theta} + g_\theta(q_{\text{des}}) \\ -(K\delta + D\dot{\delta}) \end{pmatrix} - g(q)\right) \\ &\quad - \dot{q}^T\begin{pmatrix} K_P(\theta_{\text{des}} - \theta) \\ K(\delta_{\text{des}} - \delta) \end{pmatrix} + \dot{q}^T\left(g(q) - \begin{pmatrix} g_\theta(q_{\text{des}}) \\ g_\delta(\theta_{\text{des}}) \end{pmatrix}\right) \end{aligned} \tag{17}$$

where identity (2) and the skew-symmetry of the matrix $\dot{B} - 2S$ have been used. Simplifying terms yields

$$\dot{V} = -\dot{\theta}^TK_D\dot{\theta} - \dot{\delta}^TD\dot{\delta} \leq 0 \tag{18}$$

where (9) has been utilized. When $\dot{V} = 0$, it is $\dot{q} = 0$ and the closed-loop system (1), (8) becomes

$$B\ddot{q} = \begin{pmatrix} K_P(\theta_{\text{des}} - \theta) + g_\theta(q_{\text{des}}) - g_\theta(q) \\ -(K\delta + g_\delta(\theta)) \end{pmatrix}. \tag{19}$$

In view of the previous equilibrium analysis and of (12), it is $\ddot{q} = 0$ if and only if $q = q_{\text{des}}$, or $\theta = \theta_{\text{des}}$ and $\delta = \delta_{\text{des}}$. Invoking LaSalle invariance set theorem [10], asymptotic stability of the desired state follows. Q.E.D.

For the case of no internal damping ($D = 0$), further analysis is needed to show asymptotic stability. To this purpose, we introduce two additional hypotheses:

H1: The inertia matrix B is a function of θ only.

H2: The inertia sub-matrix relative to the flexible variables $B_{\delta\delta}$ is constant.

The first hypothesis corresponds to approximating the kinetic energy of the system with that pertaining to the instantaneously undeformed arm configuration [9], [11]; this is a common engineering practice in the modeling of flexible structures, e.g., [8]. The second hypothesis is conveniently satisfied by a proper selection of boundary conditions [12] and naturally generalizes the structure of the single-link inertia matrix. Therefore, the inertia matrix takes on the following partitioned expression:

$$B = \begin{pmatrix} B_{\theta\theta}(\theta) & B_{\theta\delta}(\theta) \\ B_{\theta\delta}^T(\theta) & B_{\delta\delta} \end{pmatrix}. \tag{20}$$

As a consequence of (20), the vector of Coriolis and centrifugal forces h contains only quadratic terms in $\dot{\theta}_i\dot{\theta}_j$ and $\dot{\theta}_i\dot{\delta}_k$, for any i, j, k, and then $h = 0$ when $\dot{\theta} = 0$.

Upon these premises, asymptotic stability of the equilibrium state $q = q_{\text{des}}, \dot{q} = 0$ of system (1) under the control (8) can be proved also for the case of no internal damping.

Corollary: Under hypotheses H1 and H2, the thesis of Theorem 1 holds also for $D = 0$.

Proof: Proceeding as above leads to

$$\dot{V} = -\dot{\theta}^T K_D \dot{\theta} \leq 0 \tag{21}$$

so that $\dot{V} \equiv 0$ if and only if $\dot{\theta} \equiv 0$. In this situation, $\theta = \bar{\theta}$ and $\ddot{\theta} = 0$, and the closed-loop system (1), (8) becomes eventually

$$B_{\theta\delta}(\bar{\theta})\ddot{\delta} + g_\theta(\bar{\theta}, \delta) = K_P(\theta_{\text{des}} - \bar{\theta}) + g_\theta(\theta_{\text{des}}, \delta_{\text{des}}) \tag{22}$$

$$B_{\delta\delta}\ddot{\delta} + g_\delta(\bar{\theta}) + K\delta = 0. \tag{23}$$

A factorization of g_θ always exists so that

$$g_\theta(\bar{\theta}, \delta) = G_\theta(\bar{\theta})\delta + \gamma(\bar{\theta}) \tag{24}$$

for a constant $\bar{\theta}$. The solution $\delta(t)$ to the linear equation (23) is

$$\delta(t) = \delta_0(t) - K^{-1}g_\delta(\bar{\theta}) \tag{25}$$

where the homogeneous solution $\delta_0(t)$ is of the form

$$\delta_0(t) = \sum_{i=1}^{m} c_i u_i \cos(\omega_i t + \phi_i) \tag{26}$$

being ω_i^2 the distinct eigenfrequencies of the matrix $B_{\delta\delta}^{-1}K, u_i$ the associated right eigenvectors, and c_i constant coefficients [11]. Differentiating (26) twice, using (24) and substituting into (22) gives

$$\begin{aligned}(G_\theta(\bar{\theta}) - B_{\theta\delta}(\bar{\theta})B_{\delta\delta}^{-1}K)\delta_0(t) &= K_P(\theta_{\text{des}} - \bar{\theta}) + g_\theta(\theta_{\text{des}}, \delta_{\text{des}}) \\ &\quad - \gamma(\bar{\theta}) + G_\theta(\bar{\theta})K^{-1}g_\delta(\bar{\theta}) \\ &= \text{constant}.\end{aligned} \tag{27}$$

This set of n linear equations can be compactly rewritten as $A\delta_0(t) = b$ or, in view of (26), as

$$AU\text{col} \cdot \{c_i \cos(\omega_i t + \phi_i)\} = b \tag{28}$$

with $U = (u_1 \cdots u_m)$. Since the system eigenfrequencies ω_i are all distinct, identity (28) is a contradiction, equating a finite sum of independent time-varying functions to a constant. This means that the harmonic term (26) must vanish, and then the unique solution to (23) (i.e., when $\dot{V} = 0$) is

$$\bar{\delta} = -K^{-1}g_\delta(\bar{\theta}) \tag{29}$$

and thus $\dot{\delta} = 0$. In sum, $\dot{V} = 0$ implies $\dot{q} = 0$ and the result follows from (19) as in Theorem 1. Q.E.D.

We remark that in the present case of flexible link arms, we could not use the existence of a strict triangular form for the inertia submatrix $B_{\theta\delta}$ as in the case of elastic joint robots [3]. This basic difference has motivated the above alternate proof.

IV. Discussion

The above simple joint PD+ controller for robots with flexible links guarantees global asymptotic stability of a desired constant arm configuration in the presence of gravity. The following comments are in order.

- The control law does not require any feedback from the deflection variables, and is composed by a *linear* term plus a nominal feedforward term.
- Satisfaction of the structural assumption $\lambda_{\min}(K) > \alpha$ is not restrictive in general, and depends on the relative importance of stiffness vs. gravity. When compared with the joint elastic case [3], link stiffness is usually much smaller than transmission stiffness but the lightweight nature of the links greatly reduces also the magnitude of the gravity terms.

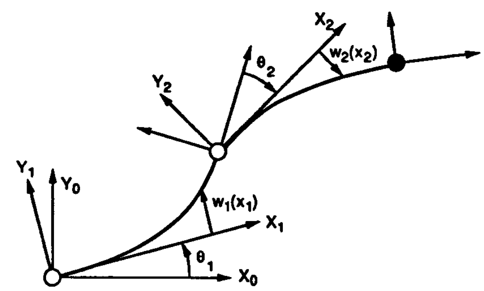

Fig. 1. A planar two-link flexible arm.

- Stability is guaranteed even in the absence of link internal damping. Physically, some small damping will always exist but the regulation transients could be very slow. If desired, a passive increase of damping can be achieved by structural modification, e.g., viscoelastic layer damping treatment [13].
- It is not necessary to compensate gravity for all configurations. In fact, the *nonlinear* control law
$$u = K_P(\theta_{\text{des}} - \theta) - K_D\dot{\theta} + g_\theta(\theta, \delta) \tag{30}$$
leads to a similar stability result. Note that, in any case, the extra term appearing either in (8) or in (30) does not include the whole gravity force appearing in the model (1).
- The knowledge of the link stiffness K and of the complete gravity term g is needed mainly for defining the steady-state deformation δ_{des}. Indeed, uncertainty in the associated model parameters produces a different asymptotically stable equilibrium state. This can be rendered arbitrarily close to the desired one by increasing K_P, provided that the arm is stiff enough.
- If the tip location is of interest, $p = \text{kin}(\theta, \delta)$, then θ_{des} can be computed by inverting for θ the direct kinematic equation
$$\text{kin}(\theta, -K^{-1}g_\delta(\theta)) = p_{\text{des}} \tag{31}$$
so as to achieve end-effector regulation at steady-state.
- All the above derivation is based on a 'generic' finite-dimensional approximation of a distributed parameter model with the assumption of small link deformation and no torsion. If this is not the case, the same result may still hold but a more complex approach would be probably needed for the analysis.

V. Case Study

In order to test the proposed controller, a planar two-link flexible arm under gravity was considered. The arm is sketched in Fig. 1 together with its frame assignments, allowing computation of kinematic quantities needed for model derivation. The following parameters were set up for the links and the tip payload:

$$\rho_1 = \rho_2 = 1.0\,\text{kg/m (link uniform density)}$$
$$\ell_1 = \ell_2 = 0.5\,\text{m (link length)}$$
$$d_1 = d_2 = 0.25\,\text{m (link center of mass)}$$
$$m_1 = m_2 = 0.5\,\text{m (link mass)}$$
$$m_{h1} = m_{h2} = 1\,\text{kg (hub mass)}$$
$$m_p = 0.1\,\text{kg (payload mass)}$$
$$J_{o1} = J_{o2} = 0.0083\,\text{kg m}^2\text{ (link inertia)}$$
$$J_{h1} = J_{h2} = 0.1\,\text{kg m}^2\text{ (hub inertia)}$$
$$J_p = 0.0005\,\text{kg m}^2\text{ (payload inertia)}$$
$$(EI)_1 = (EI)_2 = 10\,\text{N m}^2\text{ (flexural link rigidity).}$$

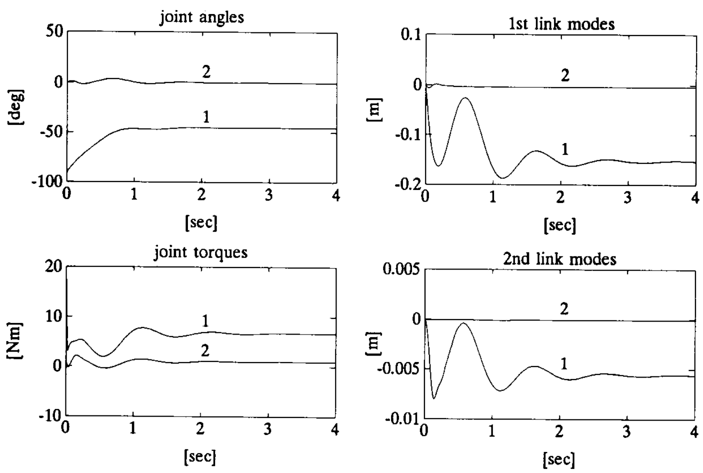

Fig. 2. Time history of joint angles and torques, and of link modal deflections.

The Lagrangian dynamic model of the arm was derived as in [8]. A modal expansion with *two* clamped-mass assumed modes was taken for each link, i.e. the link deflection w_i is expressed as

$$w_i = \sum_{j=1}^{2} \phi_{ij}(x_i)\delta_{ij}(t), \qquad i = 1, 2.$$

We obtained the natural frequencies of vibration:

$$f_{11} = 1.40\,\text{Hz} \quad f_{12} = 5.10\,\text{Hz}$$
$$f_{21} = 5.21\,\text{Hz} \quad f_{22} = 32.46\,\text{Hz}$$

the stiffness coefficients of the diagonal matrix K:

$$k_{11} = 38.79\,\text{N} \quad k_{12} = 513.37\,\text{N}$$
$$k_{21} = 536.09\,\text{N} \quad k_{22} = 20792.09\,\text{N},$$

and the coefficients related to the mode shapes:

$$\phi_{11,e} = 0.39 \qquad \phi_{12,e} = 0.36$$
$$\phi'_{11,e} = 1.34 \qquad \phi'_{12,e} = -1.38$$
$$\phi_{21,e} = 1.49 \qquad \phi_{22,e} = -0.75$$
$$\phi'_{21,e} = 4.30 \qquad \phi'_{22,e} = -15.49$$
$$v_{11} = 0.069 \qquad v_{12} = 0.12$$
$$v_{21} = 0.28 \qquad v_{22} = 0.30$$

where the primes denote spatial derivatives, the subscript e refers to evaluation of ϕ_{ij} and ϕ'_{ij} for $x_i = \ell_i$, and

$$v_{ij} = \int_0^{\ell_i} \rho_i \phi_{ij}(x_i)dx_i, \qquad i, j = 1, 2.$$

Further, the coefficients of the matrix D were related to those of K as

$$d_{ij} = 0.1\sqrt{k_{ij}}, \qquad i, j = 1, 2,$$

corresponding to relatively small internal damping of the link modes. The expressions of all terms in the model (1), i.e. $B(q)$ and $h(q, \dot{q})$, can be found in [8], except for the gravity term $g(q)$. Hence, it is reported below for completeness (standard abbreviations are used for sine and cosine):

$$g_\theta = (g_1 \quad g_2)^T \qquad g_\delta = (g_3 \quad g_4 \quad g_5 \quad g_6)^T$$

with

$$g_1 = g_{11}c_1 + (g_{12}\delta_{11} + g_{13}\delta_{12})s_1 + g_{14}c_{12} + (g_{15}\delta_{11} + g_{16}\delta_{12} + g_{17}\delta_{21} + g_{18}\delta_{22})s_{12}$$
$$g_2 = g_{21}c_{12} + (g_{22}\delta_{11} + g_{23}\delta_{12} + g_{24}\delta_{21} + g_{25}\delta_{22})s_{12}$$
$$g_3 = g_{31}c_1 + g_{32}c_{12}$$
$$g_4 = g_{41}c_1 + g_{42}c_{12}$$
$$g_5 = g_{51}c_{12}$$
$$g_6 = g_{61}c_{12},$$

where the constant coefficients are

$$g_{11} = g_0(m_1d_1 + (m_2 + m_{h2} + m_p)\ell_1)$$
$$g_{12} = -g_0((m_2 + m_{h2} + m_p)\phi_{11e} + v_{11})$$
$$g_{13} = -g_0((m_2 + m_{h2} + m_p)\phi_{12e} + v_{12})$$
$$g_{14} = g_0(m_2d_2 + m_p\ell_2)$$
$$g_{15} = -g_0(m_2d_2 + m_p\ell_2)\phi'_{11,e}$$
$$g_{16} = -g_0(m_2d_2 + m_p\ell_2)\phi'_{12,e}$$
$$g_{17} = -g_0(m_p\phi_{21,e} + v_{21})$$
$$g_{18} = -g_0(m_p\phi_{22,e} + v_{22})$$
$$g_{21} = g_0(m_2d_2 + m_p\ell_2)$$
$$g_{22} = -g_0(m_2d_2 + m_p\ell_2)\phi'_{11,e}$$
$$g_{23} = -g_0(m_2d_2 + m_p\ell_2)\phi'_{12,e}$$
$$g_{24} = -g_0(m_p\phi_{21,e} + v_{21})$$
$$g_{25} = -g_0(m_p\phi_{22,e} + v_{22})$$
$$g_{31} = g_0((m_2 + m_{h2} + m_p)\phi_{11,e} + v_{11})$$
$$g_{32} = g_0(m_2d_2 + m_p\ell_2)\phi'_{11,e}$$
$$g_{41} = g_0((m_2 + m_{h2} + m_p)\phi_{12,e} + v_{12})$$
$$g_{42} = g_0(m_2d_2 + m_p\ell_2)\phi'_{12,e}$$
$$g_{51} = g_0(m_p\phi_{21,e} + v_{21})$$
$$g_{61} = g_0(m_p\phi_{22,e} + v_{22})$$

being g_0 the gravity acceleration. It is worth noticing that the model is linear with respect to the coefficients g_{hk} (see also [8]). Also, verify that g_δ is only a function of θ, as anticipated.

The arm was initially placed in the vertical equilibrium configuration

$$\theta = (-90 \quad 0)^T \,[\text{deg}] \quad \delta = (0 \quad 0 \quad 0 \quad 0)^T \,[\text{m}].$$

The desired joint configuration was chosen

$$\theta_{\text{des}} = (-45 \quad 0)^T \,[\text{deg}].$$

From (9), with the above values of stiffness coefficients, the residual deflections at the desired state were computed as

$$\delta_{\text{des}} = (-0.15 \quad -0.0045 \quad -0.0056 \quad -0.000076)^T \,[\text{m}].$$

The PD feedback gains were chosen as

$$K_P = \text{diag}(18, 18)[\text{Nm/rad}]$$
$$K_D = \text{diag}(10, 2)[\text{N m s/rad}].$$

The resulting arm behavior under the PD+ control (8) is described by the plots in Fig. 2. It is easy to see that the desired state $(\theta_{\text{des}}, \delta_{\text{des}})$ is asymptotically reached, well within 2 s. Notice the large control effort at the start of the motion and the constant torques resulting at steady-state so as to compensate for gravity. The simulations also revealed the following facts.

- We computed a value for α in (6) a posteriori for the simulated trajectory, obtaining $\alpha = 17.67$; thus, both conditions (14) and (15) are actually satisfied. Indeed, those conditions are only sufficient, and we achieved satisfactory results even for smaller values of proportional gains.
- When increasing the gains, the system remained stable at the expense of high initial torques though. We observed, however, that too large values caused numerical instabilities, especially in association with large initial errors. A typical remedy for this inconvenience would be to impose an interpolating trajectory from the initial to the desired state even if the required motion is a point-to-point task.
- We verified that the nonlinear control law (30) leads to very similar results, not reported here for brevity; in particular, it was found that the initial control effort is reduced in view of the gravity compensation performed for the actual system configuration.

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Control of Flexible Arms with Friction in the Joints

Vicente Feliu, Kuldip S. Rattan, and H. Benjamin Brown, Jr.

***Abstract*—The control of flexible arms with friction in the joints is studied. A method to identify the dynamics of a flexible arm from its frequency response (which is strongly distorted by Coulomb's friction) is proposed. A robust control scheme that minimizes the effects of this friction is presented. The scheme consists of two nested feedback loops: an inner loop to control the motor position and an outer loop to control the tip position. It is shown that a proper design of the inner loop eliminates the effects of friction while controlling the tip position and significantly simplifies the design of the outer loop. The proposed scheme is applied to a class of lightweight flexible arms, and the experiments show that the control scheme results in a simple controller. As a result, the computations are minimized and, thus, high sampling rates may be used.**

I. Introduction

A major research effort has been made in the last five years to control flexible structures and, in particular, flexible arms. Several papers have appeared on this topic studying different aspects: Cannon and Schmitz [1] and Matsuno *et al.* [2] are examples of controlling the endpoint position using state-space techniques; Harahima and Ueshiba [3], Siciliano *et al.* [4], and Rovner and Cannon [5] used different adaptive control schemes to account for changes in the load; and Ower and Van de Vegte [6] used classical frequency-domain techniques to control a two-degree-of-freedom flexible arm. However, very little effort has been devoted to the control of flexible arms when static and dynamic frictions are present in the joints, although this is common in practice. The effects of friction are especially important in very lightweight, flexible arms or in flexible arms moving at low speeds and accelerations.

Several methods have been proposed to minimize the effects of friction in the control of dc motors. The simplest method uses a high-gain linear feedback. This method is based on the well-known property that the robustness of a closed-loop system to perturbations and changes in its parameters may be improved by increasing the

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V. Feliu is with the Department Ingenieria Electrica, Electronica y de Control, E.T.S.I. Industriales de la UNED, Madrid, Spain.

K. S. Rattan is with the Department of Electrical Engineering, Wright State University, Dayton, OH 45435.

H. B. Brown, Jr., is with the Robotics Institute, Carnegie-Mellon University, Pittsburgh, PA 15213.

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