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Explicit Dynamic Modeling of a Planar Two-Link Flexible Manipulator

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Introduction

In order to take full advantage of the benefits offered by lightweight flexible arms, it is highly desirable to have an explicit, complete, and accurate dynamic model at disposal. A planar two-link flexible arm with rotary joints subject only to bending deformations in the plane of motion is considered (torsional effects are neglected). A payload is added at the tip of the outer link, while hub inertias are included at the actuated joints. The kinematic relationships for both rigid and deflected motion can be derived according to [1]. Links are modeled as Euler-Bernoulli beams of uniform density with clamped-mass boundary conditions. A finite-dimensional model of link flexibility is obtained with two assumed modes for each link. The standard Lagrangian approach is followed and the resulting dynamic equations are explicited below. The reader is referred to [2] for the details on intermediate steps of derivation. A similar research effort has recently been produced in [3] using Kane's method.

Explicit Dynamic Model

The closed-form equations of motion for the considered flexible arm can be written as

$$\mathbf{B}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{h}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{K}\mathbf{q} = \mathbf{Q}\mathbf{u}$$

where $\mathbf{q} = (\theta_1\ \theta_2\ \delta_{11}\ \delta_{12}\ \delta_{21}\ \delta_{22})^{\mathrm{T}}$ is the vector of generalized coordinates, with θ_i being the joint angle, and δ_{ij} the variable associated with ϕ_{ij}, the mode shape j of link i. $\mathbf{B}$ is the positive-definite symmetric inertia matrix, $\mathbf{h}$ is the vector of Coriolis and centrifugal forces, and $\mathbf{K}$ is the stiffness diagonal matrix ($K_1 = K_2 = 0; K_3, \ldots, K_6 > 0$). The matrix $\mathbf{Q}$, weighting the input vector $\mathbf{u}$ of the two joint (actuator) torques, takes on the form $(\mathbf{I}_{2\times 2}\ \mathbf{O}_{2\times 4})^{\mathrm{T}}$, due to the clamped link assumptions. Structural damping can be added as $\mathbf{D}\dot{\mathbf{q}}$, where $\mathbf{D}$ is a diagonal matrix.

The resulting model is cast in a computationally advantageous form, where a set of constant coefficients appear that depend on the mechanical parameters of the arm. These are: ℓ_i, length of link i; d_i, distance of center of mass of link i from joint i axis; $\phi_{ij,e}$, mode ϕ_{ij} evaluated at the link end point (superscript $'$ denotes spatial derivative); m_i, mass of link i; m_{hi}, mass of hub i; m_p, mass of payload; J_{oi}, inertia of link i about joint i axis; J_{hi}, inertia of hub i; J_p, inertia of payload; EI_i, flexural rigidity of link i. Moreover: $v_{ij} = \int_0^{\ell_i} \rho_i \phi_{ij}(x_i) dx_i$ and $w_{ij} = \int_0^{\ell_i} \rho_i \phi_{ij}(x_i) x_i dx_i$, where ρ_i is the linear mass density of link i.

The inertia matrix turns out of the form

$$
\begin{aligned}
B_{11} &= b_{111} + b_{112}c_2 + (b_{113}t_1 + b_{114}t_2)s_2 \\
B_{12} &= b_{121} + b_{122}c_2 + (b_{123}t_1 + b_{124}t_2)s_2 \\
B_{13} &= b_{131} + b_{132}c_2 + (b_{133}t_2 + b_{134}\delta_{12})s_2 \\
B_{14} &= b_{141} + b_{142}c_2 + (b_{143}t_2 + b_{144}\delta_{11})s_2 \\
B_{15} &= b_{151} + b_{152}c_2 + b_{153}t_1 s_2 \\
B_{16} &= b_{161} + b_{162}c_2 + b_{163}t_1 s_2 \\
B_{22} &= b_{221} \\
B_{23} &= b_{231} + b_{232}c_2 + (b_{233}t_2 + b_{234}t_3)s_2 \\
B_{24} &= b_{241} + b_{242}c_2 + (b_{243}t_2 + b_{244}t_3)s_2 \\
B_{25} &= b_{251} \\
B_{26} &= b_{261} \\
B_{33} &= b_{331} + b_{332}c_2 + b_{333}t_2 s_2 \\
B_{34} &= b_{341} + b_{342}c_2 + b_{343}t_2 s_2 \\
B_{35} &= b_{351} + b_{352}c_2 + b_{353}t_3 s_2 \\
B_{36} &= b_{361} + b_{362}c_2 + b_{363}t_3 s_2 \\
B_{44} &= b_{441} + b_{442}c_2 + b_{443}t_2 s_2 \\
B_{45} &= b_{451} + b_{452}c_2 + b_{453}t_3 s_2 \\
B_{46} &= b_{461} + b_{462}c_2 + b_{463}t_3 s_2 \\
B_{55} &= b_{551} \\
B_{56} &= b_{561} \\
B_{66} &= b_{661}
\end{aligned}
$$

where $s_2 = \sin\theta_2$, $c_2 = \cos\theta_2$, and

$$
\begin{aligned}
t_1 &= t_{11}\delta_{11} + t_{12}\delta_{12} \\
t_2 &= t_{21}\delta_{21} + t_{22}\delta_{22} \\
t_3 &= t_{31}\delta_{11} + t_{32}\delta_{12}
\end{aligned}
$$

with the coefficients having the following expressions:

$$
\begin{aligned}
b_{111} &= J_{h1} + J_{o1} + J_{h2} + m_{h2}\ell_1^2 \\
&\quad + J_{o2} + m_2\ell_1^2 + J_p + m_p(\ell_1^2 + \ell_2^2) \\
b_{112} &= 2(m_2 d_2 + m_p \ell_2)\ell_1 \\
b_{113} &= 2(m_2 d_2 + m_p \ell_2) \\
b_{114} &= -2\ell_1 \\
b_{121} &= J_{h2} + J_{o2} + J_p + m_p\ell_2^2 \\
b_{122} &= (m_2 d_2 + m_p\ell_2)\ell_1 \\
b_{123} &= m_2 d_2 + m_p \ell_2 \\
b_{124} &= -\ell_1 \\
b_{131} &= w_{11} + (J_{h2} + J_{o2} + J_p + m_p\ell_2^2)\phi'_{11,e} \\
&\quad + (m_{h2} + m_2 + m_p)\ell_1\phi_{11,e} \\
b_{132} &= (m_2 d_2 + m_p\ell_2)(\phi_{11,e} + \ell_1\phi'_{11,e}) \\
b_{133} &= -(\phi_{11,e} + \ell_1\phi'_{11,e}) \\
b_{134} &= -(m_2 d_2 + m_p\ell_2)(\phi_{11,e}\phi'_{12,e} - \phi_{12,e}\phi'_{11,e}) \\
b_{141} &= w_{12} + (J_{h2} + J_{o2} + J_p + m_p\ell_2^2)\phi'_{12,e} \\
&\quad + (m_{h2} + m_2 + m_p)\ell_1\phi_{12,e} \\
b_{142} &= (m_2 d_2 + m_p\ell_2)(\phi_{12,e} + \ell_1\phi'_{12,e}) \\
b_{143} &= -(\phi_{12,e} + \ell_1\phi'_{12,e}) \\
b_{144} &= -(m_2 d_2 + m_p\ell_2)(\phi_{12,e}\phi'_{11,e} - \phi_{11,e}\phi'_{12,e}) \\
b_{151} &= w_{21} + J_p\phi'_{21,e} + m_p\ell_2\phi_{21,e} \\
b_{152} &= (v_{21} + m_p\phi_{21,e})\ell_1 \\
b_{153} &= v_{21} + m_p\phi_{21,e} \\
b_{161} &= w_{22} + J_p\phi'_{22,e} + m_p\ell_2\phi_{22,e}
\end{aligned}
$$

$$
\begin{aligned}
b_{162} &= (v_{22} + m_p\phi_{22,e})\ell_1 \\
b_{163} &= v_{22} + m_p\phi_{22,e} \\
b_{221} &= J_{h2} + J_{o2} + J_p + m_p\ell_2^2 \\
b_{231} &= (J_{h2} + J_{o2} + J_p + m_p\ell_2^2)\phi'_{11,e} \\
b_{232} &= (m_2d_2 + m_p\ell_2)\phi_{11,e} \\
b_{233} &= -\phi_{11,e} \\
b_{234} &= -(m_2d_2 + m_p\ell_2)\phi_{11,e} \\
b_{241} &= (J_{h2} + J_{o2} + J_p + m_p\ell_2^2)\phi'_{12,e} \\
b_{242} &= (m_2d_2 + m_p\ell_2)\phi_{12,e} \\
b_{243} &= -\phi_{12,e} \\
b_{244} &= -(m_2d_2 + m_p\ell_2)\phi_{12,e} \\
b_{251} &= w_{21} + J_p\phi'_{21,e} + m_p\ell_2\phi_{21,e} \\
b_{261} &= w_{22} + J_p\phi'_{22,e} + m_p\ell_2\phi_{22,e} \\
b_{331} &= m_1 \\
b_{332} &= 2(m_2d_2 + m_p\ell_2)\phi_{11,e}\phi'_{11,e} \\
b_{333} &= -2\phi_{11,e}\phi'_{11,e} \\
b_{341} &= 0 \\
b_{342} &= (m_2d_2 + m_p\ell_2)(\phi_{11,e}\phi'_{12,e} + \phi_{12,e}\phi'_{11,e}) \\
b_{343} &= -(\phi_{11,e}\phi'_{12,e} + \phi_{12,e}\phi'_{11,e}) \\
b_{351} &= (w_{21} + J_p\phi'_{21,e} + m_p\ell_2\phi_{21,e})\phi'_{11,e} \\
b_{352} &= (v_{21} + m_p\phi_{21,e})\phi_{11,e} \\
b_{353} &= -(v_{21} + m_p\phi_{21,e})\phi_{11,e} \\
b_{361} &= (w_{22} + J_p\phi'_{22,e} + m_p\ell_2\phi_{22,e})\phi'_{11,e} \\
b_{362} &= (v_{22} + m_p\phi_{22,e})\phi_{11,e} \\
b_{363} &= -(v_{22} + m_p\phi_{22,e})\phi_{11,e} \\
b_{441} &= m_1 \\
b_{442} &= 2(m_2d_2 + m_p\ell_2)\phi_{12,e}\phi'_{12,e} \\
b_{443} &= -2\phi_{12,e}\phi'_{12,e} \\
b_{451} &= (w_{21} + J_p\phi'_{21,e} + m_p\ell_2\phi_{21,e})\phi'_{12,e} \\
b_{452} &= (v_{21} + m_p\phi_{21,e})\phi_{12,e} \\
b_{453} &= -(v_{21} + m_p\phi_{21,e})\phi_{12,e} \\
b_{461} &= (w_{22} + J_p\phi'_{22,e} + m_p\ell_2\phi_{22,e})\phi'_{12,e} \\
b_{462} &= (v_{22} + m_p\phi_{22,e})\phi_{12,e} \\
b_{463} &= -(v_{22} + m_p\phi_{22,e})\phi_{12,e} \\
b_{551} &= m_2 \\
b_{561} &= 0 \\
b_{661} &= m_2
\end{aligned}
$$

$$
\begin{aligned}
t_{11} &= \phi_{11,e} - \ell_1\phi'_{11,e} \\
t_{12} &= \phi_{12,e} - \ell_1\phi'_{12,e} \\
t_{21} &= v_{21} + m_p\phi_{21,e} \\
t_{22} &= v_{22} + m_p\phi_{22,e} \\
t_{31} &= \phi'_{11,e} \\
t_{32} &= \phi'_{12,e}
\end{aligned}
$$

A complete ortho-normalization of the mode shapes has been performed, leading to convenient simplifications in the diagonal blocks of the inertia matrix relative to the deflection variables. Additionally, the non-zero stiffness coefficients in $\mathbf{K}$ take on the values $\omega_{ij}^2 m_i$, being ω_{ij} the natural angular frequency j of the associated eigenvalue problem for link i.

The components of $\mathbf{h}$ can be evaluated through the Christoffel symbols, giving

$$
\begin{aligned}
h_1 = {} & \big[(h_{101}\dot\theta_2 + h_{102}\dot\delta_{11} + h_{103}\dot\delta_{12} + h_{104}\dot\delta_{21} + h_{105}\dot\delta_{22})\dot\theta_1 \\
& + (h_{106}\dot\theta_2 + h_{107}\dot\delta_{11} + h_{108}\dot\delta_{12} + h_{109}\dot\delta_{21} + h_{110}\dot\delta_{22})\dot\theta_2 \\
& + (h_{111}\dot\delta_{21} + h_{112}\dot\delta_{22})\dot\delta_{11} + (h_{113}\dot\delta_{21} + h_{114}\dot\delta_{22})\dot\delta_{12}\big]s_2 \\
& + \big[(h_{115}\dot\theta_1 + h_{116}\dot\theta_2 + h_{117}\dot\delta_{21} + h_{118}\dot\delta_{22})t_1 \\
& + (h_{119}\dot\theta_1 + h_{120}\dot\theta_2 + h_{121}\dot\delta_{11} + h_{122}\dot\delta_{12})t_2 \\
& + h_{123}\delta_{12}\dot\delta_{11} + h_{124}\delta_{11}\dot\delta_{12}\big]\dot\theta_2 c_2
\end{aligned}
$$

$$
\begin{aligned}
h_2 = {} & (h_{201}\dot\theta_1 + h_{202}\dot\delta_{11} + h_{203}\dot\delta_{12})\dot\theta_1 s_2 \\
& + \Big\{\big[(h_{204}\dot\theta_1 + h_{205}\dot\delta_{21} + h_{206}\dot\delta_{22})t_1 \\
& + (h_{207}\dot\theta_1 + h_{208}\dot\delta_{11} + h_{209}\dot\delta_{12})t_2 \\
& + h_{210}\delta_{12}\dot\delta_{11} + h_{211}\delta_{11}\dot\delta_{12}\big]\dot\theta_1 \\
& + \big[(h_{212}\dot\delta_{11} + h_{213}\dot\delta_{12})t_2 + (h_{214}\dot\delta_{21} + h_{215}\dot\delta_{22})t_3\big]\dot\delta_{11} \\
& + \big[h_{216}\dot\delta_{12}t_2 + (h_{217}\dot\delta_{21} + h_{218}\dot\delta_{22})t_3\big]\dot\delta_{12}\Big\}c_2
\end{aligned}
$$

$$
\begin{aligned}
h_3 = {} & \big[(h_{301}\dot\theta_1 + h_{302}\dot\theta_2 + h_{303}\dot\delta_{12} + h_{304}\dot\delta_{21} + h_{305}\dot\delta_{22})\dot\theta_1 \\
& + (h_{306}\dot\theta_2 + h_{307}\dot\delta_{11} + h_{308}\dot\delta_{12} + h_{309}\dot\delta_{21} + h_{310}\dot\delta_{22})\dot\theta_2 \\
& + (h_{311}\dot\delta_{21} + h_{312}\dot\delta_{22})\dot\delta_{11} + (h_{313}\dot\delta_{21} + h_{314}\dot\delta_{22})\dot\delta_{12}\big]s_2 \\
& + \big[(h_{315}\dot\theta_1 + h_{316}\dot\theta_2 + h_{317}\dot\delta_{11} + h_{318}\dot\delta_{12})t_2 \\
& + (h_{319}\dot\theta_2 + h_{320}\dot\delta_{21} + h_{321}\dot\delta_{22})t_3 \\
& + h_{322}\delta_{12}\dot\theta_1\big]\dot\theta_2 c_2
\end{aligned}
$$

$$
\begin{aligned}
h_4 = {} & \big[(h_{401}\dot\theta_1 + h_{402}\dot\theta_2 + h_{403}\dot\delta_{11} + h_{404}\dot\delta_{21} + h_{405}\dot\delta_{22})\dot\theta_1 \\
& + (h_{406}\dot\theta_2 + h_{407}\dot\delta_{11} + h_{408}\dot\delta_{12} + h_{409}\dot\delta_{21} + h_{410}\dot\delta_{22})\dot\theta_2 \\
& + (h_{411}\dot\delta_{21} + h_{412}\dot\delta_{22})\dot\delta_{11} + (h_{413}\dot\delta_{21} + h_{414}\dot\delta_{22})\dot\delta_{12}\big]s_2 \\
& + \big[(h_{415}\dot\theta_1 + h_{416}\dot\theta_2 + h_{417}\dot\delta_{11} + h_{418}\dot\delta_{12})t_2 \\
& + (h_{419}\dot\theta_2 + h_{420}\dot\delta_{21} + h_{421}\dot\delta_{22})t_3 \\
& + h_{422}\delta_{11}\dot\theta_1\big]\dot\theta_2 c_2
\end{aligned}
$$

$$
\begin{aligned}
h_5 = {} & (h_{501}\dot\theta_1 + h_{502}\dot\delta_{11} + h_{503}\dot\delta_{12})\dot\theta_1 s_2 \\
& + \big[h_{504}t_1\dot\theta_1 + (h_{505}\dot\delta_{11} + h_{506}\dot\delta_{12})t_3\big]\dot\theta_2 c_2
\end{aligned}
$$

$$
\begin{aligned}
h_6 = {} & (h_{601}\dot\theta_1 + h_{602}\dot\delta_{11} + h_{603}\dot\delta_{12})\dot\theta_1 s_2 \\
& + \big[h_{604}t_1\dot\theta_1 + (h_{605}\dot\delta_{11} + h_{606}\dot\delta_{12})t_3\big]\dot\theta_2 c_2
\end{aligned}
$$

with the coefficients having the following expressions:

$$
\begin{aligned}
h_{101} &= -2(m_2d_2 + m_p\ell_2)\ell_1 \\
h_{102} &= 2(m_2d_2 + m_p\ell_2)(\phi_{11,e} - \ell_1\phi'_{11,e}) \\
h_{103} &= 2(m_2d_2 + m_p\ell_2)(\phi_{12,e} - \ell_1\phi'_{12,e}) \\
h_{104} &= -2(v_{21} + m_p\phi_{21,e})\ell_1 \\
h_{105} &= -2(v_{22} + m_p\phi_{22,e})\ell_1 \\
h_{106} &= -(m_2d_2 + m_p\ell_2)\ell_1 \\
h_{107} &= -(m_2d_2 + m_p\ell_2)\ell_1\phi'_{11,e}) \\
h_{108} &= -2(m_2d_2 + m_p\ell_2)\ell_1\phi'_{12,e}) \\
h_{109} &= -2(v_{21} + m_p\phi_{21,e})\ell_1 \\
h_{110} &= -2(v_{22} + m_p\phi_{22,e})\ell_1 \\
h_{111} &= -2(v_{21} + m_p\phi_{21,e})\ell_1\phi'_{11,e}
\end{aligned}
$$

$$h_{112} = -2(v_{22} + m_p\phi_{22,e})\ell_1\phi'_{11,e}$$
$$h_{113} = -2(v_{21} + m_p\phi_{21,e})\ell_1\phi'_{12,e}$$
$$h_{114} = -2(v_{22} + m_p\phi_{22,e})\ell_1\phi'_{12,e}$$
$$h_{115} = 2(m_2d_2 + m_p\ell_2)$$
$$h_{116} = m_2d_2 + m_p\ell_2$$
$$h_{117} = -(v_{21} + m_p\phi_{21,e})$$
$$h_{118} = -(v_{22} + m_p\phi_{22,e})$$
$$h_{119} = -2\ell_1$$
$$h_{120} = -\ell_1$$
$$h_{121} = -(\phi_{11,e} + \ell_1\phi'_{11,e})$$
$$h_{122} = -(\phi_{12,e} + \ell_1\phi'_{12,e})$$
$$h_{123} = -(m_2d_2 + m_p\ell_2)(\phi_{11,e}\phi'_{12,e} - \phi_{12,e}\phi'_{11,e})$$
$$h_{124} = -(m_2d_2 + m_p\ell_2)(\phi_{12,e}\phi'_{11,e} - \phi_{11,e}\phi'_{12,e})$$
$$h_{201} = (m_2d_2 + m_p\ell_2)\ell_1$$
$$h_{202} = 2(m_2d_2 + m_p\ell_2)\phi_{11,e}$$
$$h_{203} = 2(m_2d_2 + m_p\ell_2)\phi_{12,e}$$
$$h_{204} = -(m_2d_2 + m_p\ell_2)$$
$$h_{205} = -(v_{21} + m_p\phi_{21,e})$$
$$h_{206} = -(v_{22} + m_p\phi_{22,e})$$
$$h_{207} = \ell_1$$
$$h_{208} = \phi_{11,e} + \ell_1\phi'_{11,e}$$
$$h_{209} = \phi_{12,e} + \ell_1\phi'_{12,e}$$
$$h_{210} = (m_2d_2 + m_p\ell_2)(\phi_{11,e}\phi'_{12,e} - \phi_{12,e}\phi'_{11,e})$$
$$h_{211} = (m_2d_2 + m_p\ell_2)(\phi_{12,e}\phi'_{11,e} - \phi_{11,e}\phi'_{12,e})$$
$$h_{212} = \phi_{11,e}\phi'_{11,e}$$
$$h_{213} = \phi_{11,e}\phi'_{12,e} + \phi_{12,e}\phi'_{11,e})$$
$$h_{214} = (v_{21} + m_p\phi_{21,e})\phi_{11,e}$$
$$h_{215} = (v_{22} + m_p\phi_{22,e})\phi_{11,e}$$
$$h_{216} = \phi_{12,e}\phi'_{12,e}$$
$$h_{217} = (v_{21} + m_p\phi_{21,e})\phi_{12,e}$$
$$h_{218} = (v_{22} + m_p\phi_{22,e})\phi_{12,e}$$
$$h_{301} = -(m_2d_2 + m_p\ell_2)(\phi_{11,e} - \ell_1\phi'_{11,e})$$
$$h_{302} = -2(m_2d_2 + m_p\ell_2)\phi_{11,e}$$
$$h_{303} = 2(m_2d_2 + m_p\ell_2)(\phi_{12,e}\phi'_{11,e} - \phi_{11,e}\phi'_{12,e})$$
$$h_{304} = -2(v_{21} + m_p\phi_{21,e})\phi_{11,e}$$
$$h_{305} = -2(v_{22} + m_p\phi_{22,e})\phi_{11,e}$$
$$h_{306} = -(m_2d_2 + m_p\ell_2)\phi_{11,e}$$
$$h_{307} = -2(m_2d_2 + m_p\ell_2)\phi_{11,e}\phi'_{11,e}$$
$$h_{308} = -2(m_2d_2 + m_p\ell_2)\phi_{11,e}\phi'_{12,e}$$
$$h_{309} = -2(v_{21} + m_p\phi_{21,e})\phi_{11,e}$$
$$h_{310} = -2(v_{22} + m_p\phi_{22,e})\phi_{11,e}$$
$$h_{311} = -2(v_{21} + m_p\phi_{21,e})\phi_{11,e}\phi'_{11,e}$$
$$h_{312} = -2(v_{22} + m_p\phi_{22,e})\phi_{11,e}\phi'_{11,e}$$
$$h_{313} = -2(v_{21} + m_p\phi_{21,e})\phi_{11,e}\phi'_{12,e}$$
$$h_{314} = -2(v_{22} + m_p\phi_{22,e})\phi_{11,e}\phi'_{12,e}$$
$$h_{315} = -(\phi_{11,e} + \ell_1\phi'_{11,e})$$
$$h_{316} = -\phi_{11,e}$$
$$h_{317} = -2\phi_{11,e}\phi'_{11,e}$$
$$h_{318} = -(\phi_{11,e}\phi'_{12,e} + \phi_{12,e}\phi'_{11,e})$$
$$h_{319} = -(m_2d_2 + m_p\ell_2)\phi_{11,e}$$
$$h_{320} = -(v_{21} + m_p\phi_{21,e})\phi_{11,e}$$
$$h_{321} = -(v_{22} + m_p\phi_{22,e})\phi_{11,e}$$
$$h_{322} = -(m_2d_2 + m_p\ell_2)(\phi_{11,e}\phi'_{12,e} - \phi_{12,e}\phi'_{11,e})$$
$$h_{401} = -(m_2d_2 + m_p\ell_2)(\phi_{12,e} - \ell_1\phi'_{12,e})$$
$$h_{402} = -2(m_2d_2 + m_p\ell_2)\phi_{12,e}$$
$$h_{403} = 2(m_2d_2 + m_p\ell_2)(\phi_{11,e}\phi'_{12,e} - \phi_{12,e}\phi'_{11,e})$$
$$h_{404} = -2(v_{21} + m_p\phi_{21,e})\phi_{12,e}$$
$$h_{405} = -2(v_{22} + m_p\phi_{22,e})\phi_{12,e}$$
$$h_{406} = -(m_2d_2 + m_p\ell_2)\phi_{12,e}$$
$$h_{407} = -2(m_2d_2 + m_p\ell_2)\phi_{12,e}\phi'_{11,e}$$
$$h_{408} = -2(m_2d_2 + m_p\ell_2)\phi_{12,e}\phi'_{12,e}$$
$$h_{409} = -2(v_{21} + m_p\phi_{21,e})\phi_{12,e}$$
$$h_{410} = -2(v_{22} + m_p\phi_{22,e})\phi_{12,e}$$
$$h_{411} = -2(v_{21} + m_p\phi_{21,e})\phi_{12,e}\phi'_{11,e}$$
$$h_{412} = -2(v_{22} + m_p\phi_{22,e})\phi_{12,e}\phi'_{12,e}$$
$$h_{413} = -2(v_{21} + m_p\phi_{21,e})\phi_{12,e}\phi'_{11,e}$$
$$h_{414} = -2(v_{22} + m_p\phi_{22,e})\phi_{12,e}\phi'_{12,e}$$
$$h_{415} = -(\phi_{12,e} + \ell_1\phi'_{12,e})$$
$$h_{416} = -\phi_{12,e}$$
$$h_{417} = -(\phi_{11,e}\phi'_{12,e} + \phi_{12,e}\phi'_{11,e})$$
$$h_{418} = -2\phi_{12,e}\phi'_{12,e}$$
$$h_{419} = -(m_2d_2 + m_p\ell_2)\phi_{12,e}$$
$$h_{420} = -(v_{21} + m_p\phi_{21,e})\phi_{12,e}$$
$$h_{421} = -(v_{22} + m_p\phi_{22,e})\phi_{12,e}$$
$$h_{422} = -(m_2d_2 + m_p\ell_2)(\phi_{12,e}\phi'_{11,e} - \phi_{11,e}\phi'_{12,e})$$
$$h_{501} = (v_{21} + m_p\phi_{21,e})\ell_1$$
$$h_{502} = 2(v_{21} + m_p\phi_{21,e})\phi_{11,e}$$
$$h_{503} = 2(v_{21} + m_p\phi_{21,e})\phi_{12,e}$$
$$h_{504} = v_{21} + m_p\phi_{21,e}$$
$$h_{505} = -(v_{21} + m_p\phi_{21,e})\phi_{11,e}$$
$$h_{506} = -(v_{21} + m_p\phi_{21,e})\phi_{12,e}$$
$$h_{601} = (v_{22} + m_p\phi_{22,e})\ell_1$$
$$h_{602} = 2(v_{22} + m_p\phi_{22,e})\phi_{11,e}$$
$$h_{603} = 2(v_{22} + m_p\phi_{22,e})\phi_{12,e}$$
$$h_{604} = v_{22} + m_p\phi_{22,e}$$
$$h_{605} = -(v_{22} + m_p\phi_{22,e})\phi_{11,e}$$
$$h_{606} = -(v_{22} + m_p\phi_{22,e})\phi_{12,e}$$

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