2001 IEEE/ASME International Conference on
Advanced Intelligent Mechatronics Proceedings
8—12 July 2001 • Como, Italy

# Control Problems in Underactuated Manipulators

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*Abstract*— **We discuss some recent control techniques for underactuated manipulators, a special instance of mechanical systems having fewer input commands than degrees of freedom. This class includes robots with passive joints, elastic joints, or flexible links. Structural system properties are investigated showing that robots with passive joints are the most difficult to control. With reference to these, solutions are proposed for the typical problems of trajectory planning and tracking, and of set-point regulation. The relevance of nonlinear control techniques such as dynamic feedback linearization and iterative state steering is clarified through illustrative examples.**

## I. Introduction

In recent years, a remarkable effort has been devoted to the study of underactuated manipulators, a special class of second-order mechanical systems with fewer control inputs than degrees of freedom (e.g., see [1]) encompassing, among the others, rigid robots with passive joints, rigid robots with elastic joints, and robot with flexible links.

Despite of this broad definition, different underactuated systems do not share the same difficulties from the control point of view. In particular, it should be recognized that robots with passive joints raise by far the most challenging theoretical problems, typically requiring non-classical control approaches. To clarify this issue, we shall provide some inherent structural reasons for such a peculiar difference, based on the analysis of basic control properties of general underactuated manipulator dynamics.

Dynamic modeling, trajectory planning and feedback control of specific instances of underactuated mechanical systems have already been investigated; significant theoretical results can be found in [2] and [3]. However, since a general theory for these mechanisms is not yet available, only case-by-case control solutions have been obtained so far.

Motivated by this, we sketch a review of the most significant case studies found in the literature of underactuated robots with passive joints. Exploiting results from advanced nonlinear control theory, we describe in some detail two quite general approaches which have proved effective in controlling these systems, namely dynamic linearization via feedback and iterative state steering. For illustration, we also work out the application of these techniques to examples of planar robots with passive joints.

The paper is organized as follows. In Sect. II, underactuated manipulators are described in a unified framework. The typical planning and control problems are defined in Sect. III, while Sect. IV presents an analysis of the control properties. Dynamic linearization and iterative steering techniques are used in Sect. V and VI, respectively for tracking and regulation of robots with passive joints.

## II. Underactuated Manipulators

Robot dynamics can be generally described by

$$B(\theta)\ddot{\theta} + c(\theta, \dot{\theta}) + e(\theta) = G(\theta)\,\tau, \qquad (1)$$

where $\theta \in I\!R^n$ is the vector of generalized coordinates, $B(\theta)$ is the inertia matrix, $c(\theta, \dot{\theta})$ and $e(\theta)$ are respectively the vectors of velocity (Coriolis/centrifugal) and potential (gravitational/elastic) terms, and $G(\theta)$ is the matrix mapping the external forces/torques $\tau \in I\!R^m$ acting on the system into generalized forces.

When $m < n$, the robot is said to be *underactuated* (of degree $n - m$), and the mechanical system has less control inputs than generalized coordinates. If matrix $G$ has full column rank, it is easy (e.g., see [4]) to show that, by performing an input transformation and a change of coordinates, the system dynamics takes on the partitioned structure

$$\begin{pmatrix} B_{aa} & B_{ua}^T \\ B_{ua} & B_{uu} \end{pmatrix} \begin{pmatrix} \ddot{q}_a \\ \ddot{q}_u \end{pmatrix} + \begin{pmatrix} c_a \\ c_u \end{pmatrix} + \begin{pmatrix} e_a \\ e_u \end{pmatrix} = \begin{pmatrix} \tau_a \\ 0 \end{pmatrix}, \qquad (2)$$

where $\tau_a \in I\!R^m$. With a slight abuse of notation, we have kept the same symbols of eq. (1).

The new vector of generalized coordinates $q$ displays the partition in *actuated* and *unactuated* degrees of freedom, respectively $q_a \in I\!R^m$ and $q_u \in I\!R^{n-m}$. The last $n-m$ equations of the dynamics (2) represent a set of second-order differential constraints that must be satisfied for motion trajectories to be feasible.

In principle, eq. (2) includes the following types of $N$-joint robots:

a) rigid robots with $n_a$ active and $n_p = N - n_a$ passive joints:

$$n = n_a + n_p, \qquad m = n_a;$$

b) robots with $n_r$ rigid and $n_e = N - n_r$ elastic joints:

$$n = n_r + 2n_e, \qquad m = n_r + n_e;$$

c) robots with $n_r$ rigid and $n_f = N - n_r$ flexible links, each modeled by $\eta_f$ deformation modes:

$$n = n_r + (\eta_f + 1)n_f, \qquad m = n_r + n_f.$$

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Another interesting example of underactuated robot is represented by a kinematically redundant manipulator with all joints passive and forces/torques applied to the end-effector as the only available input command [4]. From a control viewpoint, however, this kind of mechanism is equivalent to a manipulator with both active and passive joints.

## III. Planning and Control Problems

As in the general case, the following problems arise when attempting the control of underactuated robots:

P1 *Trajectory planning*
Given an initial configuration $q^0$ and a final desired configuration $q^d$, compute a (dynamically feasible) trajectory that joins $q^0$ and $q^d$.

P2 *Trajectory tracking*
Given a dynamically feasible trajectory $q^d(t)$, compute a feedback control that asymptotically drives the tracking error $e = q^d - q$ to zero.

P3 *Set-point regulation*
Given a desired configuration $q^d$, compute a feedback control that makes the equilibrium state $q = q^d$, $\dot{q} = 0$ asymptotically stable.

If it is not possible to solve the assigned trajectory planning problem P1 — i.e., if no feasible trajectory exists joining $q^0$ and $q^d$ for the given system — the corresponding trajectory planning problem P2 and set-point regulation problem P3 clearly become meaningless. On the other hand, it may happen that a trajectory joining $q^0$ and $q^d$ exists, but we are not able to compute it in advance through planning; in this case, it may still be possible to solve P3, obtaining as a byproduct an asymptotic solution to P1. The study of controllability and stabilizability for the given system should clarify the situation in this respect.

## IV. Control Properties of Underactuated Robots

To simplify the analysis and the control design of underactuated robots, it is convenient to perform a partial feedback linearization of eq. (2). Solving the second equation for $\ddot{q}_u$ and substituting in the first, one finds that the (globally defined) static feedback

$$\begin{aligned} \tau_a &= \left(B_{aa} - B_{ua}^T B_{uu}^{-1} B_{ua}\right) a \\ &\quad + c_a + e_a - B_{ua}^T B_{uu}^{-1} \left(c_u + e_u\right) \end{aligned} \qquad (3)$$

yields a system in the form

$$\ddot{q}_a = a \qquad (4)$$

$$B_{uu}\, \ddot{q}_u = -B_{ua}\, a - c_u - e_u \qquad (5)$$

where the actuated degrees of freedom are now directly controlled by the new acceleration input $a$.

A preliminary step in the assessment of control properties for the underactuated system (4–5) is to check whether the second-order differential constraint (5) (with $a = \ddot{q}_a$) is integrable, either partially or completely, in the sense of [2]. If this is the case, while it is still trivially possible to control the coordinates $q_a$, full configuration controllability is lost; as a consequence, problems P1–P3 admit no solution, except for very special cases.

Equilibrium controllability can be checked in the first approximation by computing the tangent linearization of system (4–5) around the considered state. Define the equilibrium set as $\mathcal{Q} = \{q = q_e : e_u(q_e) = 0,\ \dot{q} = 0\}$. Since $c_u$ is quadratic in $\dot{q}$, the tangent linearization of the system at any point of $\mathcal{Q}$ is obtained as

$$\begin{aligned} \delta\ddot{q}_a &= a \\ B_{uu}(q_e)\, \delta\ddot{q}_u + \nabla_q^T e_u(q_e)\, \delta q &= -B_{ua}(q_e)\, a. \end{aligned}$$

It is rather immediate to see that the system is not controllable in the first approximation if $\nabla_q e_u(q_e) = 0$.

In particular, linear controllability is lost in the case of simultaneous absence of gravitational and flexibility/elasticity effects on the unactuated degrees of freedom ($e_u(q) \equiv 0$, $\forall q$). On the other hand, robots with elastic joints and/or flexible arms, or robots with passive joints subject to gravity are examples of systems that are controllable in the first approximation. In all these cases, problems P1, P2, and P3 can be solved (at least locally) by standard techniques. For example, in the presence of elastic joints, solution methods for problems P1/P2 and P3 are given, respectively, in [5] and [6]. As for robots with flexible links, the reader may refer to [7] for trajectory planning, to [8] for tracking and to [9] for set-point regulation techniques. Finally, examples of robots with passive joints in the presence of gravity are the so-called Acrobot and Pendubot, e.g., see [1], [10], [11], [12].

As an outcome of this analysis, we shall consider henceforth the most difficult case, i.e., robots with passive joints that are not subject to any kind of potential energy. As linear controllability is lost, it is necessary to resort to nonlinear controllability concepts. Among these, the most elementary is *accessibility*, which may be easily tested through the well-known Lie Algebra Rank Condition [13]; however, such characterization does not imply full controllability, because system (4–5) has a (nontrivial) drift term.

A more appropriate concept for our study is *small-time local controllability* (STLC), see [14]. If such property is violated by an underactuated manipulator with passive joints, the mechanism may still be controllable in the natural sense; nevertheless, one may safely argue that problem P1 (trajectory planning) becomes very difficult, while P2 (trajectory tracking) cannot be solved in general. In fact, roughly speaking, the lack of STLC suggests that the robot must perform maneuvers in order to achieve arbitrarily small reconfigurations. Therefore, while a trajectory joining any two given configurations may exist, its planning may be out of reach, at least with the available techniques. Similarly, asymptotic trajectory tracking becomes impossible, since it requires the possibility of recovering small errors by keeping the system close to the desired evolution. Interestingly, however, problem P3 (set-point regulation) for non-STLC systems may still be solvable; we will present an example in Sect. VI-A.

Unfortunately, only sufficient conditions [14], [15] are available for testing STLC. In particular, in the absence of gravity underactuated robots with passive joints having a single actuation ($m = 1$) violate these conditions, e.g., see [16]. In any case, even if small-time local controllability is guaranteed, there exist no systematic design method for solving P1, P2 or P3. An additional difficulty encountered in the set-point regulation problem is that, as shown in [2], robots with passive joints in the absence of gravity cannot be stabilized by smooth static feedback, for they violate the necessary condition due to Brockett [17].

The above considerations should clarify the severe theoretical difficulties arising when addressing control problems for robots with passive joints, and justify the variety of approaches taken by researchers in order to solve them on a case-by-case basis. An overview of case studies found in the related literature is given in Table I. In the following, we briefly illustrate two methods that have proven to be effective and generalizable, at least to some degree, to significant classes of underactuated manipulators: the exact linearization via dynamic feedback and the iterative steering techniques, respectively for trajectory planning/tracking and set-point regulation problems.

## V. Trajectory Planning and Tracking via Dynamic Feedback Linearization

The exact linearization technique via dynamic feedback [18] represents an effective solution to the P1 (trajectory planning) and P2 (trajectory tracking) problems, provided that a set of *linearizing* (also called *flat*) outputs $z \in I\!R^m$ exists. Such outputs have the property that the whole state and the input of the system can be written in terms of $z$ and its time derivatives. In this case, it is possible to build a dynamic compensator of the form

$$\begin{aligned}\dot{\xi} &= \alpha(\xi, q, \dot{q}) + \beta(\xi, q, \dot{q})\, v \\ a &= \gamma(\xi, q, \dot{q}) + \delta(z, q, \dot{q})\, v,\end{aligned}$$

with state $\xi \in I\!R^\nu$ and new input $v \in I\!R^m$, such that the closed-loop system is input-state-output linear and decoupled, i.e., represented by chains of integrators between $v$ and $z$.

Once the above construction has been carried out, the trajectory planning problem (P1) can be formulated and easily solved as a simple interpolation problem on the linearized system. An interesting byproduct of this approach is that linear control techniques may be applied to stabilize the linear tracking error dynamics, thus providing a straightforward solution also to the problem of tracking the planned trajectory (P2).

It should be mentioned that singularities may arise in the control design phase, essentially because dynamic feedback linearization is inherently based on model inversion. In such cases, these must be carefully kept into account and avoided when planning the trajectory via interpolation. This can be usually achieved by appropriately choosing the (re)initialization of the dynamic compensator state —actually an additional degree of freedom available in the control design.

In [19] and [20] we have shown that planar three-link (or $n$-link) robots with passive rotational third (or last) joint can be exactly linearized via feedback, with or without gravity. The linearizing output is the cartesian position of the center of percussion (CP) of the third link. In the following, we show that the same technique can be applied in the presence of a double degree of underactuation, provided that a special mechanical condition is satisfied.

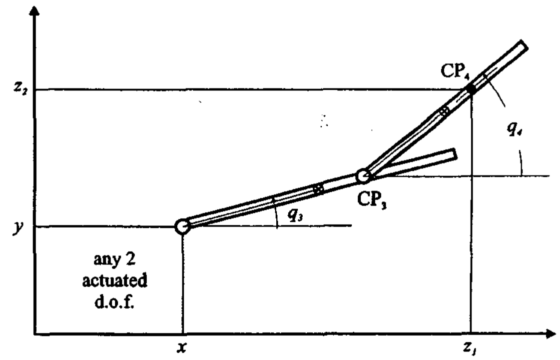


Fig. 1. An underactuated XYRR robot

### A. Example: An Underactuated XYRR Robot

An XYRR planar robot is a mechanism where the two distal joints are rotational, while the two proximal d.o.f.'s may be any combination of prismatic and rotational joints. Assume that only the first two joints are actuated, that the fourth link is hinged exactly at the center of percussion (CP$_3$) of the third link (see Fig. 1), and that the robot moves in the horizontal plane. Denote by $l_i$, $d_i$ and $k_i$ ($i = 3, 4$) respectively the length of the $i$-th link, the distance between the $i$-th joint axis and the $i$-th link center of mass, and the distance between the $i$-th joint axis and the $i$-th link center of percussion CP$_i$. In the considered case, we have

$$k_3 = \frac{I_3 + m_3 d_3^2}{m_3 d_3} = l_3, \qquad k_4 = \frac{I_4 + m_4 d_4^2}{m_4 d_4},$$

where $I_i$ represents the centroidal moment of inertia of the $i$-th link.

Choose the generalized coordinates as $q = (q_a, q_u) = (x, y, q_3, q_4)$, where $(x, y)$ are the cartesian coordinates of the base of the third link and $q_3$, $q_4$ the (absolute) orientation of the last two links w.r.t. the $x$-axis. After the partial feedback linearization procedure, the robot dynamic equations become:

$$\begin{aligned}\ddot{x} &= a_x \\ \ddot{y} &= a_y \\ l_3\ddot{q}_3 + \lambda_{34}c_{34}\ddot{q}_4 &= s_3 a_x - c_3 a_y - \lambda_{34}s_{34}\dot{q}_4^2 \\ l_3 c_{34}\ddot{q}_3 + k_4\ddot{q}_4 &= s_4 a_x - c_4 a_y + l_3 s_{34}\dot{q}_3^2,\end{aligned}$$

where we have set for compactness $s_i = \sin q_i$, $c_i = \cos q_i$, $s_{ij} = \sin(q_i - q_j)$, $c_{ij} = \cos(q_i - q_j)$ $(i, j = 3, 4)$ and $\lambda_{34} = m_4 l_3 d_4/(m_3 d_3 + m_4 l_3)$. Note that the last two equations have been conveniently scaled by constant factors. The inputs to the mechanism are the accelerations $a_x$ and $a_y$.

The linearizing outputs for the system are the cartesian coordinates of $CP_4$, the center of percussion of the fourth link (see Fig. 1):

$$\begin{aligned} z_1 &= x + l_3 c_3 + k_4 c_4 \\ z_2 &= y + l_3 s_3 + k_4 s_4. \end{aligned}$$

Following the design guidelines of [19], we obtain a dynamic compensator of dimension $\nu = 4$, with state equations

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 + \dot{q}_4^2 \xi_1 \\ \dot{\xi}_3 &= \xi_4 + 2\dot{q}_4^2 \xi_2 - \mu\, t_{34}\, \dot{q}_4\, \xi_1 \\ \dot{\xi}_4 &= u_1 + \phi\, \dot{q}_4 - \psi(\dot{q}_3 - \dot{q}_4)\dot{q}_4 \end{aligned}$$

and output equation

$$\begin{pmatrix} a_x \\ a_y \end{pmatrix} = R(q_3) \begin{pmatrix} \frac{1}{c_{34}} \left( \frac{k_4 - \lambda_{34} c_{34}}{k_4 - \lambda_{34}} \xi_1 + k_4 \dot{q}_4^2 \right) + l_3 \dot{q}_3^2 \\ u_2 \end{pmatrix},$$

where $R(q_3)$ is rotation matrix defined by the angle $q_3$, and we have set

$$\begin{aligned} t_{34} &= s_{34}/c_{34} \\ \mu &= \frac{\xi_1}{k_4 - \lambda_{34}} + \dot{q}_4^2 \\ \psi &= \mu \xi_1 / c_{34}^2 \\ \phi &= 2\dot{q}_4^3 \xi_1 - 3 t_{34}\, \mu\, \xi_2 + 3\dot{q}_4\, \xi_3 - t_{34}\, \xi_1\, \dot{\mu}. \end{aligned}$$

The auxiliary inputs $u_1$ and $u_2$ are obtained by inversion control:

$$\begin{aligned} u_1 &= c_4 v_1 + s_4 v_2 \\ u_2 &= \frac{l_3}{\psi} \left( c_4 v_2 - s_4 v_1 - \dot{q}_4\, \xi_4 + (\dot{q}_3 - \dot{q}_4)\dot{\psi} - \dot{\phi} + \psi \delta \right), \end{aligned}$$

with $(v_1, v_2)$ the new input vector and

$$\delta = t_{34} \left( \frac{l_3 + \lambda_{34}\, c_{34}}{l_3 (k_4 - \lambda_{34})} \xi_1 + \dot{q}_4^2 \right).$$

Under the action of the above dynamic compensator, the system is completely linearized and decoupled:

$$\begin{aligned} \frac{d^6 z_1}{dt^6} &= v_1 \\ \frac{d^6 z_2}{dt^6} &= v_2, \end{aligned} \qquad (6)$$

i.e., two chains of six integrators from input to output.

Planning a feasible trajectory on the equivalent representation (6) can be formulated as a smooth interpolation problem for the two outputs $z_1(t)$ and $z_2(t)$. For example, one could use polynomial functions to join the initial $z^0$ (corresponding to the starting configuration $q^0$) with the final $z^d$ (corresponding to the final configuration $q^d$), with appropriate boundary conditions on the derivatives of $z$ up to the fifth order. However, it should be considered that the above linearization procedure is valid only if the following regularity conditions are satisfied

$$c_{34} \neq 0 \qquad \text{and} \qquad \psi \neq 0$$

throughout the motion. These conditions can be easily given an interesting physical interpretation. In particular, $c_{34} \neq 0$ means that the fourth link should never become orthogonal to the third, while $\psi \neq 0$ holds as long as the acceleration $\xi_1$ of the center of percussion $CP_4$ along the fourth link axis does not vanish during the motion. Besides, being $\xi_1^2 = \ddot{z}_1^2 + \ddot{z}_2^2$, the regularity condition can be checked directly from the linearizing outputs trajectory, without actually computing $\xi_1$. In any case, one way to avoid the singularity during the motion is to reset the component $\xi_1$ of the dynamic compensator state whenever it approaches zero.

For illustration, the trajectory planning technique outlined above has been applied to generate a feasible trajectory from $q^0 = (1, 1, 0, \pi/8)$ to $q^d = (1, 2, 0, \pi/4)$ (m,m,rad,rad) for an underactuated XYRR robot with $l_3 = k_3 = 1$, $l_4 = 1$, $k_4 = 2/3$ and $\lambda_{34} = 1/3$ (m). The resulting trajectory for the center of percussion $CP_4$ of the fourth link is shown in Fig. 2, while the corresponding cartesian motion of the last two links is depicted in Fig. 3 and Fig. 4 (stroboscopic view). The two last links undergo a counterclockwise rotation of 360°. Assuming that also the first two joints are rotational, the motion of the whole manipulator appears as in Fig. 5.

As already mentioned, this approach also yields a straigthforward solution to the trajectory tracking problem: a simple linear controller (e.g., a generalized PD[5] controller) on the linearized dynamics (6) will drive the tracking error exponentially to zero. Figure 6 shows this convergence for the case of the third link base initially placed at a cartesian position corresponding to an off-trajectory start (i.e., with initial output errors $e_i = z_i^d - z_i \neq 0$, for $i = 1, 2$).

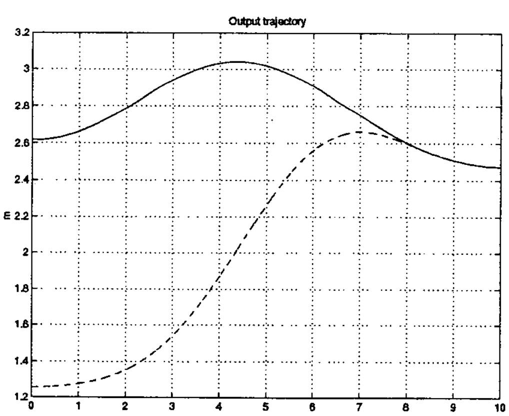


Fig. 2. Trajectory planning: linearizing outputs $z_1$ (—), $z_2$ (- -)

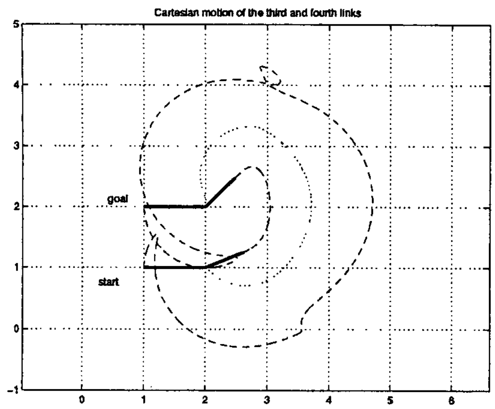


Fig. 3. Trajectory planning: cartesian motion of the last two links

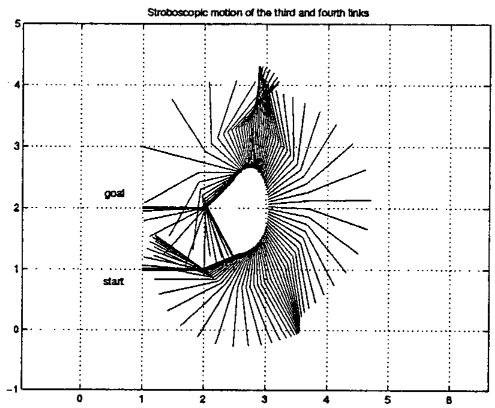


Fig. 4. Trajectory planning: stroboscopic motion of the last two links

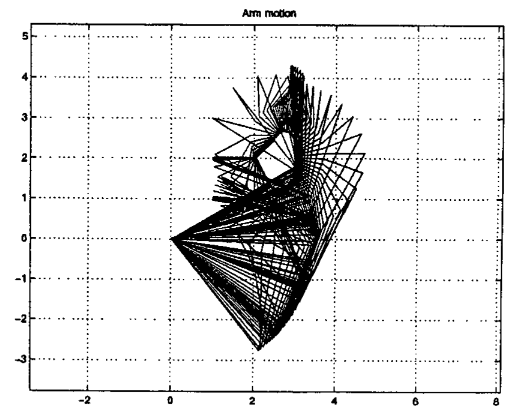


Fig. 5. Trajectory planning: stroboscopic motion of the manipulator

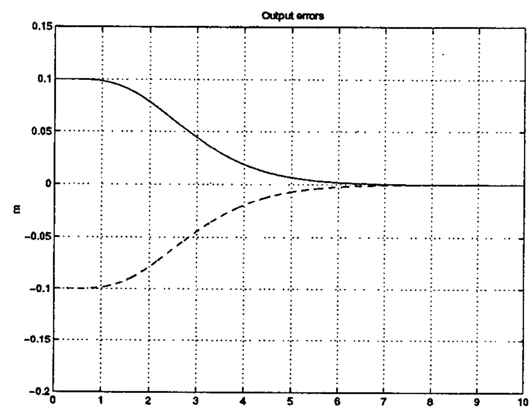


Fig. 6. Trajectory tracking: Output errors $e_1$ (—), $e_2$ (- -)

## VI. Set-Point Regulation via Iterative Steering

This technique for solving problem P3 is based on the general stabilization framework proposed in [21]. For systems in the form (4–5), it requires in principle the application of two control phases [16]. In the first, called *alignment*, the active joints $q_a$ are brought to the desired equilibrium $(q_a^d, 0)$ using, e.g., a terminal controller or a fast PD controller. At the end of this phase, the passive joints $q_u$ will have drifted to some position and velocity, and must be driven to the desired state $(q_u^d, 0)$ allowing the active joints $q_a$ to cycle over $(q_a^d, 0)$. This is achieved in the *contraction* phase by the iterative application of a finite-time open-loop controller whose task is to decrease at each iteration, of (possibly varying) period $T$, the passive joint state error $(q_u^d - q_u(kT), -\dot{q}_u(kT))$ while guaranteeing $q_a(kT) = q_a^d$ and $\dot{q}_a(kT) = 0$, with $k = 1, 2, \ldots$ . At the end of each iteration, the state of the system is measured and the parameters of the open-loop controller are accordingly updated, resulting in a sampled feedback action. The general results of [21] indicate how to choose the open-loop controller so as to obtain asymptotic stability of the desired equilibrium, with exponential rate of convergence.

One difficulty in applying the conceptual approach outlined above to system (4–5) lies in the derivation of a suitable open-loop controller; this is essentially due to the presence in the dynamic equations of a drift term which makes their forward integration impossible. As proposed in [16], an useful tool to this end is the *nilpotent approximation* of the system, which is by construction polynomial and triangular (and hence forward-integrable) but retains the controllability properties of the original dynamics. On the approximate system, it is possible to compute an appropriate open-loop controller which satisfies the conditions of [21]. However, one may find that this controller works only from certain *contraction* regions of the passive joint state space. In this case, it may be necessary to perform an additional intermediate phase, called *transition*, between alignment and contraction, so as to bring $(q_u, \dot{q}_u)$ from the value attained at the end of the first phase to a state belonging to one of the contraction regions. The design of the transition phase depends on the specific mechanism under consideration.

In [16], we presented a complete solution to the P3 prob-

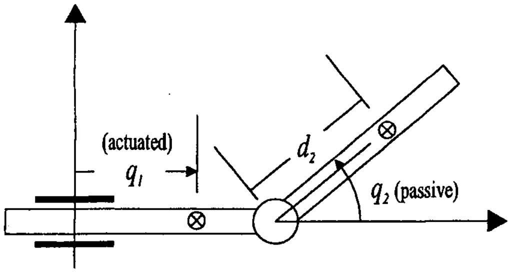


Fig. 7. An underactuated PR robot

lem for a 2R planar robot with passive second joint. Hereafter, we sketch the application of the same technique to an underactuated PR robot (detailed in [22]). This system does not satisfy the weakest available sufficient conditions for STLC, as given in [4].

### *A. Example: An Underactuated PR Robot*

A planar PR robot with a passive second joint[1] is shown in Fig. 7. After partial feedback linearization, the system dynamics takes the form:

$$\begin{aligned} \ddot{q}_1 &= a \\ \ddot{q}_2 &= \frac{1}{k_2} \sin q_2 \, a, \end{aligned}$$

where $q_1$, $q_2$ are the generalized coordinates and we have set $k_2 = (I_2 + m_2 d_2^2)/m_2 d_2$, with $m_2$, $I_2$ respectively the mass and the centroidal moment of inertia of the second link, and $d_2$ the distance between the second joint axis and the center of mass of the second link.

For the alignment phase, we can use a simple PD controller

$$a = k_p(q_1^d - q_1) - k_d \dot{q}_1, \qquad k_p, k_d > 0$$

to bring the first joint to the desired position. Denoting by $(q_{2k}, \dot{q}_{2k})$, the passive joint state at the beginning of the $k$-th iteration ($k = 1, 2, \ldots$), the contraction phase is obtained by the iterated application of the polynomial open-loop controller for a period $T_k$

$$a(t) = \frac{A_k}{T_k^2} \left(42\lambda^5 - 105\lambda^4 + 90\lambda^3 - 30\lambda^2 + 3\lambda\right)$$

where $\lambda = t/T_k$ and

$$T_k = (1 - \eta_1) \frac{q_{2d} - q_{2k}}{\dot{q}_{2k}} \tag{7}$$

$$A_k = \sqrt{\frac{T_k(1 - \eta_2)\dot{q}_{2k}}{k_2^2 \beta \sin 2q_{2k}}}, \tag{8}$$

being $\beta = 3/80080$, and $\eta_1$, $\eta_2 \in (0, 1)$ are the chosen contraction rates. The above open-loop controller has been designed on the basis of the nilpotent approximation of the system computed at $(q_k, \dot{q}_k)$:

$$\begin{aligned} \dot{\zeta}_1 &= 1 \\ \dot{\zeta}_2 &= a \\ \dot{\zeta}_3 &= -\zeta_2 \\ \dot{\zeta}_4 &= -\frac{k_2 \dot{q}_{2k}^2}{4 \cos q_{2k}} \zeta_1^2 - \frac{1}{2} \zeta_3. \end{aligned}$$

This local approximation is expressed in a new state $\zeta$, related to the original state $(q, \dot{q})$ through a change of coordinates based on the structure of the system Lie Algebra.

For illustration, the set-point regulation technique described above has been applied to stabilize an underactuated PR robot having $k_2 = 2.5$ m at the configuration $q^d = (0, \pi/4)$ starting from an initial configuration $q^0 = (1, -\pi/4)$ (m,rad). Figure 8 shows the joint evolution during the alignment, transition and contraction phases. Note how the second joint velocity is kept constant at the end of the alignment phase (by setting $a = 0$) until $q_2$ enters the contraction region, implicitly defined by the conditions $0 < T_k < \infty$ and $0 < A_k < \infty$ in eqs. (7–8). The acceleration command $a$ and the actual input force command $\tau_a$ on the active prismatic joint are reported in Fig. 9.

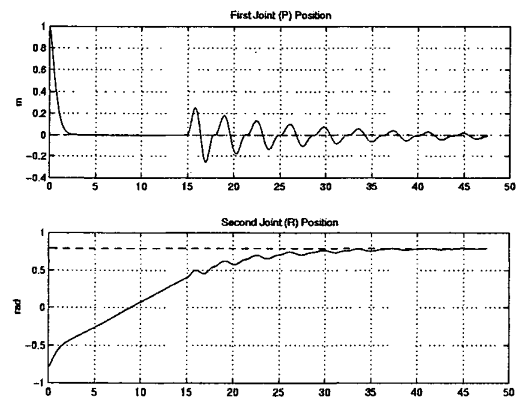


Fig. 8. Set-point regulation: Joint evolutions

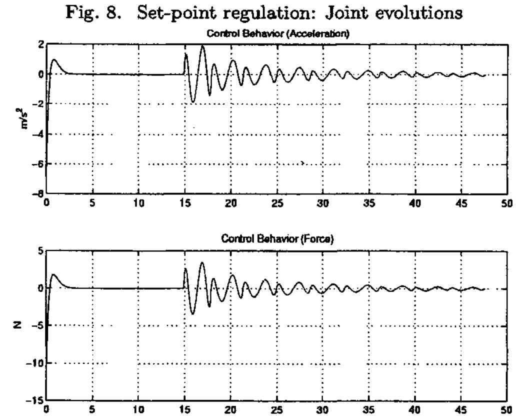


Fig. 9. Set-point regulation: first joint acceleration $a$ and force $\tau_a$

[1]Based on the results in [2], it is immediate to show that the same manipulator with a passive *first* joint is integrable, in the sense that the second-order differential constraint (2) turns out to be holonomic. Therefore, such a mechanism would not be controllable.

| Robot | Actuation | Notes | Trajectory Planning (P1) and Tracking (P2) | Set-point Regulation (P3) |
|---|---|---|---|---|
| 2R | second joint passive | | open | periodic/Poincarè [23] iterative steering [16] |
| 2R | first joint passive | integrable [2] | – | – |
| 2R+g | second joint passive | Pendubot | open | energetic [10] |
| 2R+g | first joint passive | Acrobot | open | energetic [11] iterative steering [12] |
| PR | second joint passive | | open | iterative steering [here,22] |
| PR | first joint passive | integrable [2] | – | – |
| RP | second joint passive | | open | iterative steering [24] |
| RP | first joint passive | integrable [24] | – | – |
| 3R | third joint passive | STLC | elementary maneuvers (rot/trans) [25] dynamic feedback linearization [19] | vanishing trajectory [26] |
| 3R+g | third joint passive | STLC | dynamic feedback linearization [20] | extension of [11] |
| $n$R | last joint passive | | extension of [19] | extension of [19] |
| 4R | last two joints passive | $CP_3$ hinged | elementary maneuvers (rot/trans) [27] dynamic feedback linearization [here] | open |

TABLE I

UNDERACTUATED PLANAR MANIPULATORS WITH PASSIVE JOINTS IN THE LITERATURE


*Acknowledgments*

This work was partially supported by MURST within the *MISTRAL* project.


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