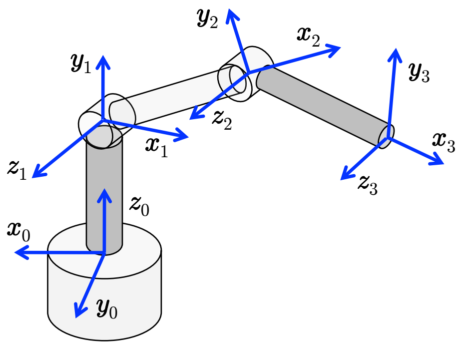
Robotics 2

Midterm test in classroom – March 29, 2017

Exercise 1

For the 3R spatial robot in Fig. 1, determine the symbolic expression of the elements of the inertia matrix $\boldsymbol{B}(\boldsymbol{q})$ using the recursive algorithm with moving frames for the computation of the kinetic energy of the links. The coordinates $\boldsymbol{q} \in \mathbb{R}^3$ to be used are those of the Denavit-Hartenberg (DH) convention.



i	α_i	d_i	a_i	θ_i
1	$\pi/2$	L_1	0	q_1
2	0	0	L_2	q_2
3	0	0	L_3	q_3

Figure 1: An elbow-type 3R robot, with associated DH frames and table of parameters.

Besides the masses m_i, $i = 1, 2, 3$, of the links, the other constant dynamic parameters are specified as

$$^1\boldsymbol{r}_{1,c1} = \begin{pmatrix} A \\ -F \\ 0 \end{pmatrix}, \qquad ^2\boldsymbol{r}_{2,c2} = \begin{pmatrix} -C \\ 0 \\ 0 \end{pmatrix}, \qquad ^3\boldsymbol{r}_{3,c3} = \begin{pmatrix} -D \\ 0 \\ E \end{pmatrix}, \tag{1}$$

where A, C, D, E and F take positive values, and

$$^i\boldsymbol{I}_{ci} = \begin{pmatrix} I_{xx,i} & 0 & 0 \\ 0 & I_{yy,i} & 0 \\ 0 & 0 & I_{zz,i} \end{pmatrix}, \qquad i = 1, 2, 3. \tag{2}$$

Once $\boldsymbol{B}(\boldsymbol{q})$ has been obtained, define a set of dynamic coefficients that linearly parametrize the inertia matrix and is of the smallest possible cardinality.

Exercise 2

When the Jacobian $\boldsymbol{J}(\boldsymbol{q})$ is a $m \times n$ matrix with full row rank m, its weighted pseudoinverse $\boldsymbol{J}^{\#}_W(\boldsymbol{q})$ takes the explicit form

$$\boldsymbol{J}^{\#}_W(\boldsymbol{q}) = \boldsymbol{W}^{-1}\boldsymbol{J}^T(\boldsymbol{q})\left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{W}^{-1}\boldsymbol{J}^T(\boldsymbol{q})\right)^{-1}, \tag{3}$$

where $\boldsymbol{W}$ is a $n \times n$, symmetric, and positive definite matrix. The matrix $\boldsymbol{J}^{\#}_W$ in (3) satisfies three of the four identities that uniquely define a pseudoinverse. Prove that the weighted pseudoinverse takes the following more general form, which holds true even when the Jacobian $\boldsymbol{J}(\boldsymbol{q})$ loses rank:

$$\boldsymbol{J}^{\#}_W(\boldsymbol{q}) = \boldsymbol{W}^{-1/2}\left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{W}^{-1/2}\right)^{\#}. \tag{4}$$

Exercise 3

Consider a 4R planar robot with all links of equal length $\ell = 0.5$ [m]. The robot is stretched along the $\boldsymbol{x}_0$ axis, in the DH configuration $\boldsymbol{q} = \boldsymbol{0}$. The end-effector of the robot should execute an instantaneous linear velocity $\boldsymbol{v} = \begin{pmatrix} 0 & 10 \end{pmatrix}^T$ [m/s]. The joint velocities are bounded as $|\dot{q}_i| \leq V_i$, $i = 1, \dots, 4$, with $V_1 = 4$, $V_2 = 2$, and $V_3 = V_4 = 1$ [rad/s]. Find a feasible joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^4$ that executes the given Cartesian task, either as such or in a scaled way, while satisfying the hard bounds on joint velocity. Scale the task velocity $\boldsymbol{v}$ only if strictly needed. A solution with a lower norm is preferred.
Hint: The following useful expressions hold for the pseudoinverse of a block matrix $\boldsymbol{A}$ (with a submatrix $\boldsymbol{O}$ of zeros) and of a vector $\boldsymbol{u} \neq \boldsymbol{0}$:

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{B} & \boldsymbol{O} \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{A}^{\#} = \begin{pmatrix} \boldsymbol{B}^{\#} \\ \boldsymbol{O}^T \end{pmatrix}, \qquad \boldsymbol{A} = \begin{pmatrix} \boldsymbol{B} \\ \boldsymbol{O} \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{A}^{\#} = \begin{pmatrix} \boldsymbol{B}^{\#} & \boldsymbol{O}^T \end{pmatrix}, \tag{5}$$

$$\boldsymbol{u} \in \mathbb{R}^n \quad \Rightarrow \quad \boldsymbol{u}^{\#} = \left(\boldsymbol{u}^T \boldsymbol{u}\right)^{-1} \boldsymbol{u}^T = \frac{\boldsymbol{u}^T}{\|\boldsymbol{u}\|^2}, \qquad \left(\boldsymbol{u}^T\right)^{\#} = \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|^2}. \tag{6}$$

Exercise 4

Consider a 3R planar robot with links of equal length $\ell = 1$ [m]. The primary task for this robot is to execute an instantaneous Cartesian velocity $\boldsymbol{v} \in \mathbb{R}^2$ with its end-effector. Denote the associated task Jacobian as $\boldsymbol{J}(\boldsymbol{q})$.

- When the robot is in the configuration $\boldsymbol{q}_0 = \begin{pmatrix} \pi/2 & \pi/3 & -2\pi/3 \end{pmatrix}^T$, use the Reduced Gradient (RG) method to determine the joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^3$ that realizes a desired Cartesian velocity $\boldsymbol{v} = \begin{pmatrix} 1 & -\sqrt{3} \end{pmatrix}^T$ [m/s] while maximizing the objective function

$$H(\boldsymbol{q}) = \sin^2 q_2 + \sin^2 q_3. \tag{7}$$

- As an auxiliary task, the robot should move so as to always keep the position $\boldsymbol{p}_2 = (x_2, y_2)$ of the endpoint of its second link on the circle defined by

$$x_2^2 + (y_2 - 1.5)^2 = 0.75. \tag{8}$$

  Determine the Jacobian $\boldsymbol{J}_a(\boldsymbol{q})$ associated to this auxiliary task. When the robot is in the configuration $\boldsymbol{q}_0$ defined above, is the robot in an algorithmic singularity? Can the two requested primary and auxiliary tasks be executed at the same time?

Exercise 5

Consider a 2R planar robot with the nominal kinematics expressed by the DH parameters in Tab. 1.

i	α_i	d_i	a_i	θ_i
1	0	0	L_1	q_1
2	0	0	L_2	q_2

Table 1: Nominal kinematic parameters of a 2R planar robot.

In order to improve the robot accuracy, we would like to perform a large number of calibration experiments in which the robot end-effector position $\boldsymbol{p} \in \mathbb{R}^2$ (in the plane of motion) is measured by an external laser system. To recover the residual errors of the nominal direct kinematics with respect to the external measurements, it is assumed that we require only the fine tuning of the parameters $\boldsymbol{a} \in \mathbb{R}^2$ and of the encoder measurements $\boldsymbol{\theta} \in \mathbb{R}^2$, whose adjustments will be performed simultaneously. What will be the expression of the 2×4 regressor matrix $\boldsymbol{\Phi}$ for a single calibration experiment?

[270 minutes (4.5 hours); open books, but no computer or smartphone]

Solution

March 29, 2017

Exercise 1

From the table of DH parameters given in Fig. 1, we compute the needed rotation matrices

$$
{}^0\boldsymbol{R}_1(q_1) = \begin{pmatrix} c_1 & 0 & s_1 \\ s_1 & 0 & -c_1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad {}^1\boldsymbol{R}_2(q_2) = \begin{pmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad {}^2\boldsymbol{R}_3(q_3) = \begin{pmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{pmatrix},
$$

with the usual shorthand notation for trigonometric quantities, e.g., $s_2 = \sin q_2$, $c_{23} = \cos(q_2 + q_3)$. Similarly, the following kinematic quantities will be used:

$$
{}^1\boldsymbol{r}_{0,1} = \begin{pmatrix} 0 \\ L_1 \\ 0 \end{pmatrix}, \qquad {}^2\boldsymbol{r}_{1,2} = \begin{pmatrix} L_2 \\ 0 \\ 0 \end{pmatrix}, \qquad {}^3\boldsymbol{r}_{2,3} = \begin{pmatrix} L_3 \\ 0 \\ 0 \end{pmatrix}.
$$

The moving frames algorithm is initialized with ${}^0\boldsymbol{\omega}_0 = \boldsymbol{0}$ and ${}^0\boldsymbol{v}_0 = \boldsymbol{0}$. Also, ${}^i\boldsymbol{z}_i = \boldsymbol{z}_0 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$ for $i = 0, 1, 2$ (joint axis 1, 2, and 3, respectively).

Link $i = 1$

$$
{}^1\boldsymbol{\omega}_1 = {}^0\boldsymbol{R}_1^T(q_1)\,{}^0\boldsymbol{\omega}_1 = {}^0\boldsymbol{R}_1^T(q_1)\left({}^0\boldsymbol{\omega}_0 + \dot{q}_1\,{}^0\boldsymbol{z}_0\right) = {}^0\boldsymbol{R}_1^T(q_1)\begin{pmatrix} 0 \\ 0 \\ \dot{q}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ \dot{q}_1 \\ 0 \end{pmatrix}
$$

$$
\begin{aligned}
{}^1\boldsymbol{v}_1 = {}^0\boldsymbol{R}_1^T(q_1)\,{}^0\boldsymbol{v}_1 &= {}^0\boldsymbol{R}_1^T(q_1)\left({}^0\boldsymbol{v}_0 + {}^0\boldsymbol{\omega}_1 \times {}^0\boldsymbol{r}_{0,1}\right) \\
&= {}^0\boldsymbol{R}_1^T(q_1)\,{}^0\boldsymbol{\omega}_1 \times {}^0\boldsymbol{R}_1^T(q_1)\,{}^0\boldsymbol{r}_{0,1} = {}^1\boldsymbol{\omega}_1 \times {}^1\boldsymbol{r}_{0,1} = \begin{pmatrix} 0 \\ \dot{q}_1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ L_1 \\ 0 \end{pmatrix} = \boldsymbol{0}
\end{aligned}
$$

$$
{}^1\boldsymbol{v}_{c1} = {}^1\boldsymbol{v}_1 + {}^1\boldsymbol{\omega}_1 \times {}^1\boldsymbol{r}_{1,c1} = \begin{pmatrix} 0 \\ \dot{q}_1 \\ 0 \end{pmatrix} \times \begin{pmatrix} A \\ -F \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -A\dot{q}_1 \end{pmatrix}
$$

$$
\Rightarrow \quad T_1 = \frac{1}{2}\, m_1 \left\|{}^1\boldsymbol{v}_{c1}\right\|^2 + \frac{1}{2}\,{}^1\boldsymbol{\omega}_1^T\,{}^1\boldsymbol{I}_{c1}\,{}^1\boldsymbol{\omega}_1 = \frac{1}{2}\left(I_{yy,1} + m_1 A^2\right)\dot{q}_1^2.
$$

Link $i = 2$

$$
{}^2\boldsymbol{\omega}_2 = {}^1\boldsymbol{R}_2^T(q_2)\,{}^1\boldsymbol{\omega}_2 = {}^1\boldsymbol{R}_2^T(q_2)\left({}^1\boldsymbol{\omega}_1 + \dot{q}_2\,{}^1\boldsymbol{z}_1\right) = {}^1\boldsymbol{R}_2^T(q_2)\begin{pmatrix} 0 \\ \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} = \begin{pmatrix} s_2\,\dot{q}_1 \\ c_2\,\dot{q}_1 \\ \dot{q}_2 \end{pmatrix}
$$

$$
\begin{aligned}
{}^2\boldsymbol{v}_2 = {}^1\boldsymbol{R}_2^T(q_2)\,{}^1\boldsymbol{v}_2 &= {}^1\boldsymbol{R}_2^T(q_2)\left({}^1\boldsymbol{v}_1 + {}^1\boldsymbol{\omega}_2 \times {}^1\boldsymbol{r}_{1,2}\right) \\
&= {}^1\boldsymbol{R}_2^T(q_2)\,{}^1\boldsymbol{\omega}_2 \times {}^1\boldsymbol{R}_2^T(q_2)\,{}^1\boldsymbol{r}_{1,2} = {}^2\boldsymbol{\omega}_2 \times {}^2\boldsymbol{r}_{1,2} = \begin{pmatrix} s_2\,\dot{q}_1 \\ c_2\,\dot{q}_1 \\ \dot{q}_2 \end{pmatrix} \times \begin{pmatrix} L_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ L_2\,\dot{q}_2 \\ -L_2 c_2\,\dot{q}_1 \end{pmatrix}
\end{aligned}
$$

$$
{}^2\boldsymbol{v}_{c2} = {}^2\boldsymbol{v}_2 + {}^2\boldsymbol{\omega}_2 \times {}^2\boldsymbol{r}_{2,c2} = \begin{pmatrix} 0 \\ L_2\,\dot{q}_2 \\ -L_2 c_2\,\dot{q}_1 \end{pmatrix} + \begin{pmatrix} s_2\,\dot{q}_1 \\ c_2\,\dot{q}_1 \\ \dot{q}_2 \end{pmatrix} \times \begin{pmatrix} -C \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ (L_2 - C)\,\dot{q}_2 \\ -(L_2 - C)\,c_2\,\dot{q}_1 \end{pmatrix}
$$

$$
\begin{aligned}
\Rightarrow \quad T_2 &= \tfrac{1}{2}\, m_2 \left\|{}^2\boldsymbol{v}_{c2}\right\|^2 + \tfrac{1}{2}\,{}^2\boldsymbol{\omega}_2^T\,{}^2\boldsymbol{I}_{c2}\,{}^2\boldsymbol{\omega}_2 \\
&= \tfrac{1}{2}\left(I_{xx,2} s_2^2 + I_{yy,2} c_2^2 + m_2 (L_2 - C)^2 c_2^2\right)\dot{q}_1^2 + \tfrac{1}{2}\left(I_{zz,2} + m_2 (L_2 - C)^2\right)\dot{q}_2^2.
\end{aligned}
$$

Link $i = 3$

$$^3\boldsymbol{\omega}_3 = {}^2\boldsymbol{R}_3^T(q_3)^2\boldsymbol{\omega}_3 = {}^2\boldsymbol{R}_3^T(q_3)\left({}^2\boldsymbol{\omega}_2 + \dot{q}_3{}^2\boldsymbol{z}_2\right) = {}^2\boldsymbol{R}_3^T(q_3)\begin{pmatrix} s_2\,\dot{q}_1 \\ c_2\,\dot{q}_1 \\ \dot{q}_2+\dot{q}_3 \end{pmatrix} = \begin{pmatrix} s_{23}\,\dot{q}_1 \\ c_{23}\,\dot{q}_1 \\ \dot{q}_2+\dot{q}_3 \end{pmatrix}$$

$$\begin{aligned} ^3\boldsymbol{v}_3 \;&=\; {}^2\boldsymbol{R}_3^T(q_3)^2\boldsymbol{v}_3 = {}^2\boldsymbol{R}_3^T(q_3)\left({}^2\boldsymbol{v}_2 + {}^2\boldsymbol{\omega}_3\times{}^2\boldsymbol{r}_{2,3}\right) = {}^2\boldsymbol{R}_3^T(q_3)^2\boldsymbol{v}_2 + {}^2\boldsymbol{R}_3^T(q_3)^2\boldsymbol{\omega}_3\times{}^2\boldsymbol{R}_3^T(q_3)^2\boldsymbol{r}_{2,3} \\ &=\; {}^2\boldsymbol{R}_3^T(q_3)^2\boldsymbol{v}_2 + {}^3\boldsymbol{\omega}_3\times{}^3\boldsymbol{r}_{2,3} = \begin{pmatrix} L_2 s_3\,\dot{q}_2 \\ L_2 c_3\,\dot{q}_2 \\ -L_2 c_2\,\dot{q}_1 \end{pmatrix} + \begin{pmatrix} s_{23}\,\dot{q}_1 \\ c_{23}\,\dot{q}_1 \\ \dot{q}_2+\dot{q}_3 \end{pmatrix}\times\begin{pmatrix} L_3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} L_2 s_3\,\dot{q}_2 \\ L_2 c_3\,\dot{q}_2 + L_3\,(\dot{q}_2+\dot{q}_3) \\ -\left(L_2 c_2 + L_3 c_{23}\right)\,\dot{q}_1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} ^3\boldsymbol{v}_{c3} = {}^3\boldsymbol{v}_3 + {}^3\boldsymbol{\omega}_3\times{}^3\boldsymbol{r}_{3,c3} \;&=\; \begin{pmatrix} L_2 s_3\,\dot{q}_2 \\ L_2 c_3\,\dot{q}_2 + L_3\,(\dot{q}_2+\dot{q}_3) \\ -\left(L_2 c_2 + L_3 c_{23}\right)\,\dot{q}_1 \end{pmatrix} + \begin{pmatrix} s_{23}\,\dot{q}_1 \\ c_{23}\,\dot{q}_1 \\ \dot{q}_2+\dot{q}_3 \end{pmatrix}\times\begin{pmatrix} -D \\ 0 \\ E \end{pmatrix} \\ &=\; \begin{pmatrix} L_2 s_3\,\dot{q}_2 + E c_{23}\,\dot{q}_1 \\ L_2 c_3\,\dot{q}_2 + (L_3 - D)\,(\dot{q}_2+\dot{q}_3) - E s_{23}\,\dot{q}_1 \\ -\left(L_2 c_2 + (L_3 - D)\,c_{23}\right)\,\dot{q}_1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \Rightarrow\quad T_3 \;&=\; \tfrac{1}{2}\,m_3\left\|{}^3\boldsymbol{v}_{c3}\right\|^2 + \tfrac{1}{2}\,{}^3\boldsymbol{\omega}_3^T\,{}^3\boldsymbol{I}_{c3}{}^3\boldsymbol{\omega}_3 \\ &=\; \tfrac{1}{2}\,m_3\Big(E^2\dot{q}_1^2 + \left(L_2c_2 + (L_3-D)\,c_{23}\right)^2\dot{q}_1^2 + L_2^2\dot{q}_2^2 + (L_3-D)^2\,(\dot{q}_2+\dot{q}_3)^2 \\ &\qquad\quad + 2\,(L_3-D)\,(L_2c_3\,\dot{q}_2 - E s_{23}\,\dot{q}_1)\,(\dot{q}_2+\dot{q}_3) - 2EL_2s_2\,\dot{q}_1\dot{q}_2\Big) \\ &\quad + \tfrac{1}{2}\left(I_{xx,3}s_{23}^2 + I_{yy,3}c_{23}^2\right)\dot{q}_1^2 + \tfrac{1}{2}\,I_{zz,3}\,(\dot{q}_2+\dot{q}_3)^2\,. \end{aligned}$$

Robot inertia matrix

From

$$T = \sum_{i=1}^{3} T_i = \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}} = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} b_{ij}(\boldsymbol{q})\,\dot{q}_i\dot{q}_j,$$

we obtain the elements b_{ij} of the symmetric inertia matrix $\boldsymbol{B}(\boldsymbol{q})$ as

$$\begin{aligned} b_{11} &= I_{yy,1} + m_1A^2 + I_{xx,2}s_2^2 + \left(I_{yy,2} + m_2(L_2-C)^2\right)c_2^2 \\ &\quad + m_3E^2 + I_{xx,3}\,s_{23}^2 + I_{yy,3}\,c_{23}^2 + m_3\left(L_2c_2 + (L_3-D)\,c_{23}\right)^2 \\ b_{12} &= -m_3E\left(L_2s_2 + (L_3-D)\,s_{23}\right) \\ b_{13} &= -m_3E\,(L_3-D)\,s_{23} \\ b_{21} &= b_{12} \\ b_{22} &= I_{zz,2} + m_2(L_2-C)^2 + I_{zz,3} + m_3\,(L_3-D)^2 + m_3L_2^2 + 2\,m_3L_2\,(L_3-D)\,c_3 \\ b_{23} &= I_{zz,3} + m_3\,(L_3-D)^2 + m_3L_2\,(L_3-D)\,c_3 \\ b_{31} &= b_{13} \\ b_{32} &= b_{23} \\ b_{33} &= I_{zz,3} + m_3\,(L_3-D)^2\,. \end{aligned}$$

Minimal parametrization

Reorganizing the squares of trigonometric functions, the element b_{11} can be also rewritten as

$$\begin{aligned} b_{11} = I_{yy,1} &+ m_1A^2 + I_{yy,2} + m_2(L_2-C)^2 + I_{yy,3} + m_3\,(L_3-D)^2 + m_3\left(L_2^2 + E^2\right) \\ &+ \left(I_{xx,2} - I_{yy,2} - m_2(L_2-C)^2 - m_3L_2^2\right)s_2^2 \\ &+ \left(I_{xx,3} - I_{yy,3} - m_3\,(L_3-D)^2\right)s_{23}^2 + 2\,m_3L_2\,(L_3-D)\,c_2c_{23}. \end{aligned} \tag{9}$$

Using the expression (9) for b_{11}, and introducing constant dynamic coefficients a_i ($i = 1, \ldots, 8$), the inertia matrix $\boldsymbol{B}(\boldsymbol{q})$ takes the more compact, linearly parametrized form

$$\boldsymbol{B}(\boldsymbol{q}) = \begin{pmatrix} a_1 + a_2 s_2^2 + a_3 s_{23}^2 + 2a_4 c_2 c_{23} & -a_7 s_2 - a_8 s_{23} & -a_8 s_{23} \\ -a_7 s_2 - a_8 s_{23} & a_5 + 2a_4 c_3 & a_6 + a_4 c_3 \\ -a_8 s_{23} & a_6 + a_4 c_3 & a_6 \end{pmatrix}. \tag{10}$$

Note that the most notable simplification occurs when $E = 0$. In this case, it follows that $a_7 = a_8 = 0$ and the inertia matrix becomes block diagonal.

Exercise 2

One needs to verify that the expression in (4) satisfies the three identities (dropping dependency on $\boldsymbol{q}$):

$$(i)\;\; \boldsymbol{J}\boldsymbol{J}_W^{\#}\boldsymbol{J} = \boldsymbol{J}, \qquad (ii)\;\; \boldsymbol{J}_W^{\#}\boldsymbol{J}\boldsymbol{J}_W^{\#} = \boldsymbol{J}_W^{\#}, \qquad (iii)\;\; \left(\boldsymbol{J}\boldsymbol{J}_W^{\#}\right)^T = \boldsymbol{J}\boldsymbol{J}_W^{\#}.$$

For (i), using the similar property (i) of the pseudoinverse of $\boldsymbol{J}\boldsymbol{W}^{-1/2}$,

$$\begin{aligned} \boldsymbol{J}\boldsymbol{J}_W^{\#}\boldsymbol{J} &= \boldsymbol{J}\left(\boldsymbol{W}^{-1/2}\left(\boldsymbol{J}\boldsymbol{W}^{-1/2}\right)^{\#}\right)\boldsymbol{J} = \boldsymbol{J}\left(\boldsymbol{W}^{-1/2}\left(\boldsymbol{J}\boldsymbol{W}^{-1/2}\right)^{\#}\right)\boldsymbol{J}\cdot\left(\boldsymbol{W}^{-1/2}\boldsymbol{W}^{1/2}\right) \\ &= \left(\boldsymbol{J}\boldsymbol{W}^{-1/2}\left(\boldsymbol{J}\boldsymbol{W}^{-1/2}\right)^{\#}\boldsymbol{J}\boldsymbol{W}^{-1/2}\right)\boldsymbol{W}^{1/2} = \left(\boldsymbol{J}\boldsymbol{W}^{-1/2}\right)\boldsymbol{W}^{1/2} = \boldsymbol{J}. \end{aligned}$$

For (ii), using the property (ii) in the definition of the pseudoinverse of $\boldsymbol{J}\boldsymbol{W}^{-1/2}$,

$$\begin{aligned} \boldsymbol{J}_W^{\#}\boldsymbol{J}\boldsymbol{J}_W^{\#} &= \left(\boldsymbol{W}^{-1/2}\left(\boldsymbol{J}\boldsymbol{W}^{-1/2}\right)^{\#}\right)\boldsymbol{J}\left(\boldsymbol{W}^{-1/2}\left(\boldsymbol{J}\boldsymbol{W}^{-1/2}\right)^{\#}\right) \\ &= \boldsymbol{W}^{-1/2}\left(\left(\boldsymbol{J}\boldsymbol{W}^{-1/2}\right)^{\#}\boldsymbol{J}\boldsymbol{W}^{-1/2}\left(\boldsymbol{J}\boldsymbol{W}^{-1/2}\right)^{\#}\right) = \boldsymbol{W}^{-1/2}\left(\boldsymbol{J}\boldsymbol{W}^{-1/2}\right)^{\#} = \boldsymbol{J}_W^{\#}. \end{aligned}$$

Finally for (iii), the symmetry of the matrix

$$\boldsymbol{J}\boldsymbol{J}_W^{\#} = \boldsymbol{J}\left(\boldsymbol{W}^{-1/2}\left(\boldsymbol{J}\boldsymbol{W}^{-1/2}\right)^{\#}\right) = \left(\boldsymbol{J}\boldsymbol{W}^{-1/2}\right)\left(\boldsymbol{J}\boldsymbol{W}^{-1/2}\right)^{\#}$$

follows from the same property of symmetry holding for the pseudoinverse of $\boldsymbol{J}\boldsymbol{W}^{-1/2}$.

Exercise 3

The task Jacobian of the planar 4R robot is given by

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -\ell\,(s_1 + s_{12} + s_{123} + s_{1234}) & -\ell\,(s_{12} + s_{123} + s_{1234}) & -\ell\,(s_{123} + s_{1234}) & -\ell\, s_{1234} \\ \ell\,(c_1 + c_{12} + c_{123} + c_{1234}) & \ell\,(c_{12} + c_{123} + c_{1234}) & \ell\,(c_{123} + c_{1234}) & \ell\, c_{1234} \end{pmatrix} \tag{11}$$

with

$$\boldsymbol{v} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}, \qquad \boldsymbol{v} \in \mathbb{R}^2, \qquad \dot{\boldsymbol{q}} \in \mathbb{R}^4.$$

When $\boldsymbol{q} = \boldsymbol{0}$ and for $\ell = 0.5$ [m], the Jacobian becomes

$$\boldsymbol{J} := \boldsymbol{J}(\boldsymbol{0}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 1.5 & 1 & 0.5 \end{pmatrix}$$

and is clearly not full rank. However, the desired task velocity lies in the range of $\boldsymbol{J}$,

$$\boldsymbol{v} = \begin{pmatrix} 0 \\ 10 \end{pmatrix} \in \mathcal{R}\{\boldsymbol{J}\} = \alpha\begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

so that it will be realizable at least in direction, possibly in a scaled form in case the joint velocity bounds cannot be satisfied.

Looking for a minimum norm joint velocity, to start with, we derive first the pseudoinverse solution. Using the hints in the text, it is easy to compute the pseudoinverse of $\boldsymbol{J}$ without resorting to a SVD. We have

$$\dot{\boldsymbol{q}}_{PS} = \boldsymbol{J}^{\#}\boldsymbol{v} = \begin{pmatrix} 0 & \frac{2}{7.5} \\ 0 & \frac{1.5}{7.5} \\ 0 & \frac{1}{7.5} \\ 0 & \frac{0.5}{7.5} \end{pmatrix} \begin{pmatrix} 0 \\ 10 \end{pmatrix} = \begin{pmatrix} 2.6667 \\ 2.0000 \\ 1.3333 \\ 0.6667 \end{pmatrix} \text{ [rad/s]}.$$

The third joint velocity violates the maximum bound, $\dot{q}_{PS,3} = 1.3333 > 1 = V_3$, so this is not a feasible solution. Thus, we search for an equivalent but feasible solution by using the SNS (Saturation in the Null Space) method, which is particularly simple to apply here.

In step 1 of the SNS method, we saturate the (single) overdriven joint by setting $\dot{q}_3 = V_3 = 1$ [rad/s]. Then, the original task is modified by removing the saturated contribution of the third joint velocity (discarding the associated column of $\boldsymbol{J}$). We rewrite this as

$$\boldsymbol{v}_1 = \boldsymbol{v} - \boldsymbol{J}_3 V_3 = \begin{pmatrix} 0 \\ 10 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 9 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 1.5 & 0.5 \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_4 \end{pmatrix} = \boldsymbol{J}_{-3}\, \dot{\boldsymbol{q}}_{-3},$$

where $\boldsymbol{J}_{-i}$ is the Jacobian obtained by deleting the ith column and, similarly, $\dot{\boldsymbol{q}}_{-i}$ is the vector of joint velocity without the ith component. We recompute next the contribution of the remaining active joints, by pseudoinverting the $\boldsymbol{J}_{-3}$ matrix for the modified task. We obtain

$$\dot{\boldsymbol{q}}_{PS_{-3}} = \boldsymbol{J}_{-3}^{\#}\boldsymbol{v}_1 = \begin{pmatrix} 0 & \frac{2}{6.5} \\ 0 & \frac{1.5}{6.5} \\ 0 & \frac{0.5}{6.5} \end{pmatrix} \begin{pmatrix} 0 \\ 9 \end{pmatrix} = \begin{pmatrix} 2.7692 \\ 2.0769 \\ 0.6923 \end{pmatrix} \text{ [rad/s]}, \qquad (\text{with the additional } \dot{q}_3 = 1 = V_3).$$

The second joint velocity violates the maximum bound, $\dot{q}_{PS_{-3},2} = 2.0769 > 2 = V_2$, so this is not yet a feasible solution and we proceed with the SNS method.

In step 2, we saturate also the second overdriven joint by setting $\dot{q}_2 = V_2 = 2$ [rad/s]. The original task is modified by removing both saturated contributions of the second and third joint velocities (discarding the two associated columns of $\boldsymbol{J}$). We rewrite this as

$$\boldsymbol{v}_2 = \boldsymbol{v} - \boldsymbol{J}_2 V_2 - \boldsymbol{J}_3 V_3 = \begin{pmatrix} 0 \\ 10 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2 & 0.5 \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_4 \end{pmatrix} = \boldsymbol{J}_{-23}\, \dot{\boldsymbol{q}}_{-23},$$

with obvious notation. We recompute next the contribution of the remaining active joints, by pseudoinverting the (now square, but still singular) $\boldsymbol{J}_{-23}$ matrix for the modified task. We obtain

$$\dot{\boldsymbol{q}}_{PS_{-23}} = \boldsymbol{J}_{-23}^{\#}\boldsymbol{v}_2 = \begin{pmatrix} 0 & \frac{2}{4.25} \\ 0 & \frac{0.5}{4.25} \end{pmatrix} \begin{pmatrix} 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 2.8235 \\ 0.7059 \end{pmatrix} \text{ [rad/s]}, \quad (\text{with } \dot{q}_2 = 2 = V_2,\ \dot{q}_3 = 1 = V_3).$$

All bounds are now satisfied and the obtained joint velocity is feasible. Recomposing the complete joint velocity vector, we have the solution

$$\dot{\boldsymbol{q}}^* = \begin{pmatrix} 2.8235 \\ 2.0000 \\ 1.0000 \\ 0.7059 \end{pmatrix} \text{ [rad/s]}, \qquad \text{with } \boldsymbol{J}\dot{\boldsymbol{q}}^* = \boldsymbol{v} \text{ and } \|\dot{\boldsymbol{q}}^*\| = 3.6702.$$

Therefore, there was no need to scale the original task velocity $\boldsymbol{v}$ in order to find a feasible joint velocity solution. Note that the solution $\dot{\boldsymbol{q}}^*$ can be rewritten in the general form

$$\dot{\boldsymbol{q}}^* = (\boldsymbol{J}\boldsymbol{W})^{\#}\, \boldsymbol{v} + \Big(\boldsymbol{I} - (\boldsymbol{J}\boldsymbol{W})^{\#}\, \boldsymbol{J}\Big)\, \dot{\boldsymbol{q}}_N,$$

with

$$\boldsymbol{W} = \text{diag}\{1,0,0,1\} \text{ (the selected active joints)}, \quad \dot{\boldsymbol{q}}_N = \begin{pmatrix} 0 \\ V_2 \\ V_3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} \text{(the saturated joint velocities)},$$

which explains also the name given to the method. Indeed, this solution is not unique. However, the SNS method (actually a variant of it, Opt-SNS, which is not needed in this simple case) guarantees also that a feasible minimum norm solution is obtained. For example, another feasible solution is obtained by saturating the first joint velocity to its maximum value ($\dot{q}_1 = V_1 = 4$ [rad/s]) and adjusting the other three joint velocities accordingly. We have

$$\dot{\boldsymbol{q}}_{PS_{-1}} = \boldsymbol{J}^{\#}_{-1}\left(\boldsymbol{v} - \boldsymbol{J}_1 V_1\right) = \begin{pmatrix} 0 & \frac{1.5}{3.5} \\ 0 & \frac{1}{3.5} \\ 0 & \frac{0.5}{3.5} \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0.8571 \\ 0.5714 \\ 0.2857 \end{pmatrix} \text{ [rad/s]}, \qquad (\text{with } \dot{q}_1 = 4 = V_1).$$

The complete solution $\dot{\boldsymbol{q}}^{\diamond}$ is feasible, but has a larger norm than the SNS solution $\dot{\boldsymbol{q}}^{*}$:

$$\dot{\boldsymbol{q}}^{\diamond} = \begin{pmatrix} 4.0000 \\ 0.8571 \\ 0.5714 \\ 0.2857 \end{pmatrix} \text{ [rad/s]}, \qquad \text{with } \boldsymbol{J}\dot{\boldsymbol{q}}^{\diamond} = \boldsymbol{v} \text{ and } \|\dot{\boldsymbol{q}}^{\diamond}\| = 4.1404.$$

Exercise 4

Reduced Gradient

The Jacobian of the primary task is similar to that of the previous exercise, see (11)

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -\ell\left(s_1 + s_{12} + s_{123}\right) & -\ell\left(s_{12} + s_{123}\right) & -\ell s_{123} \\ \ell\left(c_1 + c_{12} + c_{123}\right) & \ell\left(c_{12} + c_{123}\right) & \ell c_{123} \end{pmatrix}. \tag{12}$$

When evaluated at $\boldsymbol{q} = \boldsymbol{q}_0 = \begin{pmatrix} \pi/2 & \pi/3 & -2\pi/3 \end{pmatrix}^T$ and for $\ell = 1$ [m], the Jacobian (12) becomes

$$\boldsymbol{J} := \boldsymbol{J}(\boldsymbol{q}_0) = \begin{pmatrix} -2 & -1 & -0.5 \\ 0 & 0 & \sqrt{3}/2 \end{pmatrix}$$

and is clearly full row rank. However, for the purpose of designing a RG solution, we need to extract from $\boldsymbol{J}$ a non-singular 2×2 submatrix $\boldsymbol{J}_a$, and not every selection will work. In fact, the three possible alternatives (i.e., deleting respectively column 1, 2, or 3) yield

$$\det \boldsymbol{J}_{-1} = -.8660, \qquad \det \boldsymbol{J}_{-2} = -1.7321, \qquad \det \boldsymbol{J}_{-3} = 0 \text{ (singular!)}.$$

We will choose as $\boldsymbol{J}_a$ the minor with the largest determinant (presumably, the best conditioned solution). Thus, $\boldsymbol{q}_a = (q_1, q_3)$, $\boldsymbol{q}_b = q_2$. Accordingly, after a reordering of variables obtained by the unitary matrix $\boldsymbol{T}$ (with $\boldsymbol{T}^{-1} = \boldsymbol{T}^T$)

$$\boldsymbol{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \boldsymbol{q} \rightarrow \boldsymbol{T}\dot{\boldsymbol{q}} = \begin{pmatrix} \dot{q}_1 \\ \dot{q}_3 \\ \hline \dot{q}_2 \end{pmatrix} = \begin{pmatrix} \dot{\boldsymbol{q}}_a \\ \dot{\boldsymbol{q}}_b \end{pmatrix}, \quad \boldsymbol{J} \rightarrow \boldsymbol{J}\boldsymbol{T} = \begin{pmatrix} \boldsymbol{J}_1 & \boldsymbol{J}_3 \,|\, \boldsymbol{J}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{J}_a & \boldsymbol{J}_b \end{pmatrix},$$

we have

$$\boldsymbol{J}_a = \begin{pmatrix} -2 & -0.5 \\ 0 & \sqrt{3}/2 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{J}_a^{-1} = \begin{pmatrix} -0.5 & -0.2887 \\ 0 & 1.1547 \end{pmatrix}, \qquad \boldsymbol{J}_b = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

The gradient of the objective function H in (7) evaluated at $\boldsymbol{q} = \boldsymbol{q}_0$ is

$$\nabla_{\boldsymbol{q}} H(\boldsymbol{q}) = \begin{pmatrix} 0 \\ 2\sin q_2 \cos q_2 \\ 2\sin q_3 \cos q_3 \end{pmatrix} \qquad \Rightarrow \qquad \nabla_{\boldsymbol{q}} H := \nabla_{\boldsymbol{q}} H(\boldsymbol{q}_0) = \begin{pmatrix} 0 \\ 0.8660 \\ 0.8660 \end{pmatrix}.$$

The reduced gradient is thus

$$\nabla_{\boldsymbol{q}_b} H' := \nabla_{\boldsymbol{q}_b} H'(\boldsymbol{q}_0) = \begin{pmatrix} -\left(\boldsymbol{J}_a^{-1}\boldsymbol{J}_b\right)^T & 1 \end{pmatrix} \boldsymbol{T} \nabla_{\boldsymbol{q}} H = \begin{pmatrix} -0.5 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0.8660 \\ 0.8660 \end{pmatrix} = 0.8660.$$

The solution to the first item is thus

$$\dot{\boldsymbol{q}}_{RG} = \boldsymbol{T}^T \begin{pmatrix} \dot{\boldsymbol{q}}_a \\ \dot{\boldsymbol{q}}_b \end{pmatrix} = \boldsymbol{T}^T \begin{pmatrix} \boldsymbol{J}_a^{-1}\left(\boldsymbol{v} - \boldsymbol{J}_b \nabla_{\boldsymbol{q}_b} H'\right) \\ \nabla_{\boldsymbol{q}_b} H' \end{pmatrix} = \begin{pmatrix} -0.4330 \\ 0.8660 \\ -2.0000 \end{pmatrix} \text{ [rad/s]}.$$

Task augmentation

We consider next the auxiliary task of keeping the endpoint of the second robot link on the circle (8). The endpoint position is

$$\boldsymbol{p}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \ell(c_1 + c_{12}) \\ \ell(s_1 + s_{12}) \end{pmatrix}$$

and its associated Jacobian is

$$\boldsymbol{J}_2(\boldsymbol{q}) = \frac{\partial \boldsymbol{p}_2(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -\ell\,(s_1 + s_{12}) & -\ell s_{12} & 0 \\ \ell\,(c_1 + c_{12}) & \ell c_{12} & 0 \end{pmatrix}. \tag{13}$$

We first note that when the robot is in the configuration $\boldsymbol{q}_0 = \begin{pmatrix} \pi/2 & \pi/3 & -2\pi/3 \end{pmatrix}^T$, the position $\boldsymbol{p}_2$ satisfies already the constraint (8), see Fig. 2. Thus, the auxiliary task should constrain the joint velocities so that $\boldsymbol{p}_2$ (when moving or not) will remain on the assigned circle.

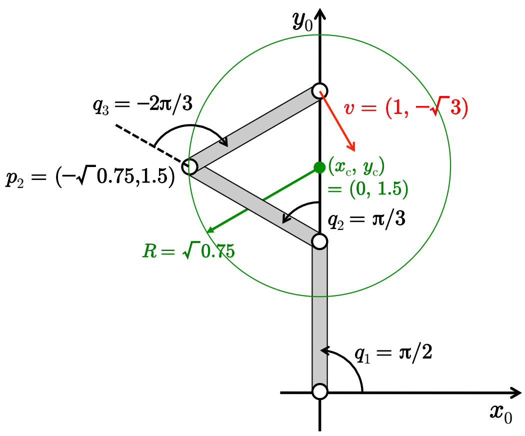


Figure 2: The primary task (in red) and the auxiliary task (in green) for the 3R planar robot.

Differentiating (8) with respect to time yields

$$2x_2\,\dot{x}_2 + 2(y_2 - 1.5)\,\dot{y}_2 = 0.$$

Rearranging this equation so as to isolate the velocity $\dot{\boldsymbol{p}}_2$ and using (13) leads to

$$\begin{pmatrix} 2x_2 & 2(y_2 - 1.5) \end{pmatrix} \begin{pmatrix} \dot{x}_2 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 2x_2 & 2(y_2 - 1.5) \end{pmatrix} \boldsymbol{J}_2(\boldsymbol{q})\dot{\boldsymbol{q}} = 0.$$

The auxiliary task Jacobian is then the 1×3 row vector

$$\begin{aligned}\boldsymbol{J}_a(\boldsymbol{q}) &= \begin{pmatrix} 2x_2 & 2(y_2 - 1.5) \end{pmatrix} \boldsymbol{J}_2(\boldsymbol{q}) = \begin{pmatrix} 2\ell(c_1 + c_{12}) & 2(\ell(s_1 + s_{12}) - 1.5) \end{pmatrix} \begin{pmatrix} -\ell\,(s_1 + s_{12}) & -\ell s_{12} & 0 \\ \ell\,(c_1 + c_{12}) & \ell c_{12} & 0 \end{pmatrix} \\ &= \begin{pmatrix} -3\ell\,(c_1 + c_{12}) & -2\ell^2 s_2 - 3\ell c_{12} & 0 \end{pmatrix}.\end{aligned} \tag{14}$$

Augmenting the primary task Jacobian (12) with the auxiliary task Jacobian (14) leads to a square, 3×3 *extended* Jacobian $\boldsymbol{J}_e(\boldsymbol{q})$ and to an extended task vector $\boldsymbol{v}_e \in \mathbb{R}^3$:

$$\boldsymbol{J}_e(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{J}(\boldsymbol{q}) \\ \boldsymbol{J}_a(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} -\ell\,(s_1 + s_{12} + s_{123}) & -\ell\,(s_{12} + s_{123}) & -\ell s_{123} \\ \ell\,(c_1 + c_{12} + c_{123}) & \ell\,(c_{12} + c_{123}) & \ell c_{123} \\ -3\ell\,(c_1 + c_{12}) & -2\ell^2 s_2 - 3\ell c_{12} & 0 \end{pmatrix}, \quad \boldsymbol{v}_e = \begin{pmatrix} \boldsymbol{v} \\ 0 \end{pmatrix}, \quad \boldsymbol{J}_e(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{v}_e.$$

At the configuration $\boldsymbol{q}_0$, using also $\ell = 1$, we have

$$\boldsymbol{J}_e := \boldsymbol{J}_e(\boldsymbol{q}_0) = \begin{pmatrix} -2 & -1 & -0.5 \\ 0 & 0 & 0.8660 \\ 2.5981 & 0.8660 & 0 \end{pmatrix}. \tag{15}$$

It is easy to see that $\boldsymbol{q}_0$ is not a singularity for the extended task, and thus in particular not an algorithmic singularity. In fact,

$$\mathrm{rank}(\boldsymbol{J}) = 2, \qquad \mathrm{rank}(\boldsymbol{J}_a) = 1, \qquad \text{and } \mathrm{rank}(\boldsymbol{J}_e) = 3 = 2 + 1.$$

Therefore, in this configuration the robot can realize any generic extended task velocity $\boldsymbol{v}_e \in \mathbb{R}^3$ (in particular, with an arbitrary top part $\boldsymbol{v} \in \mathbb{R}^2$). Therefore, the joint velocity

$$\dot{\boldsymbol{q}}^{\dagger} = \boldsymbol{J}_e^{-1}\boldsymbol{v}_e = \boldsymbol{J}_e^{-1} \begin{pmatrix} 1 \\ -\sqrt{3} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} \text{ [rad/s]}$$

will instantaneously realize both tasks at the same time. For the particular numerical value assigned as $\boldsymbol{v}$, the simple rotation of the third link around its joint axis will realize the primary task. In this case, the endpoint of the second link will remain at rest, thus satisfying in a trivial way the auxiliary task. In general, with a different desired value of $\boldsymbol{v}$, all robot joints will move so as to realize the extended task, instantaneously keeping the endpoint of the second link on the circle (i.e., its velocity will be tangential to the circle in the current position). For instance,

$$\dot{\boldsymbol{q}}^{\dagger\prime} = \boldsymbol{J}_e^{-1}\boldsymbol{v}_e' = \boldsymbol{J}_e^{-1} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1.5774 \\ -4.7321 \\ 1.1547 \end{pmatrix} \text{ [rad/s]}.$$

Exercise 5

For the direct kinematics, using the homogeneous transformations defined through Tab. 1, we obtain

$$\begin{aligned}{}^0\boldsymbol{p}_{0,2} &= {}^0\boldsymbol{A}_1(q_1)\,{}^1\boldsymbol{p}_{1,2} = {}^0\boldsymbol{A}_1(q_1)\left({}^1\boldsymbol{A}_2(q_2)\,{}^2\boldsymbol{p}_{2,2}\right) \\ &= {}^0\boldsymbol{A}_1(q_1) \begin{pmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_2c_2 \\ a_2s_2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a_1c_1 + a_2c_{12} \\ a_1s_1 + a_2s_{12} \\ 0 \\ 1 \end{pmatrix}.\end{aligned}$$

The planar position $\boldsymbol{p} \in \mathbb{R}^2$ is given by the (x, y) components extracted from the position vector ${}^0\boldsymbol{p}_{0,2}$. Thus, from

$$\boldsymbol{p} = \begin{pmatrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_1 + a_2 s_{12} \end{pmatrix} = \boldsymbol{f}(\boldsymbol{a}, \boldsymbol{\theta})$$

we obtain

$$\boldsymbol{J_a} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{a}} = \begin{pmatrix} c_1 & c_{12} \\ s_1 & s_{12} \end{pmatrix}, \qquad \boldsymbol{J_\theta} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\theta}} = \begin{pmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \end{pmatrix}.$$

The regressor equation is then

$$\Delta \boldsymbol{p} = \boldsymbol{J_a} \Delta \boldsymbol{a} + \boldsymbol{J_\theta} \Delta \boldsymbol{\theta} = \boldsymbol{\Phi} \Delta \boldsymbol{\phi}, \qquad \Delta \boldsymbol{p} = \boldsymbol{p}_m - \boldsymbol{p} = \boldsymbol{p}_m - \boldsymbol{f}(\boldsymbol{a}_n, \boldsymbol{\theta}_n),$$

with

$$\Delta \boldsymbol{\phi} = \begin{pmatrix} \Delta \boldsymbol{a} \\ \Delta \boldsymbol{\theta} \end{pmatrix} \in \mathbb{R}^4, \qquad \boldsymbol{\Phi} = \begin{pmatrix} \boldsymbol{J_a}(\boldsymbol{\theta}) & \boldsymbol{J_\theta}(\boldsymbol{a}, \boldsymbol{\theta}) \end{pmatrix}_{\boldsymbol{a}=\boldsymbol{a}_n, \boldsymbol{\theta}=\boldsymbol{\theta}_n}, \qquad \boldsymbol{a}_n = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}, \quad \boldsymbol{\theta}_n = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}.$$

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