Robotics II

January 9, 2013

Exercise 1

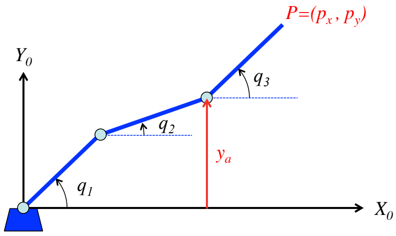

Figure 1: A $3R$ planar robot with unitary link lengths and two sets of task variables

Consider the $3R$ planar robot of Fig. 1, having links of unitary length and with the generalized coordinates defined therein. This robot is redundant for the task of positioning its end-effector at $\boldsymbol{p} = (p_x, p_y)$, as well as for the task of imposing a value to the second link end-point height y_a.

- **a)** For each *separate* task, define the associated task Jacobian and its singularities.
- **b)** Characterize the so-called *algorithmic* singularities (configurations where each task can be executed separately, but not both tasks simultaneously).
- **c)** For the simultaneous execution of both tasks, provide the expression of an inverse differential kinematic solution at the velocity level, based on a *task-priority* strategy that assigns higher priority to the end-effector position task.

Exercise 2

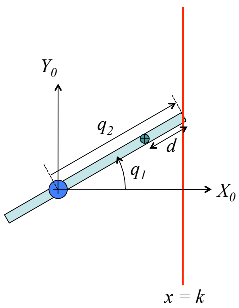

Figure 2: A RP robot moving on a horizontal plane with its end-effector constrained on a line

The end-effector of the RP robot in Fig. 2 is constrained to move on the Cartesian line $x = k$, with $k > 0$. For this operative condition, derive the expression of the *constrained* robot dynamics (in this case, two second-order differential equations, with a dynamically consistent projection matrix acting on forces/torques so as to automatically satisfy the motion constraint in any admissible robot state).

[210 minutes; open books]

Solutions

January 9, 2013

Exercise 1

Being the generalized coordinates q_i ($i = 1, 2, 3$) the absolute angles of the links w.r.t. the $\boldsymbol{x}_0$ axis, the end-effector position is expressed as

$$\boldsymbol{p} = \begin{pmatrix} \cos q_1 + \cos q_2 + \cos q_3 \\ \sin q_1 + \sin q_2 + \sin q_3 \end{pmatrix} = \boldsymbol{f}_1(\boldsymbol{q})$$

The associated task Jacobian is

$$\boldsymbol{J}_1(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}_1}{\partial \boldsymbol{q}} = \begin{pmatrix} -\sin q_1 & -\sin q_2 & -\sin q_3 \\ \cos q_1 & \cos q_2 & \cos q_3 \end{pmatrix}$$

and is singular if and only if

$$\sin(q_2 - q_1) = \sin(q_3 - q_2) = 0, \qquad (\Rightarrow \ \sin(q_3 - q_1) = 0) \tag{1}$$

or, in terms of Denavit-Hartenberg relative link angles $\theta_i = q_i - q_{i-1}$ (for $i = 2.3$), when $\sin\theta_2 = \sin\theta_3 = 0$. This occurs only when all three links are folded or stretched along a common radial line originating at the robot base.

The height y_a of the end-point of the second link and its associated task Jacobian are given by

$$y_a = \sin q_1 + \sin q_2 = f_2(\boldsymbol{q}) \quad \Rightarrow \quad \boldsymbol{J}_2(\boldsymbol{q}) = \frac{\partial f_2}{\partial \boldsymbol{q}} = \begin{pmatrix} \cos q_1 & \cos q_2 & 0 \end{pmatrix}.$$

This Jacobian is singular if and only if

$$\cos q_1 = \cos q_2 = 0, \tag{2}$$

namely when the first two links are either folded or stretched *and* the end-point of the second link is on the $\boldsymbol{y}_0$ axis.

When considering the two tasks together, the *Extended* Jacobian is square

$$\boldsymbol{J}_E(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{J}_1(\boldsymbol{q}) \\ \boldsymbol{J}_2(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} -\sin q_1 & -\sin q_2 & -\sin q_3 \\ \cos q_1 & \cos q_2 & \cos q_3 \\ \cos q_1 & \cos q_2 & 0 \end{pmatrix}.$$

Algorithmic singularities will occur when both $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$ are full (row) rank, but

$$\det \boldsymbol{J}_E = -\cos q_3 \cdot \sin(q_2 - q_1) = 0. \tag{3}$$

Comparing eqs. (1–2) with (3), this happens when

- the third link is vertical ($\cos q_3 = 0$), while the first two are not; or,
- the first two links are aligned ($\sin(q_2 - q_1) = 0$) but not vertical, and the third link is not aligned with the first two.

Indeed, the above are only particular conditions for singularity of the Extended Jacobian. In fact, $\boldsymbol{J}_E$ is not invertible as soon as the third link is vertical and/or the first two links are aligned, no matter what is the situation of the other links.

Let $\boldsymbol{v}_d \in \mathbb{R}^2$ be a desired velocity for the robot end-effector and $\dot{y}_{a,d}$ a desired height variation rate for the end-point of the second link. An inverse solution of the form

$$\dot{\boldsymbol{q}} = \boldsymbol{J}_E^{-1}(\boldsymbol{q}) \begin{pmatrix} \boldsymbol{v}_d \\ \dot{y}_{a,d} \end{pmatrix}$$

will blow out as soon as a singularity occurs for $\boldsymbol{J}_E$. A task-priority solution, with the first task (of dimension $m_1 = 2$) of higher priority than the second one (of dimension $m_2 = 1$), is given by

$$\dot{\boldsymbol{q}} = \boldsymbol{J}_1^{\#}(\boldsymbol{q})\,\boldsymbol{v}_d + \left(\boldsymbol{J}_2(\boldsymbol{q})\left(\boldsymbol{I} - \boldsymbol{J}_1^{\#}(\boldsymbol{q})\boldsymbol{J}_1(\boldsymbol{q})\right)\right)^{\#}\left(\dot{y}_{a,d} - \boldsymbol{J}_2(\boldsymbol{q})\boldsymbol{J}_1^{\#}(\boldsymbol{q})\,\boldsymbol{v}_d\right). \tag{4}$$

This will guarantee perfect execution of the first task even when $\boldsymbol{J}_E$ is singular (i.e., eq. (3) holds), provided that eq. (1) is *not* satisfied (in particular, in algorithmic singularities, where eq. (2) is *not* satisfied too).

Using the properties of projection matrices (symmetry and idempotency), and being the matrix $\boldsymbol{J}_2(\boldsymbol{I} - \boldsymbol{J}_1^{\#}\boldsymbol{J}_1)$ a row vector in our case, the solution (4) can also be rewritten as

$$\dot{\boldsymbol{q}} = \boldsymbol{J}_1^{\#}(\boldsymbol{q})\,\boldsymbol{v}_d + \alpha\left(\boldsymbol{I} - \boldsymbol{J}_1^{\#}(\boldsymbol{q})\boldsymbol{J}_1(\boldsymbol{q})\right)\boldsymbol{J}_2^T(\boldsymbol{q}),$$

with the scalar

$$\alpha = \alpha(\boldsymbol{q}, \boldsymbol{v}_d, \dot{y}_{a,d}) = \frac{\dot{y}_{a,d} - \boldsymbol{J}_2(\boldsymbol{q})\boldsymbol{J}_1^{\#}(\boldsymbol{q})\,\boldsymbol{v}_d}{\boldsymbol{J}_2(\boldsymbol{q})\left(\boldsymbol{I} - \boldsymbol{J}_1^{\#}(\boldsymbol{q})\boldsymbol{J}_1(\boldsymbol{q})\right)\boldsymbol{J}_2^T(\boldsymbol{q})}.$$

Exercise 2

Following the Lagrangian approach, with multipliers $\boldsymbol{\lambda}$ used to weigh the holonomic constraints $\boldsymbol{h}(\boldsymbol{q}) = \boldsymbol{0}$, the dynamic equations (in the absence of gravity) take the form

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{u} + \boldsymbol{A}^T(\boldsymbol{q})\boldsymbol{\lambda} \qquad s.t. \qquad \boldsymbol{h}(\boldsymbol{q}) = \boldsymbol{0},$$

with $\boldsymbol{A}(\boldsymbol{q}) = \partial\boldsymbol{h}(\boldsymbol{q})/\partial\boldsymbol{q}$. By further elaboration, one can eliminate the multipliers (the forces that arise when attempting to violate the constraints) and obtain the so-called *constrained* robot dynamics in the form

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} = \left(\boldsymbol{I} - \boldsymbol{A}^T(\boldsymbol{q})\left(\boldsymbol{A}_B^{\#}(\boldsymbol{q})\right)^T\right)(\boldsymbol{u} - \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})) - \boldsymbol{B}(\boldsymbol{q})\boldsymbol{A}_B^{\#}(\boldsymbol{q})\dot{\boldsymbol{A}}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

where

$$\boldsymbol{A}_B^{\#}(\boldsymbol{q}) = \boldsymbol{B}^{-1}(\boldsymbol{q})\boldsymbol{A}^T(\boldsymbol{q})\left(\boldsymbol{A}(\boldsymbol{q})\boldsymbol{B}^{-1}(\boldsymbol{q})\boldsymbol{A}^T(\boldsymbol{q})\right)^{-1}$$

is the (dynamically consistent) pseudoinverse of $\boldsymbol{A}$, weighted by the robot inertia matrix.

We need thus to provide the robot inertia matrix $\boldsymbol{B}$, the Coriolis and centrifugal vector $\boldsymbol{c}$, the matrix $\boldsymbol{A}$ and its time derivative $\dot{\boldsymbol{A}}$. The kinetic energy[1] is

$$T = T_1 + T_2 = \frac{1}{2}I_1\dot{q}_1^2 + \frac{1}{2}\left(I_2\dot{q}_1^2 + m_2\boldsymbol{v}_{c2}^T\boldsymbol{v}_{c2}\right).$$

[1] For simplicity, it is assumed that the first link has its center of mass on the axis of the first joint. Otherwise, if the center of mass is at a distance d_{c1}, simply replace I_1 by $I_1 + m_1 d_{c1}^2$ in the following.

Since

$$\boldsymbol{p}_{c2} = \begin{pmatrix} (q_2 - d)\cos q_1 \\ (q_2 - d)\sin q_1 \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2} = \begin{pmatrix} -(q_2 - d)\sin q_1 \dot{q}_1 + \dot{q}_2 \cos q_1 \\ (q_2 - d)\cos q_1 \dot{q}_1 + \dot{q}_2 \sin q_1 \end{pmatrix},$$

it follows

$$T = \frac{1}{2}\left(I_1 + I_2 + m_2(q_2 - d)^2\right)\dot{q}_1^2 + \frac{1}{2} m_2 \dot{q}_2^2 = \frac{1}{2}\dot{\boldsymbol{q}}^T \begin{pmatrix} I_1 + I_2 + m_2(q_2 - d)^2 & 0 \\ 0 & m_2 \end{pmatrix} \dot{\boldsymbol{q}} = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}}.$$

From the inertia matrix, using the Christoffel symbols, we obtain

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} 2m_2(q_2 - d)\dot{q}_1\dot{q}_2 \\ -m_2(q_2 - d)\dot{q}_1^2 \end{pmatrix}.$$

The (scalar) Cartesian constraint on the end-effector is

$$h(\boldsymbol{q}) = q_2 \cos q_1 - k = 0.$$

Thus,

$$\boldsymbol{A}(\boldsymbol{q}) = \frac{\partial h(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -q_2 \sin q_1 & \cos q_1 \end{pmatrix}$$

and

$$\dot{\boldsymbol{A}}(\boldsymbol{q}) = \begin{pmatrix} -\dot{q}_2 \sin q_1 - q_2 \cos q_1 \dot{q}_1 & -\sin q_1 \dot{q}_1 \end{pmatrix}.$$

Since q_2 is never allowed to go to zero (by the constraint $x = k > 0$ on the end-effector), matrix $\boldsymbol{A}$ has always full rank and all expressions in the constrained dynamics hold without singularities. For instance, the dynamically consistent weighted pseudoinverse takes the final expression

$$\boldsymbol{A}_B^{\#}(\boldsymbol{q}) = \frac{m_2(I_1 + I_2 + m_2(q_2 - d)^2)}{I_1 + I_2 + m_2 q_2^2 + m_2 d(d - 2q_2)\cos^2 q_1} \begin{pmatrix} -\dfrac{q_2 \sin q_1}{I_1 + I_2 + m_2(q_2 - d)^2} \\ \dfrac{\cos q_1}{m_2} \end{pmatrix}.$$

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