Robotics II

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Consider a 2R planar robot with equal links of unitary length, uniformly distributed link masses, and moving in the horizontal plane. During *free motion*, the robot dynamics is

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{\tau},$$

where

$$\boldsymbol{B}(\boldsymbol{q}) = \begin{pmatrix} a_1 + 2a_2 \cos q_2 & a_3 + a_2 \cos q_2 \\ a_3 + a_2 \cos q_2 & a_3 \end{pmatrix} \qquad \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} -a_2 \sin q_2 \, \dot{q}_2(\dot{q}_2 + 2\dot{q}_1) \\ a_2 \sin q_2 \, \dot{q}_1^2 \end{pmatrix},$$

and with the dynamic coefficients a_1, a_2, and a_3 being known (their numerical value is not essential in the following).

In the configuration $\boldsymbol{q} = (0, \pi/4)$ [rad] and with $\dot{\boldsymbol{q}} = (0, -\pi/2)$ [rad/s], the *robot is hit* by an instantaneous planar Cartesian force $\boldsymbol{F} = (-10, 0)$ [N] applied at the midpoint of the second link.

1. Determine the explicit expression of the feedback control law for the joint torque vector $\boldsymbol{\tau}$ that minimizes in norm the resulting instantaneous joint acceleration $\ddot{\boldsymbol{q}}$. Provide the associated expression of $\|\ddot{\boldsymbol{q}}\|$.
2. What are the robot sensing requirements for achieving this result? How could we design a dynamic control law for the torque $\boldsymbol{\tau}$ that approximates this same robot behavior when having only joint position and velocity measurements available?

[120 minutes; open books]

Solution

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When the 2R planar robot is hit at a given state $(\boldsymbol{q}, \dot{\boldsymbol{q}})$ by a force $\boldsymbol{F}$, its dynamics becomes

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{\tau} + \boldsymbol{J}_k^T(\boldsymbol{q})\boldsymbol{F}, \tag{1}$$

where the Jacobian $\boldsymbol{J}_k(\boldsymbol{q})$ is associated to the collision point, in the present case the midpoint of the second link. Taking into account the kinematic data and the collision configuration $\boldsymbol{q} = (0, \pi/4)$, we have

$$\begin{aligned}\boldsymbol{J}_K = \boldsymbol{J}_k(\boldsymbol{q})|_{\boldsymbol{q}=(0,\pi/4)} &= \left.\begin{pmatrix} -\sin q_1 - 0.5\sin(q_1+q_2) & -0.5\sin(q_1+q_2) \\ \cos q_1 + 0.5\cos(q_1+q_2) & 0.5\cos(q_1+q_2) \end{pmatrix}\right|_{\boldsymbol{q}=(0,\pi/4)} \\ &= \begin{pmatrix} -\dfrac{\sqrt{2}}{4} & -\dfrac{\sqrt{2}}{4} \\ 1+\dfrac{\sqrt{2}}{4} & \dfrac{\sqrt{2}}{4} \end{pmatrix}.\end{aligned}$$

The instantaneous joint acceleration $\ddot{\boldsymbol{q}}$ in response to the collision force $\boldsymbol{F}$ and to an applied torque $\boldsymbol{\tau}$ at the joints will be

$$\ddot{\boldsymbol{q}} = \boldsymbol{B}^{-1}(\boldsymbol{q})\left(\boldsymbol{\tau} + \boldsymbol{J}_k^T(\boldsymbol{q})\boldsymbol{F} - \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})\right).$$

Therefore, the control torque

$$\boldsymbol{\tau} = \boldsymbol{\tau}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{F}) = \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{J}_k^T(\boldsymbol{q})\boldsymbol{F} \tag{2}$$

will produce $\ddot{\boldsymbol{q}} = \boldsymbol{0}$, which is the minimum possible acceleration ($\|\ddot{\boldsymbol{q}}\| = 0$, and the robot will continue its motion unperturbed by the collision). In order to be realizable, this control law should have access to a (direct or indirect) instantaneous measure of $\boldsymbol{F}$, beyond measuring the internal state of the robot. Moreover, the collision point (assumed to be at the midpoint of the second link in the formulation) should also be known exactly in order to use the correct $\boldsymbol{J}_k^T(\boldsymbol{q})$. These two requirements could be matched in principle if there is a camera observing the scene and/or a surface touch sensor capable of measuring the collision force (if different from the assumed one).

If this can be accomplished, the actual value of the control torque at the collision instant is computed using

$$\begin{aligned}\boldsymbol{c} = \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})|_{\boldsymbol{q}=(0,\pi/4),\dot{\boldsymbol{q}}=(0,-\pi/2)} &= \left.\begin{pmatrix} -a_2 \sin q_2\, \dot{q}_2(\dot{q}_2 + 2\dot{q}_1) \\ a_2 \sin q_2\, \dot{q}_1^2 \end{pmatrix}\right|_{\boldsymbol{q}=(0,\pi/4),\dot{\boldsymbol{q}}=(0,-\pi/2)} \\ &= \begin{pmatrix} -a_2 \dfrac{\sqrt{2}}{2}\left(\dfrac{\pi}{2}\right)^2 \\ 0 \end{pmatrix}\end{aligned}$$

as

$$\boldsymbol{\tau} = \boldsymbol{c} - \boldsymbol{J}_K^T\boldsymbol{F} = \begin{pmatrix} -a_2 \dfrac{\sqrt{2}}{2}\left(\dfrac{\pi}{2}\right)^2 \\ 0 \end{pmatrix} - \begin{pmatrix} -\dfrac{\sqrt{2}}{4} & * \\ -\dfrac{\sqrt{2}}{4} & * \end{pmatrix}\begin{pmatrix} -10 \\ 0 \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} 5 - a_2\left(\dfrac{\pi}{2}\right)^2 \\ 5 \end{pmatrix}, \tag{3}$$

where the dynamic coefficient a_2 is known.

Indeed, the above situation is quite restrictive. In order to get rid of extra sensing and measurements, we could use the concept of model-based residuals to estimate at once the global quantity

$$\tau_k = \boldsymbol{J}_k(\boldsymbol{q})^T \boldsymbol{F}$$

needed in the control action (2). In fact, we can generate on line the following (here, two-dimensional) residual signal $\boldsymbol{r}$

$$\boldsymbol{r} = \boldsymbol{K}_I \left(\boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}} - \int \left(\boldsymbol{C}^T(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{r} + \boldsymbol{\tau} \right) dt \right) \tag{4}$$

for $\boldsymbol{K}_I > 0$, typically diagonal, and where the matrix $\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ is a factorization of the Coriolis and centrifugal terms that satisfies

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}, \text{ s.t. } \dot{\boldsymbol{B}}(\boldsymbol{q}) - 2\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \text{ is skew-symmetric} \;\;\Rightarrow\;\; \dot{\boldsymbol{B}}(\boldsymbol{q}) = \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{C}^T(\boldsymbol{q}, \dot{\boldsymbol{q}}). \tag{5}$$

Such a matrix always exists and can be computed, e.g., using the Christoffels symbols. From (4), using the dynamics (1) and the property (5), it follows that the evolution of $\boldsymbol{r}$ satisfies

$$\begin{aligned}
\dot{\boldsymbol{r}} &= \boldsymbol{K}_I \left(\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{B}}(\boldsymbol{q})\dot{\boldsymbol{q}} - \boldsymbol{C}^T(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{r} - \boldsymbol{\tau} \right) \\
&= \boldsymbol{K}_I \left(\boldsymbol{\tau} + \boldsymbol{\tau}_k - \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \dot{\boldsymbol{B}}(\boldsymbol{q})\dot{\boldsymbol{q}} - \boldsymbol{C}^T(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{r} - \boldsymbol{\tau} \right) \\
&= \boldsymbol{K}_I \left(\boldsymbol{\tau}_k - \boldsymbol{r} \right).
\end{aligned}$$

Therefore, $\boldsymbol{r}$ is a low-pass stable first-order filter of the unknown signal $\boldsymbol{\tau}_k$ with bandwidth going to infinity for increasingly larger $\boldsymbol{K}_I$. Its evaluation in (4) requires only the proprioceptive measurements $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ (beside, and as before, a reasonable accuracy for the dynamic model of the robot) and the available of the command $\boldsymbol{\tau}$ being applied to the robot.

As a result, the control law (2) can be replaced by

$$\tau = \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{r}, \tag{6}$$

and the resulting robot behavior approximated at a low expense.

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