Robotics 2

Robots with kinematic redundancy

Part 1: Fundamentals

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Redundant robots

- direct kinematics of the task $r = f(q)$

$$f : \mathcal{Q} \rightarrow \mathcal{R}$$

joint space (dim $\mathcal{Q} = N$)

task space (dim $\mathcal{R} = M$)

- a robot is (kinematically) redundant for the task if $N > M$
  (more degrees of freedom than strictly needed for executing the task)
- r may contain the position and/or the orientation of the end-effector or, more in general, be any parameterization of the task (even not in the Cartesian workspace)
- "redundancy" of a robot is thus a relative concept, i.e., it holds with respect to a given task

Some E-E tasks and their dimensions

TASKS [for the robot end-effector (E-E)]	dimension M
position in the plane	2
position in 3D space	3
orientation in the plane	1
pointing in 3D space	2
position and orientation in 3D space	6

a planar robot with $N = 3$ joints is redundant for the task of positioning its E-E in the plane ($M = 2$), but NOT for the task of positioning AND orienting the E-E in the plane ($M = 3$)

Typical cases of redundant robots

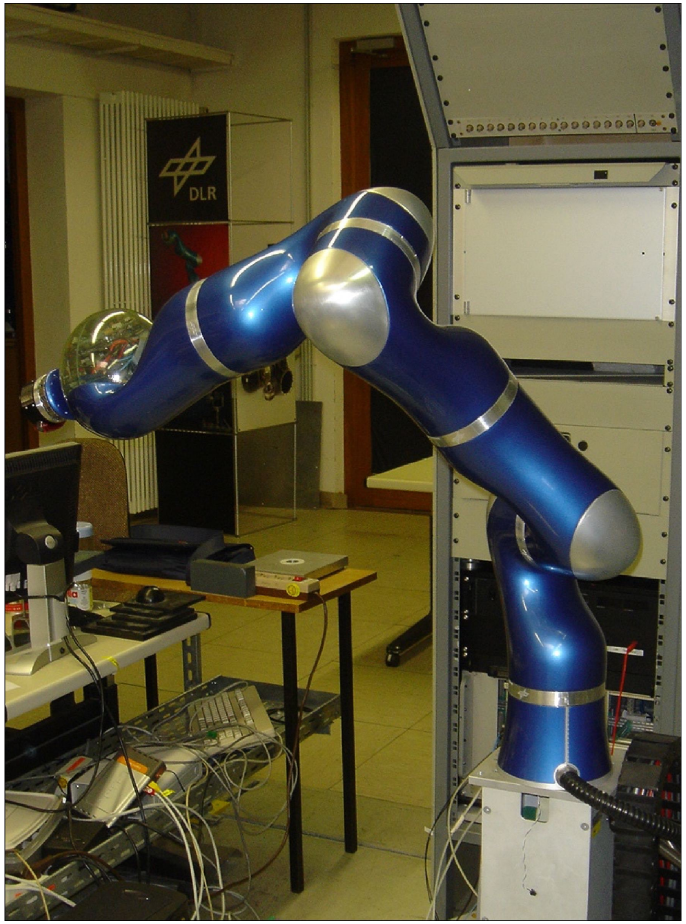
- 6R robot mounted on a linear track/rail
  - 7 dofs for positioning and orienting its end-effector in 3D space
- 6-dof robot used for arc welding tasks
  - task does not prescribe the final roll angle of the welding gun
- dexterous robotic hands
- multiple cooperating manipulators
- manipulator on a mobile (wheeled or legged) base
- humanoid robots, team of mobile robots ...
- "kinematic" redundancy is not the only type…
  - redundancy of components (actuators, sensors)
  - redundancy in the control/supervision architecture

Uses of robot redundancy

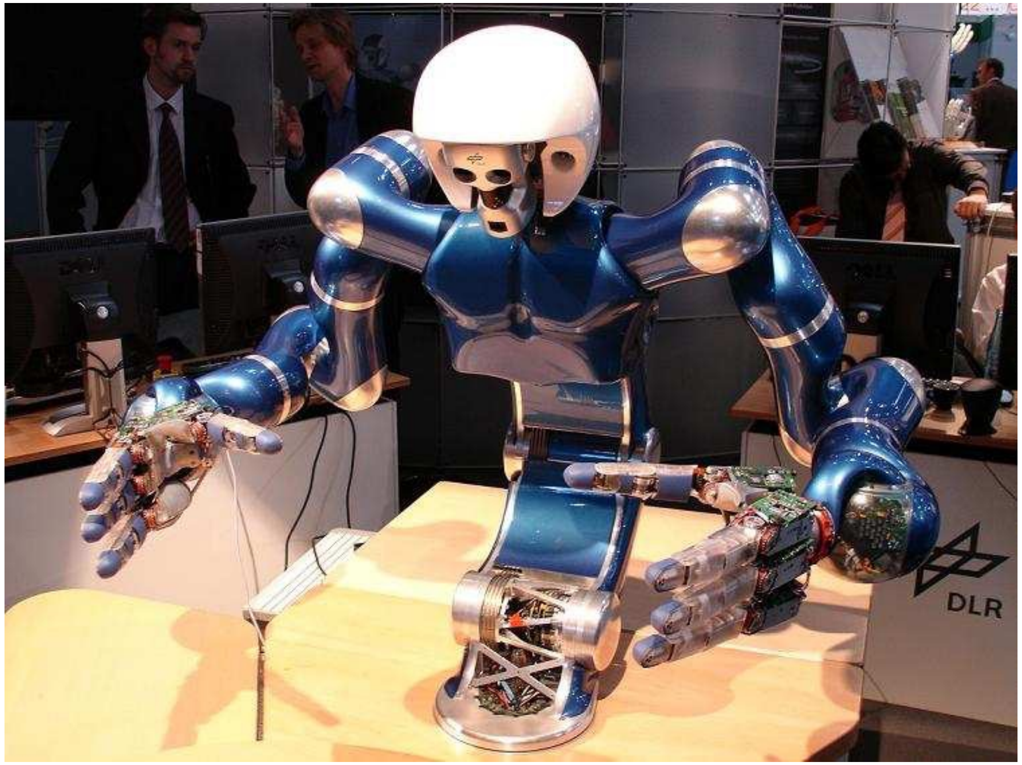
- avoid collision with obstacles (in Cartesian space) …
- … or kinematic singularities (in joint space)
- stay within the admissible joint ranges
- increase manipulability in specified directions
- uniformly distribute/limit joint velocities and/or accelerations
- minimize energy consumption or needed motion torques
- optimize execution time (on a given Cartesian path)
- increase dependability with respect to actuation faults
- ...

all objectives should be quantitatively "measurable"

DLR robots: LWR-III and Justin



7R LWR-III lightweight manipulator:
elastic joints (HD), joint torque sensing,
13.5 kg weight = payload



Justin two-arm upper-body humanoid:
43R actuated =
two arms (2×7) + torso (3*)
+ head (2) + two hands (2×12),
45 kg weight

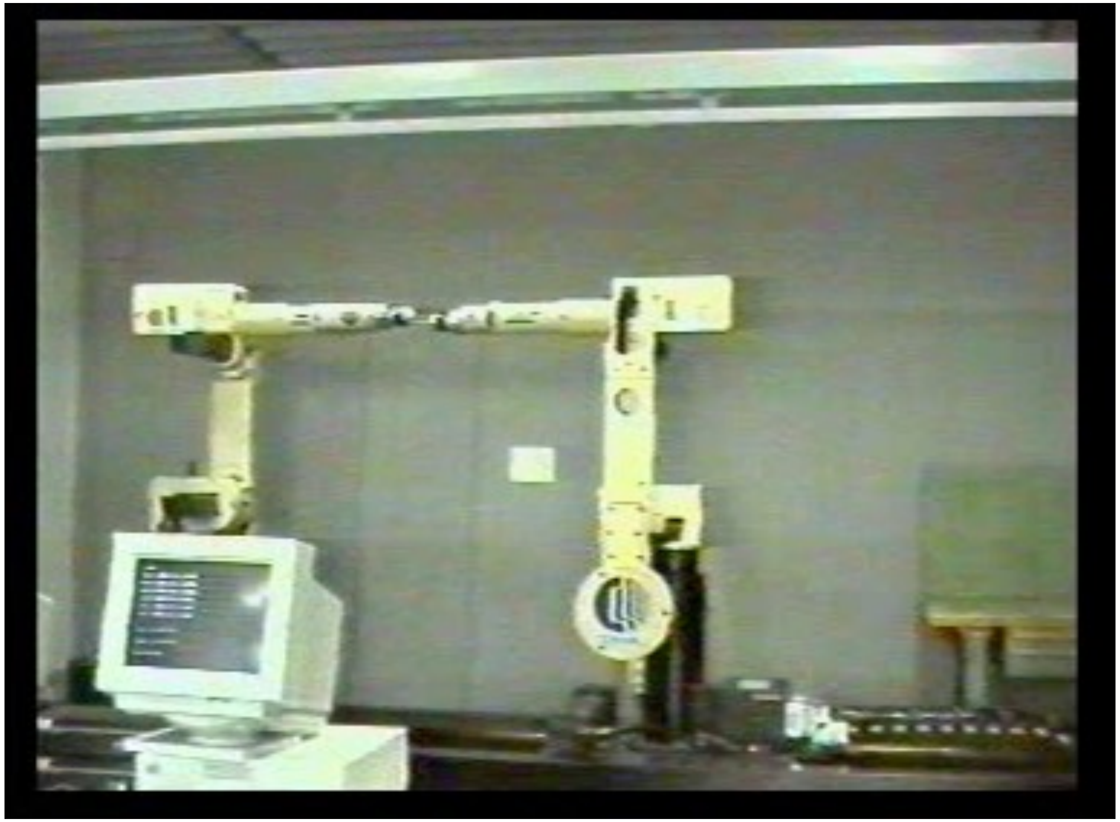
* = one joint is dependent on the motion of the other two

Justin carrying a trailer

video

motion planning for DLR Justin robot in the configuration space, avoiding Cartesian obstacles and using robot redundancy

Dual-arm redundancy



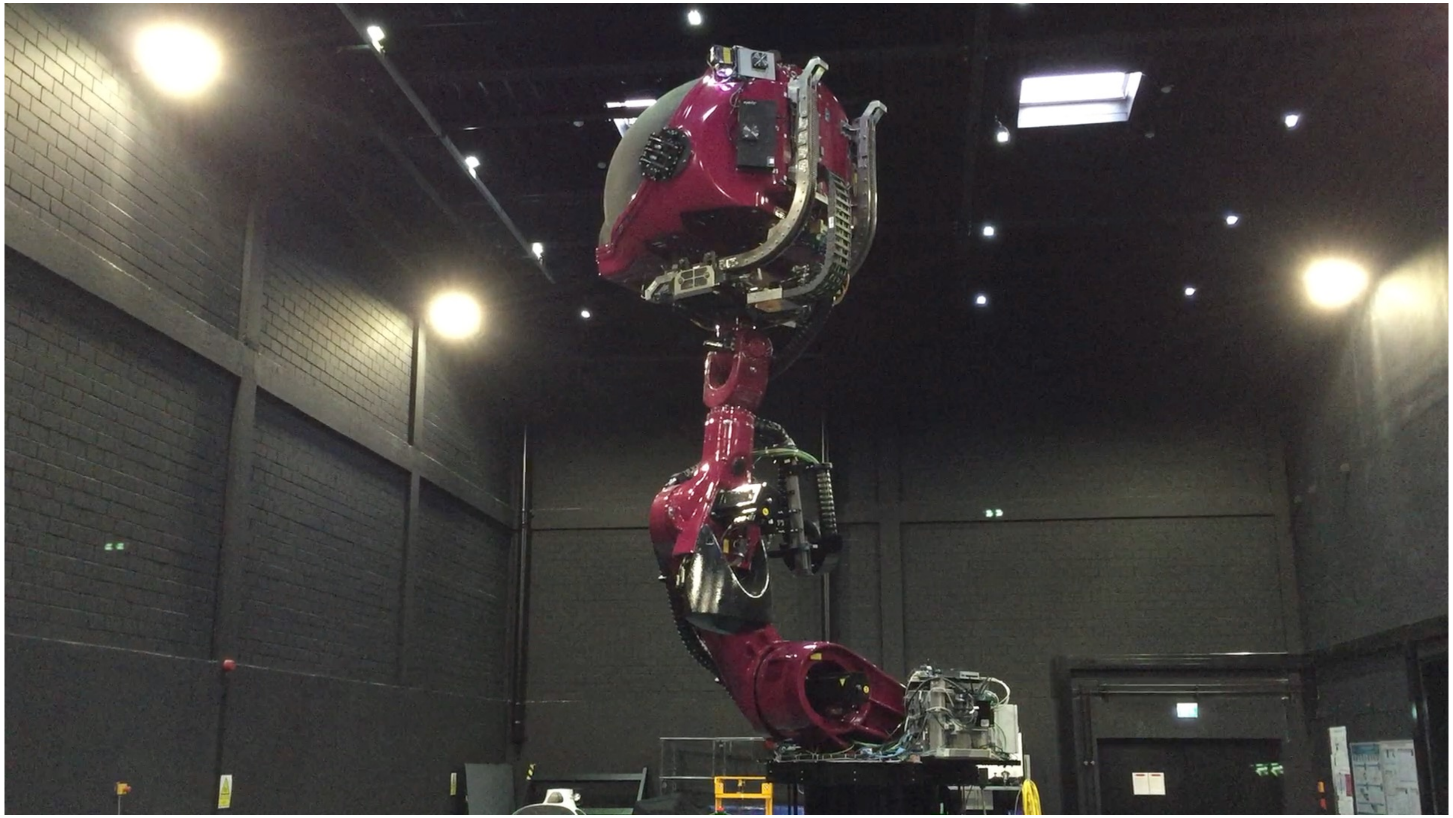
video

DIS, Uni Napoli

two 6R Comau robots, one mounted on a linear track (+1P)
coordinated 6D motion using the null-space of the right-side robot ($N - M = 1$)

Motion cueing from redundancy

video

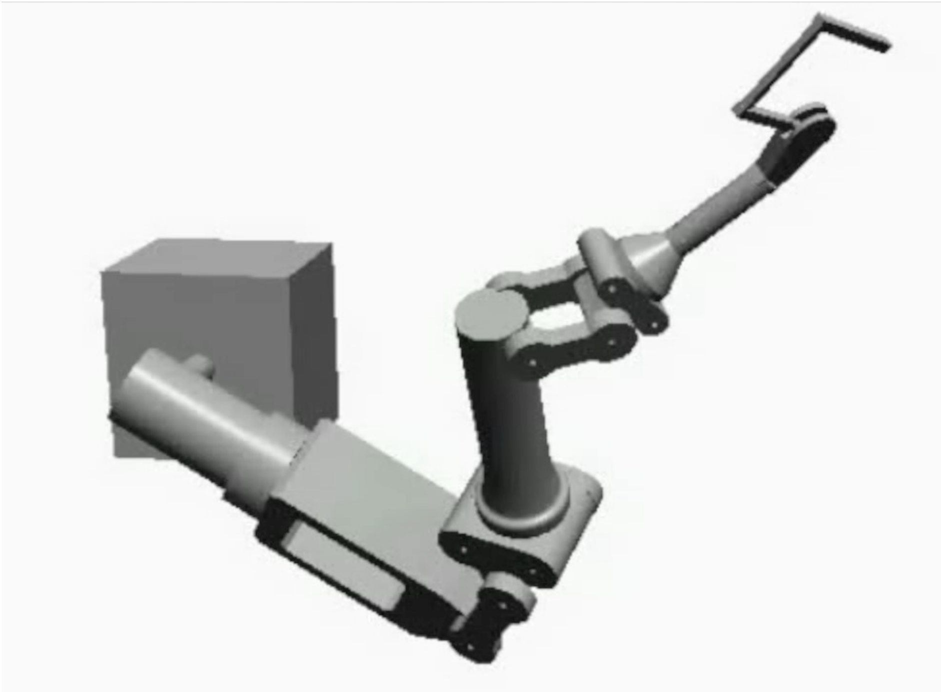


Max Planck Institute for Biological Cybernetics, Tübingen

a 6R KUKA KR500 mounted on a linear track (+1P) with a sliding cabin (+1R), used as a dynamic emulation platform for human perception ($N - M = 2$)

Self-motion

video



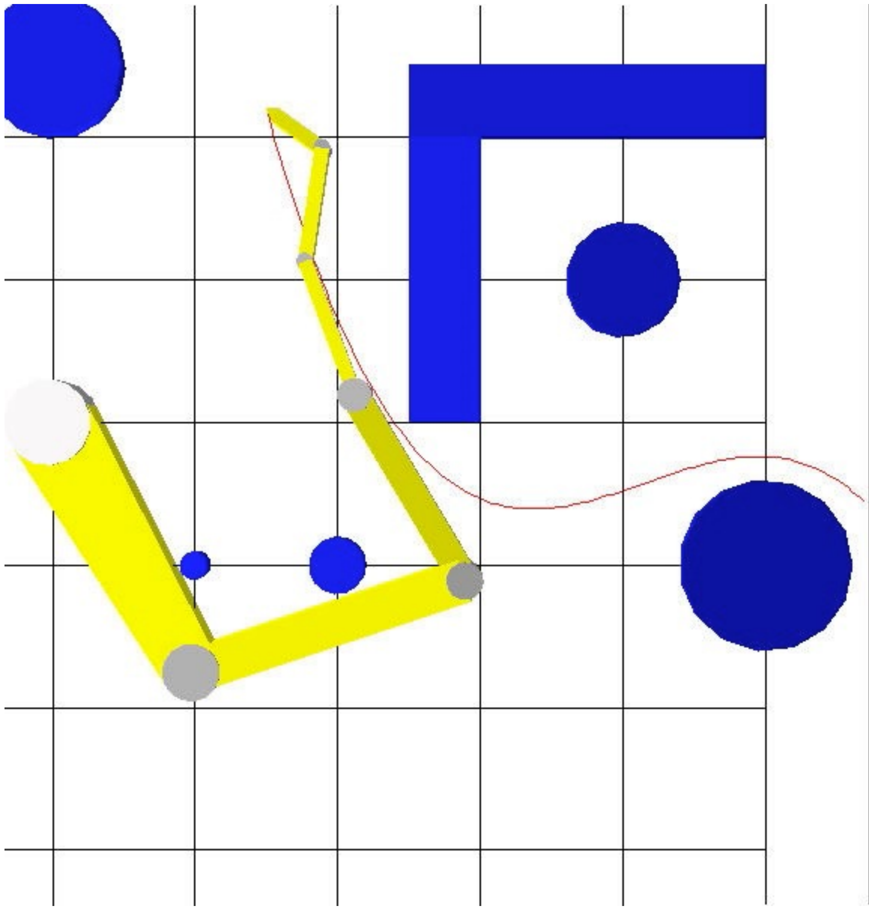
8R Dexter: self-motion with constant 6D pose of E-E ($N - M = 2$)

video

Nakamura's Lab, Uni Tokyo

6R robot with spherical shoulder in compliant tasks for the Cartesian E-E position ($N - M = 3$)

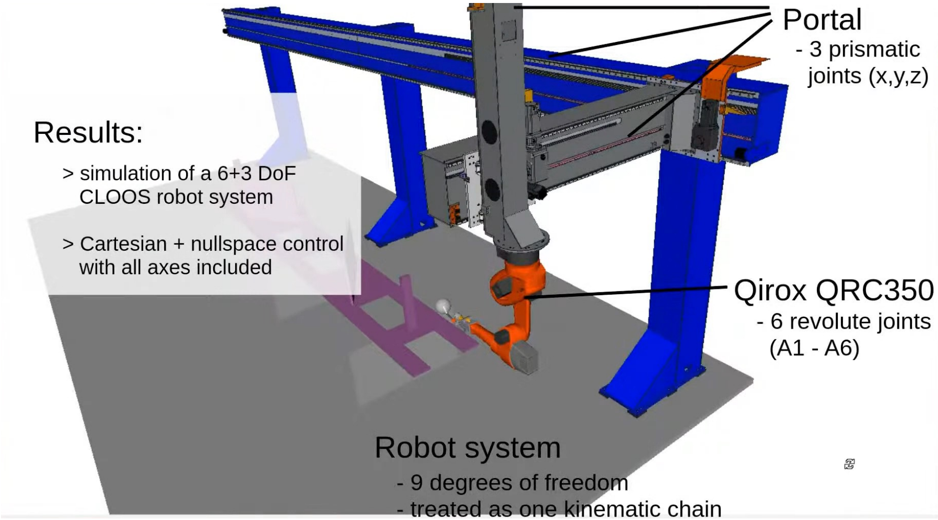
Obstacle avoidance

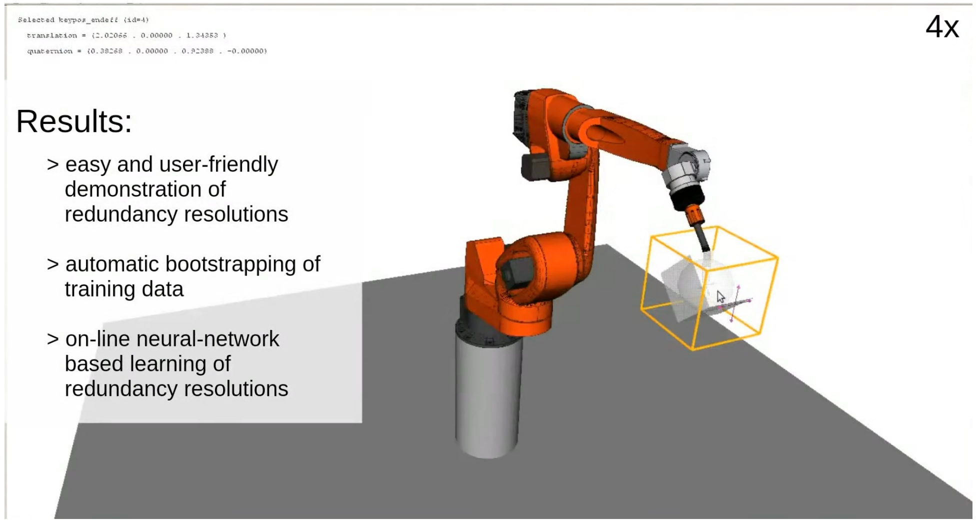


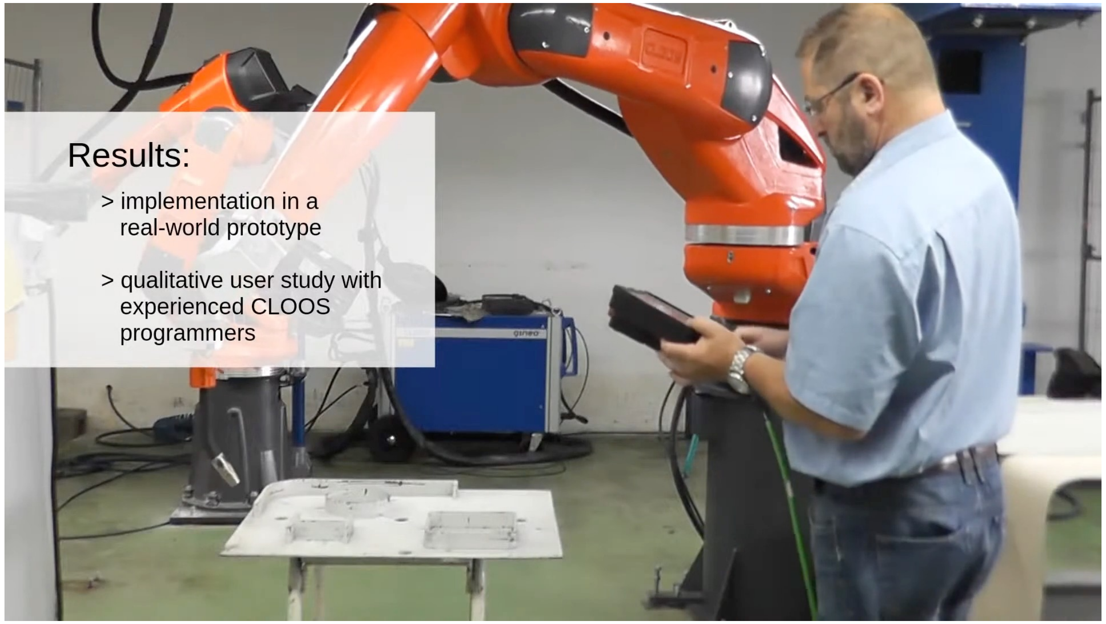
video

6R planar arm moving on a given geometric path for the E-E ($N - M = 4$)

An Echord++ industrial experiment





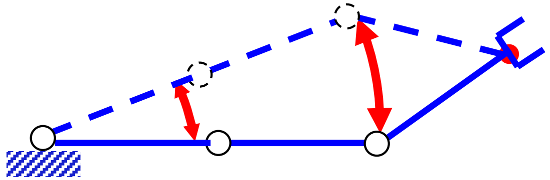


3 clips
of a single video

Inverse kinematics problem

- find $q(t)$ that realizes the task: $f\big(q(t)\big) = r(t)$ (at all times t)
- infinite solutions exist when the robot is redundant
  (even for $r(t) = r$ = constant)

$N = 3 > 2 = M$

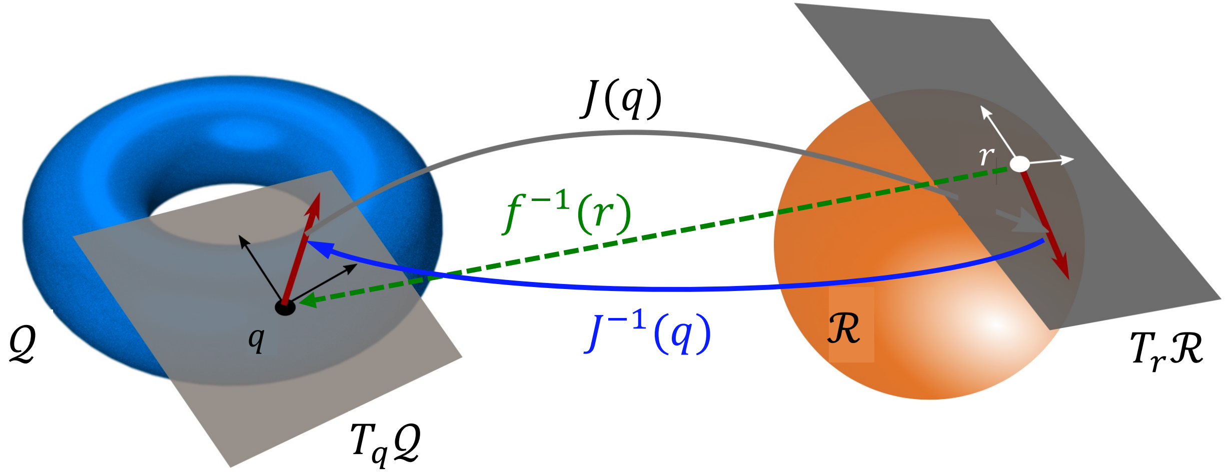


- the robot arm may have "internal displacements" that are unobservable at the task level (e.g., not contributing to E-E motion)
  - these joint displacements can be chosen so as to improve/optimize in some way the behavior of the robotic system
- self-motion: an arm reconfiguration in the joint space that does not change/affect the value of the task variables r
- solutions are mainly sought at differential level (e.g., velocity)

Inverse kinematics at velocity level

working with tangent spaces to manifolds

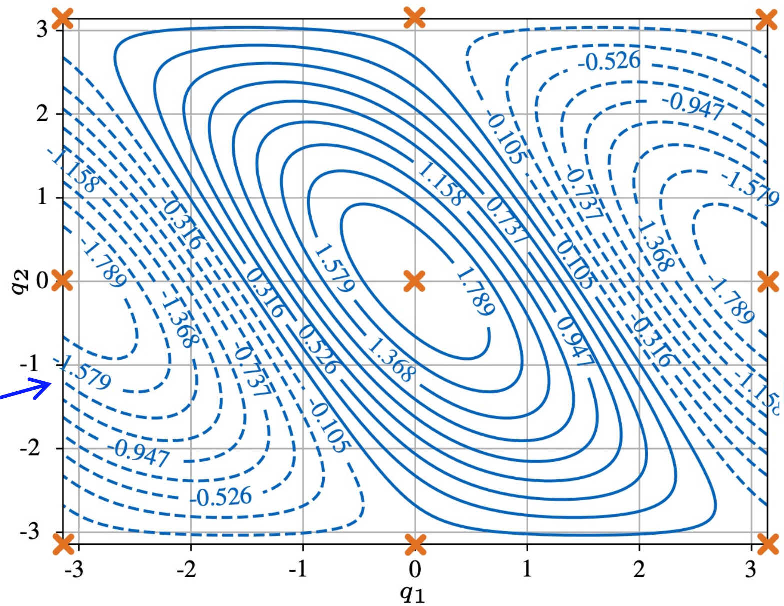
non-redundant case:
2R planar robot
in E-E position task
$(N = M = 2)$



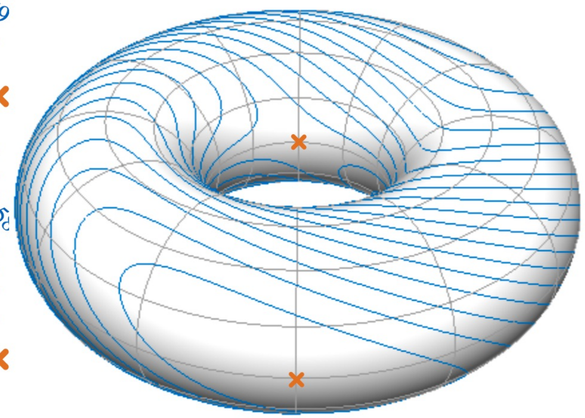
the map between the tangent spaces is linear!

redundant case:
2R planar robot
in x-position only task
$(N = 2,\ M = 1)$

iso-level curves
correspond to
$q \in \mathcal{Q}: f_x(q) = x$



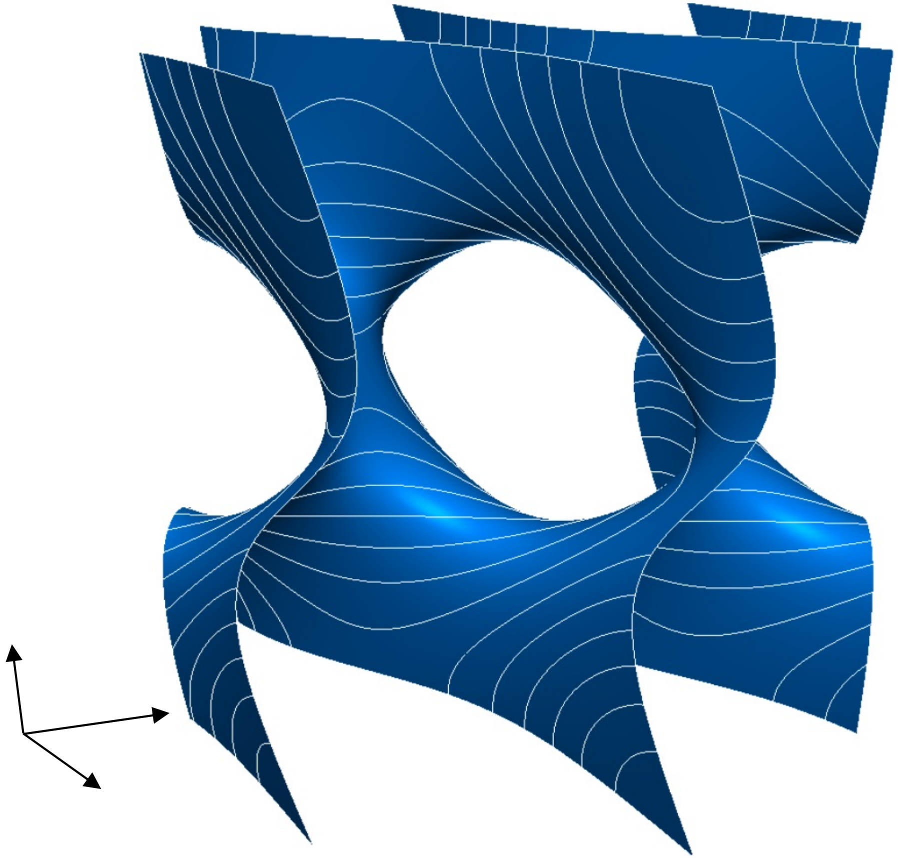
local ($\mathbb{R}^2$) vs global (torus)
representation of the
configuration manifold $\mathcal{Q}$



Self-motion manifold

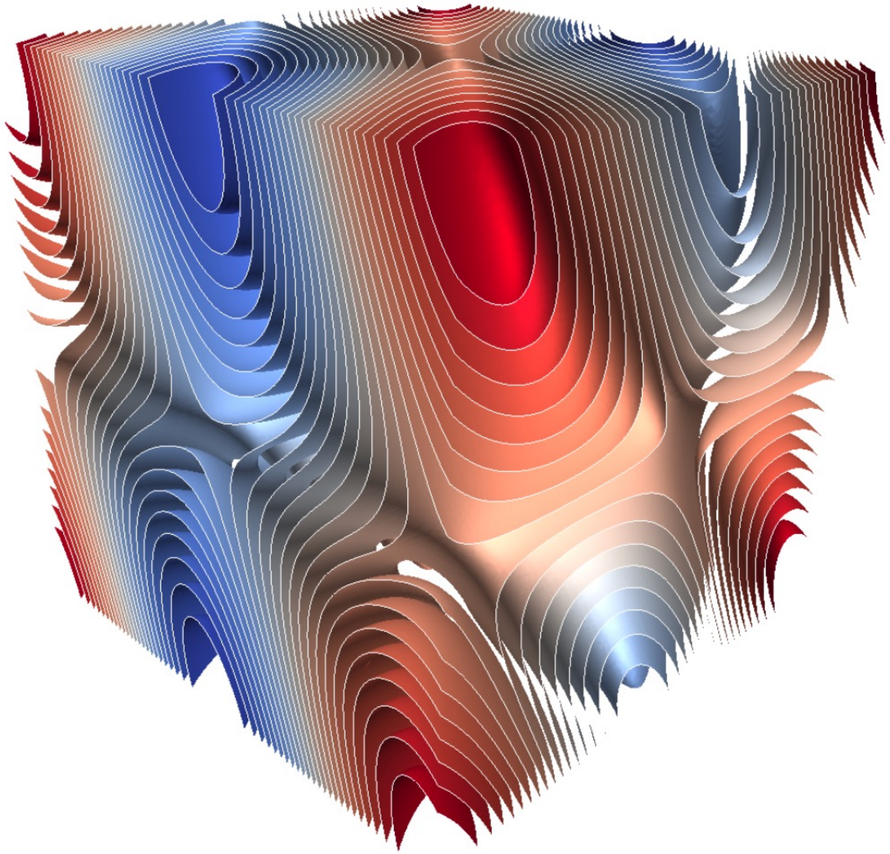
for a 3R planar robot ($N = 3$) in x-position task ($M = 1$)

one leaf in the 3-dimensional configuration space Q



using the local representation in $\mathbb{R}^3$ of manifold Q
(boundaries at $q_i = \pm\pi,\ i = 1{,}2{,}3$, coincide!)

multiple leaves



pictures courtesy of A. Albu-Schäffer, A. Sachtler:
"Redundancy resolution at position level,"
IEEE Transactions on Robotics, 2023

Redundancy resolution

via optimization of an objective function

Local methods

given $\dot{r}(t)$ and $q(t)$, $t = kT_s$

optimization of $H(q, \dot{q})$

$\dot{q}(kT_s)$ ← ON-LINE

$$q((k+1)T_s) = q(kT_s) + T_s\,\dot{q}(kT_s)$$

discrete-time form

Global methods

given $r(t)$, $t \in [t_0,\, t_0 + T]$, $q(t_0)$

optimization of $\int_{t_0}^{t_0+T} H(q, \dot{q})dt$

$$q(t),\ t \in [t_0,\, t_0 + T]$$

often, expressed
in parametric form

OFF-LINE

relatively EASY
(an LQ problem)

⟷

quite DIFFICULT
(nonlinear TPBV problems arise)

Local resolution methods

three classes of methods for solving $\dot{r} = J(q)\dot{q}$

1 Jacobian-based methods (here, analytic Jacobian in general!)

among the infinite solutions, one is chosen, e.g., that minimizes a suitable (possibly weighted) norm

2 null-space methods

a term is added to the previous solution so as not to affect execution of the task trajectory, i.e., belonging to the null-space $\mathcal{N}(J(q))$

3 task augmentation methods

redundancy is reduced/eliminated by adding $S \leq N - M$ further auxiliary tasks (when $S = N - M$, the problem has been "squared")

$$r = f(q) \Rightarrow \dot{r} = J(q)\dot{q}$$

1 Jacobian-based methods

we look for a solution to $\dot{r} = J(q)\dot{q}$ in the form

$$J = \underbrace{\boxed{}}_{N}\}M \qquad \dot{q} = K(q)\dot{r} \qquad K = \underbrace{\boxed{}}_{M}\}N$$

minimum requirement for K: $J(q)K(q)J(q) = J(q)$

(➡ K = generalized inverse of J)

$$\forall \dot{r} \in \mathcal{R}(J(q)) \Rightarrow \quad J(q)[K(q)\dot{r}] = J(q)K(q)J(q)\dot{q} = J(q)\dot{q} = \dot{r}$$

example:

if $J = [J_a \;\; J_b]$, $\det(J_a) \neq 0$, one such generalized inverse of J is $K_r = \begin{pmatrix} J_a^{-1} \\ 0 \end{pmatrix}$

(actually, this is a stronger right-inverse)

Pseudoinverse

$$\dot{q} = J^{\#}(q)\dot{r}$$

... a very common choice: $K = J^{\#}$

- $J^{\#}$ always exists, and is the unique matrix satisfying

$$J\, J^{\#}\, J = J \qquad J^{\#}\, J\, J^{\#} = J^{\#}$$
$$\left(J\, J^{\#}\right)^T = J\, J^{\#} \qquad \left(J^{\#}\, J\right)^T = J^{\#}\, J$$

- if J is full (row) rank, $J^{\#} = J^T (J\, J^T)^{-1}$; else, it is computed numerically using the SVD (Singular Value Decomposition) of J (`pinv` of Matlab)
- the pseudo-inverse solution is the only joint velocity that minimizes the norm $\|\dot{q}\|^2 = \dot{q}^T \dot{q}$ among all joint velocities that minimize the task error norm $\|\dot{r} - J(q)\dot{q}\|^2$
- if the task is feasible ($\dot{r} \in \mathcal{R}(J(q))$), there will be no task error

Weighted pseudoinverse

$$\dot{q} = J_W^{\#}(q)\dot{r}$$

another choice: $K = J_W^{\#}$

- the solution $\dot{q}$ minimizes the weighted norm

$$\|\dot{q}\|_W^2 = \dot{q}^T W \dot{q}$$

$W > 0$, symmetric (often diagonal)

- if J is full (row) rank, $J_W^{\#} = W^{-1}J^T(JW^{-1}J^T)^{-1}$
- large weight $W_i \Rightarrow$ small $\dot{q}_i$
  - larger weights for proximity joints (carrying/moving more "mass")
  - weights chosen proportionally to the inverse of the joint ranges
- it is NOT a "pseudoinverse" (4th relation does not hold), but it shares similar properties

Singular Value Decomposition (SVD)

- the `SVD` routine of Matlab applied to J provides two orthonormal matrices $U_{M\times M}$ and $V_{N\times N}$, and a matrix $\Sigma_{M\times N}$ of the form

$$\Sigma = \left(\begin{array}{cccc|c} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \ddots & & 0_{M\times(N-M)} \\ & & & \sigma_M & \end{array}\right) \qquad \begin{array}{c} \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_\rho > 0 \\ \sigma_{\rho+1} = \cdots = \sigma_M = 0 \\ \text{singular values of } J \end{array}$$

where $\rho = \text{rank}(J) \leq M$, so that their product is

$$\boxed{J = U\Sigma V^T}$$

- the columns of U are eigenvectors of $J\,J^T$ (associated to its non-negative eigenvalues σ_i^2), the columns of V are eigenvectors of $J^T J$
- the last $N-\rho$ columns of V are a basis for the null space of J

$$Jv_i = \sigma_i u_i \quad (\mathrm{i} = 1, \cdots, \rho) \qquad\qquad \boxed{Jv_i = 0 \quad (\mathrm{i} = \rho + 1, \cdots, N)}$$

Computation of pseudoinverses

- show that the pseudoinverse of J is equal to

$$J = U\Sigma V^T \quad \Rightarrow \quad J^\# = V\Sigma^\# U^T \qquad \Sigma^\# = \left(\begin{array}{cccc} \frac{1}{\sigma_1} & & & \\ & \ddots & & \\ & & \frac{1}{\sigma_\rho} & \\ & & & 0_{(M-\rho)\times(M-\rho)} \\ \hline & & 0_{(N-M)\times M} & \end{array}\right)$$

for any rank ρ of J

- show that matrix $J_W^\#$ appears when solving the constrained linear-quadratic (LQ) optimization problem (with $W > 0$, symmetric, and assuming J of full rank)

$$\min \frac{1}{2} \|\dot{q}\|_W^2 \quad \text{s.t.} \quad J(q)\dot{q} - \dot{r} = 0$$

and that the pseudoinverse is a particular case for $W = I$

- show that a weighted pseudoinverse of J can be computed by $\mathtt{SVD/pinv}$ as

$$J_{aux} = JW^{-1/2} \qquad J_W^\# = W^{-1/2}\,\mathtt{pinv}(J_{aux})$$

applies equally to square and non-square matrices

Singularity robustness

Damped Least Squares (DLS)

unconstrained minimization of a **suitable** objective function

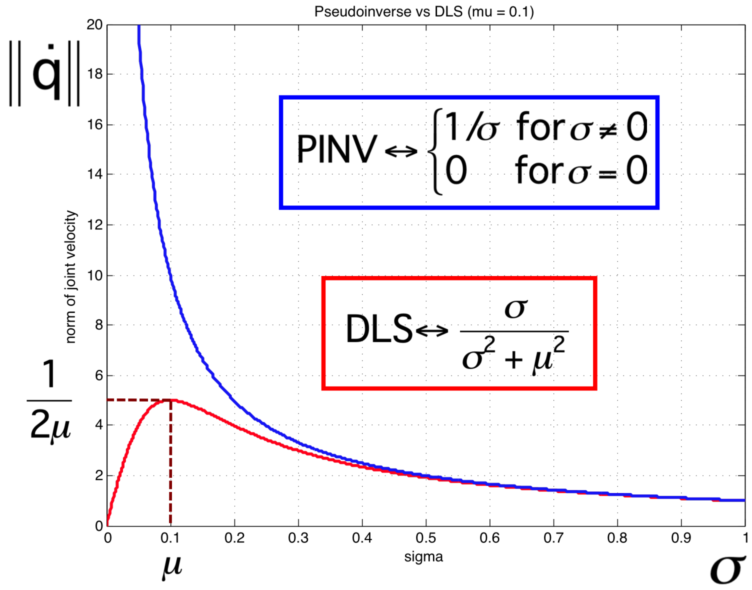
$$\min_{\dot{q}} H(\dot{q}) = \frac{\mu^2}{2}\|\dot{q}\|^2 + \frac{1}{2}\|\dot{r} - J\dot{q}\|^2$$

compromise between large joint velocity and task accuracy

SOLUTION $\dot{q} = J_{DLS}(q)\dot{r} = J^T(J\,J^T + \mu^2 I_M)^{-1}\dot{r}$

- induces a robust behavior when crossing singularities, but in its basic version gives always a task error $\dot{e} = \mu^2(J\,J^T + \mu^2 I_M)^{-1}\dot{r}$ (as for $N = M$)
  - J_{DLS} is not a generalized inverse K
  - using SVD: $J = U\,\Sigma\,V^T \Rightarrow J_{DLS} = V\Sigma_{DLS}U^T$, $\Sigma_{DLS} = \left(\begin{array}{c|c} \underbrace{\boxed{diag\left\{\dfrac{\sigma_i}{\sigma_i^2+\mu^2}\right\}}}_{\rho\times\rho} & 0_{(M-\rho)\times(M-\rho)} \\ \hline 0_{(N-M)\times\rho} & 0_{(N-M)\times(M-\rho)} \end{array}\right)$
  - choice of a variable damping factor $\mu^2(q) \geq 0$, function of the minimum singular value $\sigma_\rho(q) > 0$ of J ≅ a measure of distance from a singularity (if $\rho = M$) or of further loss of rank (when $\rho < M$)
  - numerical filtering: introduces damping only/mostly in non-feasible directions for the task (see Maciejewski and Klein, *J of Rob Syst*, 1988)

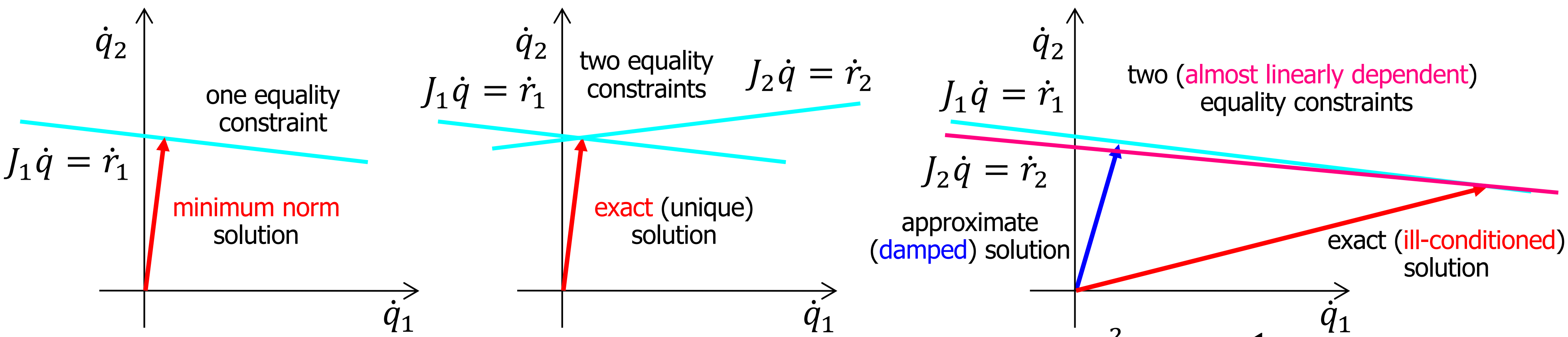
Behavior of DLS solution



a. comparison of joint velocity norm with PINV (pseudoinverse) or DLS solutions

- in a task direction along a vector u of U, when the associated singular value $\sigma \longrightarrow 0$
- PINV goes to infinity (and then is 0 at $\sigma = 0$)
- DLS peaks a value of $1/2\mu$ at $\sigma = \mu$ (and then smoothly goes to 0...)

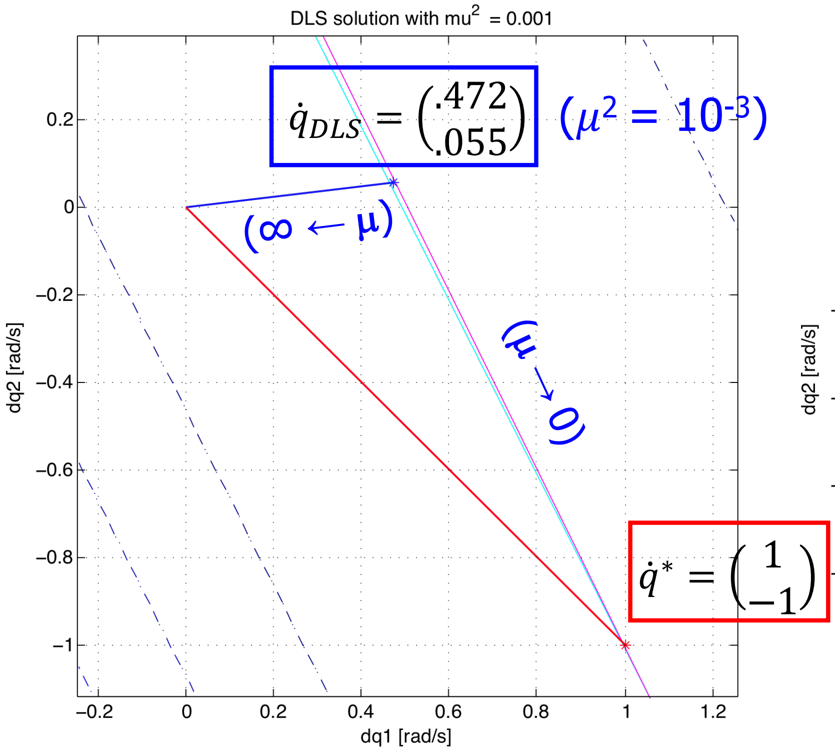
b. graphical interpretation of "damping" effect (here $M = N = 2$, for simplicity)

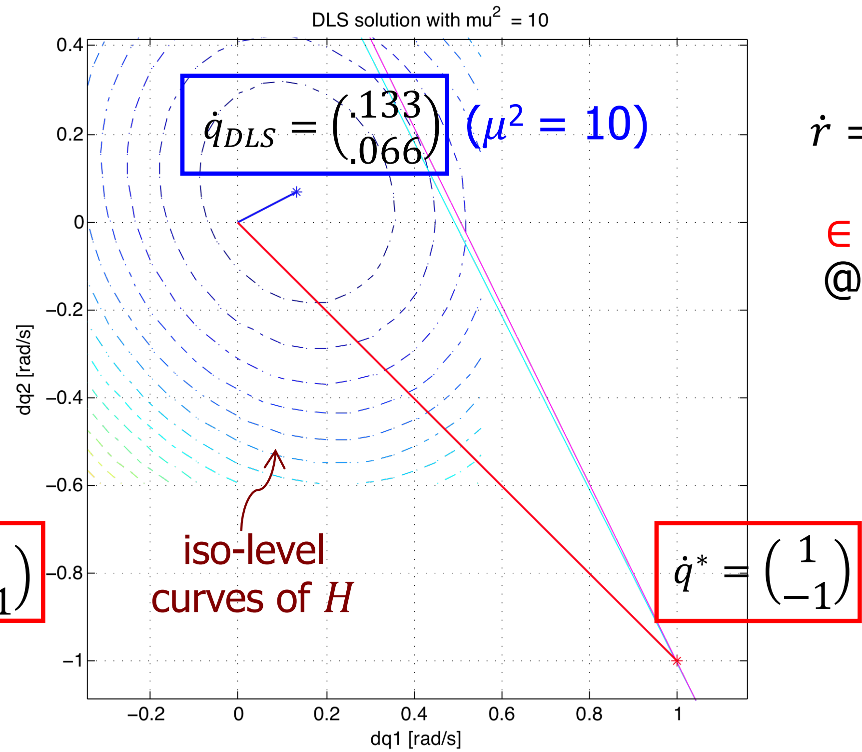


$$H(\dot{q}) = \frac{\mu^2}{2}\|\dot{q}\|^2 + \frac{1}{2}\|\dot{r} - J\dot{q}\|^2$$

Numerical example of DLS solution

planar 2R arm, unit links, close to (stretched) singular configuration $q_1 = 45°$, $q_2 = 1.5°$)





$$\dot{r} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$\in \mathcal{R}(J)$ even @singularity!

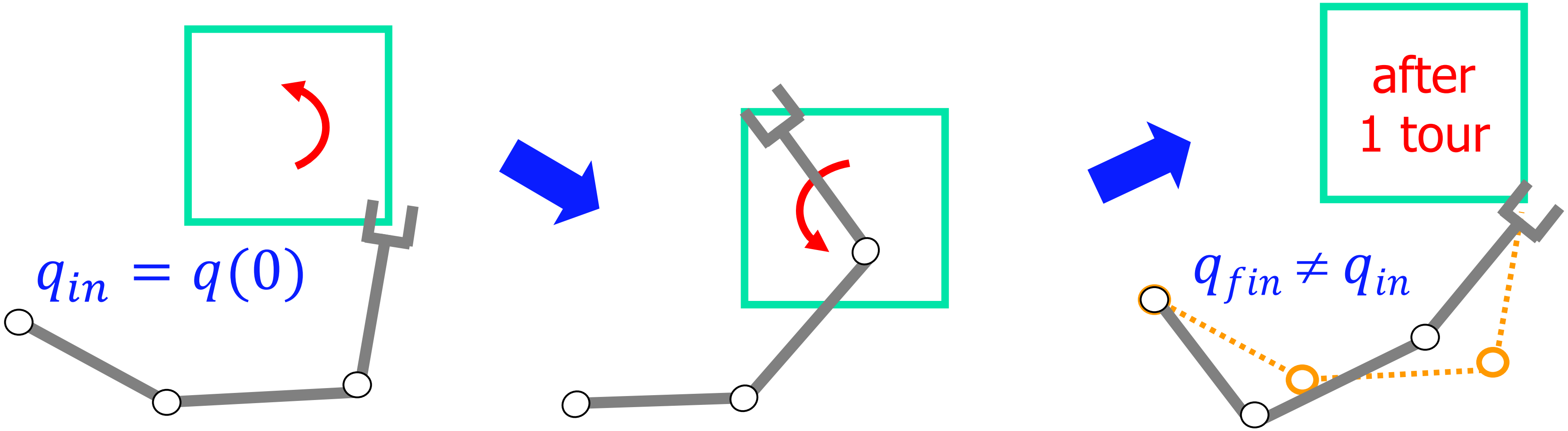
exact solution (μ=0)

$$H = \frac{\mu^2}{2}\|\dot{q}\|^2 + \frac{1}{2}\|\dot{r} - J\dot{q}\|^2$$

μ^2	0	10^{-4}	10^{-3}	10^{-2}	10
$\|\dot{q}\|$	$\sqrt{2}$	.8954	.4755	.4467	.1490
$\|\dot{e}\|$	0	$6.6 \cdot 10^{-3}$	$1.4 \cdot 10^{-2}$	$1.6 \cdot 10^{-2}$	.6668
H_{min}	0	$7.7 \cdot 10^{-5}$	$2.2 \cdot 10^{-4}$	$1.2 \cdot 10^{-3}$	$3.4 \cdot 10^{-1}$

Limits of Jacobian-based methods

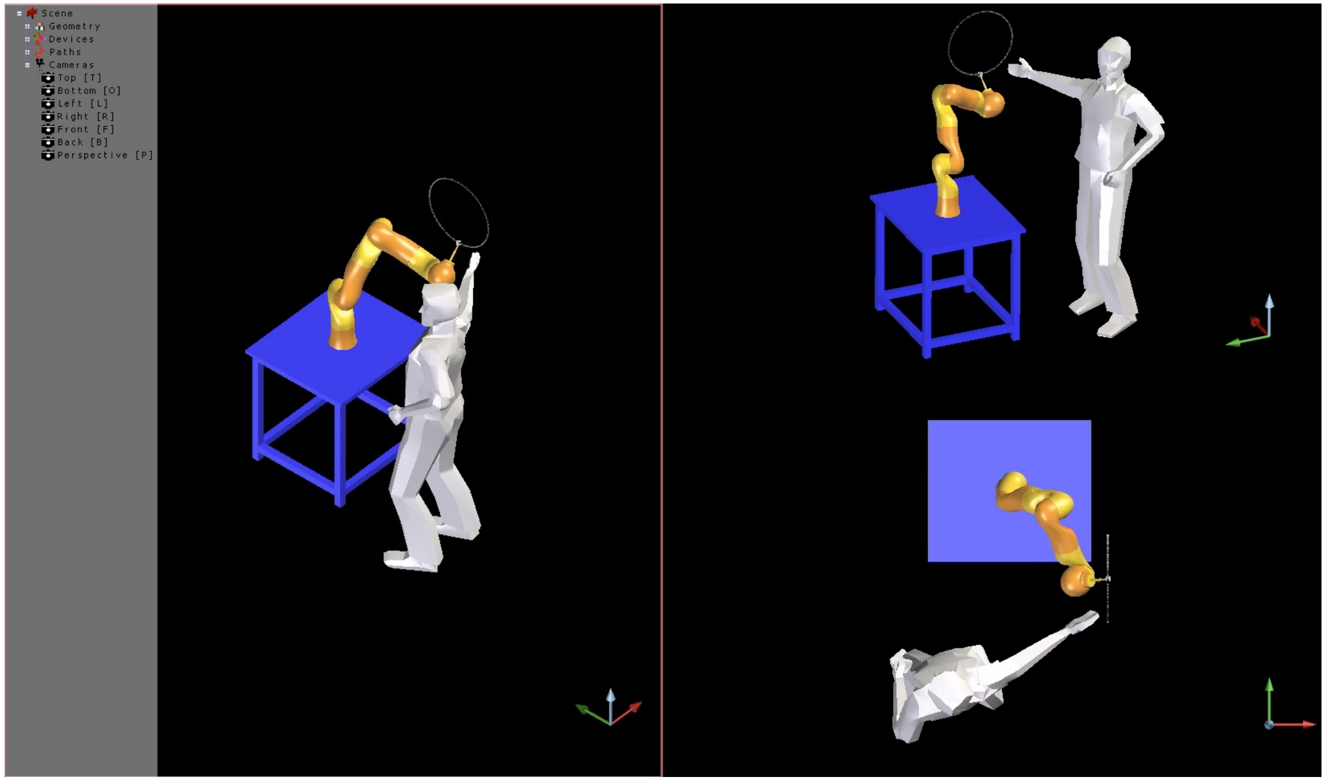
- no guarantee that singularities are globally avoided during task execution
  - despite joint velocities are kept to a minimum, this is only a local property and "avalanche" phenomena may occur
- typically lead to non-repeatable motion in the joint space
  - cyclic motions in task space do not map to cyclic motions in joint space



$$q(t) = q(0) + \int_0^t K\big(q(\tau)\big)\dot{r}(\tau)\, d\tau$$

Drift with Jacobian pseudoinverse

- a 7R KUKA LWR4 robot moves in the vicinity of a human operator
- we command a cyclic Cartesian path (only in position, $M = 3$), to be repeated several times using the pseudoinverse solution
- (unexpected) collision of a link occurs during the third cycle ...



video

2 Null-space methods

general solution of $J\dot{q} = \dot{r}$

$$\dot{q} = \underbrace{J^{\#}\dot{r}}_{\text{a particular solution (here, the pseudoinverse) in } \mathcal{R}(J^T)} + \underbrace{\left(I - J^{\#}J\right)\dot{q}_0}_{\text{"orthogonal" projection of } \dot{q}_0 \text{ in } \mathcal{N}(J)}$$

all solutions of the associated homogeneous equation $J\dot{q} = 0$ (self-motions)

a particular solution (here, the pseudoinverse) in $\mathcal{R}(J^T)$

"orthogonal" projection of $\dot{q}_0$ in $\mathcal{N}(J)$

properties of projector $[I - J^{\#}J]$

- symmetric
- idempotent: $[I - J^{\#}J]^2 = [I - J^{\#}J]$
- $[I - J^{\#}J]^{\#} = [I - J^{\#}J]$
- $J^{\#}\dot{r}$ is orthogonal to $[I - J^{\#}J]\dot{q}_0$

even more in general…

$$\dot{q} = K_1\dot{r} + (I - K_2J)\dot{q}_0$$

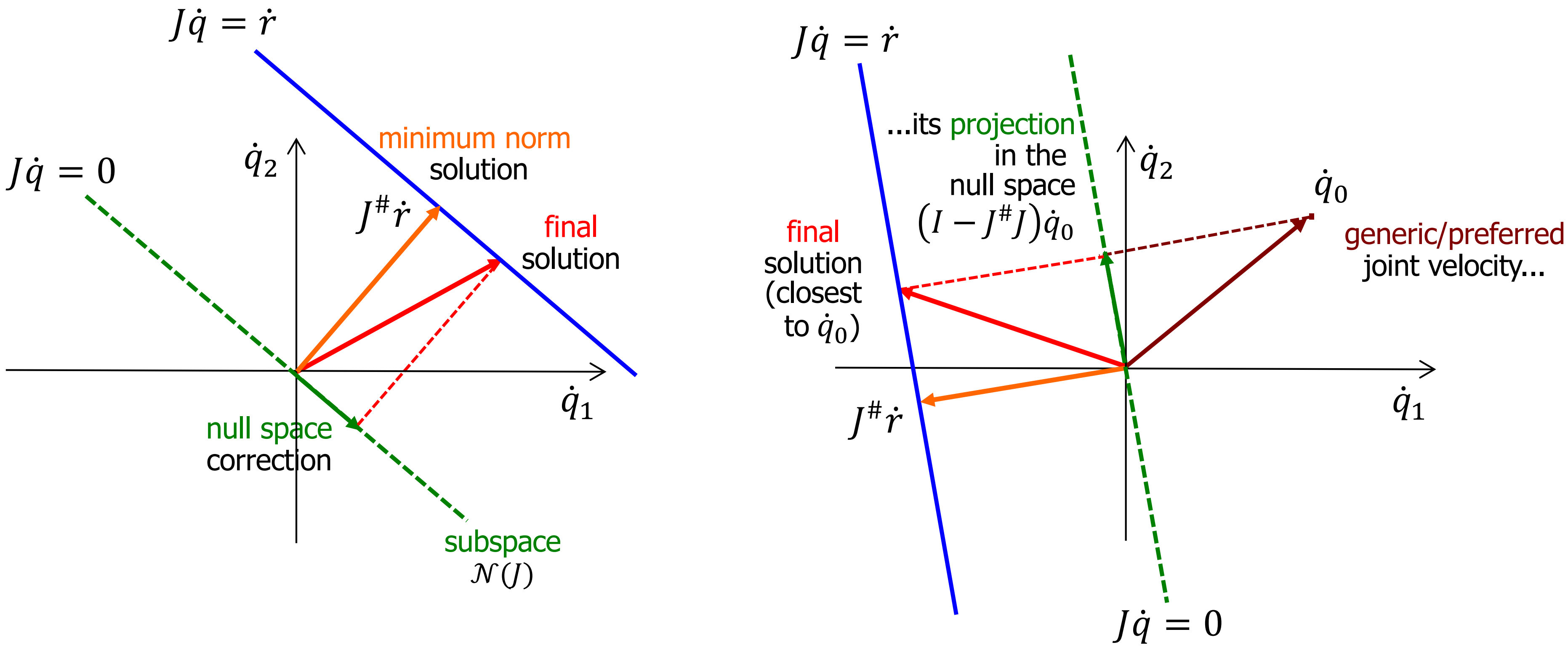
K_1, K_2 generalized inverses of J $(JK_iJ = J)$

... but with less nice properties!

how do we choose $\dot{q}_0$?

Geometric view on Jacobian null space

in the space of velocity commands



a correction is added to the original pseudoinverse (minimum norm) solution
i) which is in the null space of the Jacobian
ii) and possibly satisfies additional criteria or objectives

Linear-Quadratic Optimization
generalities

$$\min_x H(x) = \frac{1}{2}(x - x_0)^T W(x - x_0)$$

$$\text{s.t.} \quad \underbrace{J}_{M\times N} x = y$$

$x \in \mathbb{R}^N$

$W > 0$ (symmetric)

$y \in \mathbb{R}^M$

$\text{rank}(J) = \rho(J) = M$

$L(x, \lambda) = H(x) + \lambda^T (Jx - y)$ ← Lagrangian (with multipliers λ)

necessary conditions

$$\nabla_x L = \left(\frac{\partial L}{\partial x}\right)^T = W(x - x_0) + J^T\lambda = 0 \quad \Rightarrow \quad \boxed{x = x_0 - W^{-1}J^T\lambda}$$

$$\nabla_\lambda L = \left(\frac{\partial L}{\partial \lambda}\right)^T = Jx - y = 0 \quad \Rightarrow \quad \boxed{Jx_0 - JW^{-1}J^T\lambda - y = 0}$$

\+ sufficient condition for a minimum

$$\nabla_x^2 L = W > 0$$

$$\boxed{\lambda = \underbrace{(JW^{-1}J^T)^{-1}}_{M\times M \text{ invertible}}(Jx_0 - y)} \quad \Rightarrow \quad x = x_0 + W^{-1}J^T(JW^{-1}J^T)^{-1}(y - Jx_0)$$

Linear-Quadratic Optimization

application to robot redundancy resolution

PROBLEM

$$\min_{\dot q} H(\dot q) = \frac{1}{2}(\dot q - \dot q_0)^T W (\dot q - \dot q_0)$$

$$\text{s.t.} \quad J\dot q = \dot r$$

$\dot q_0$ is a "privileged" joint velocity

SOLUTION

$$\dot q = \dot q_0 + \underbrace{W^{-1}J^T(JW^{-1}J^T)^{-1}}_{J_W^{\#}}(\dot r - J\dot q_0)$$

$$\boxed{\dot q = \underbrace{J_W^{\#}\dot r}_{\text{minimum weighted norm solution (for } \dot q_0 = 0)} + \underbrace{\left(I - J_W^{\#}J\right)}_{\text{"projection" matrix in the null-space } \mathcal{N}(J)}\dot q_0}$$

minimum weighted norm solution (for $\dot q_0 = 0$)

"projection" matrix in the null-space $\mathcal{N}(J)$

Projected Gradient (PG)

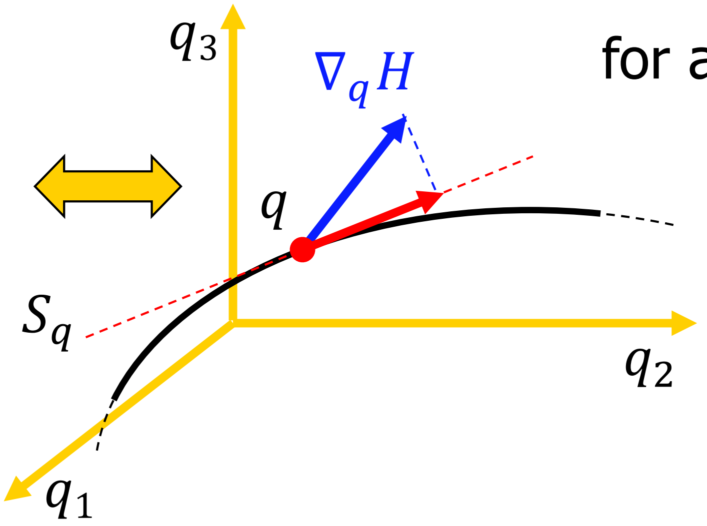
$$\dot{q} = J^{\#}\dot{r} + \left(I - J^{\#}J\right)\dot{q}_0$$

the choice $\dot{q}_0 = \nabla_q H(q)$ → differentiable objective function

realizes one step of a constrained optimization algorithm

while executing the time-varying task $r(t)$
the robot tries to increase the value of $H(q)$

projected
gradient



for a fixed $\bar{r}$: $S_q = \{q \in \mathbb{R}^N : f(q) = \bar{r}\}$

$$\Rightarrow \quad \dot{q} = \left(I - J^{\#}J\right)\nabla_q H$$

N-dimensional $\boxed{\left(I - J^{\#}J\right)\nabla_q H = 0}$

is a necessary condition
of constrained optimality

Typical objective functions $H(q)$

- manipulability (maximize the "distance" from singularities)

$$H_{\text{man}}(q) = \sqrt{\det[J(q)J^T(q)]}$$

- joint range (minimize the "distance" from the mid points of the joint ranges)

$$q_i \in [q_{m,i}, q_{M,i}] \qquad \bar{q}_i = \frac{q_{M,i} + q_{m,i}}{2}$$

$$H_{\text{range}}(q) = \frac{1}{2N} \sum_{i=1}^{N} \left(\frac{q_i - \bar{q}_i}{q_{M,i} - q_{m,i}} \right)^2$$

$$\dot{q}_0 = - \nabla_q H(q)$$

- obstacle avoidance (maximize the minimum distance to Cartesian obstacles)

also known as "clearance"

$$H_{\text{obs}}(q) = \min_{\substack{a \in \text{robot} \\ b \in \text{obstacles}}} \|a(q) - b\|^2$$

potential difficulties due to non-differentiability (this is a max-min problem)

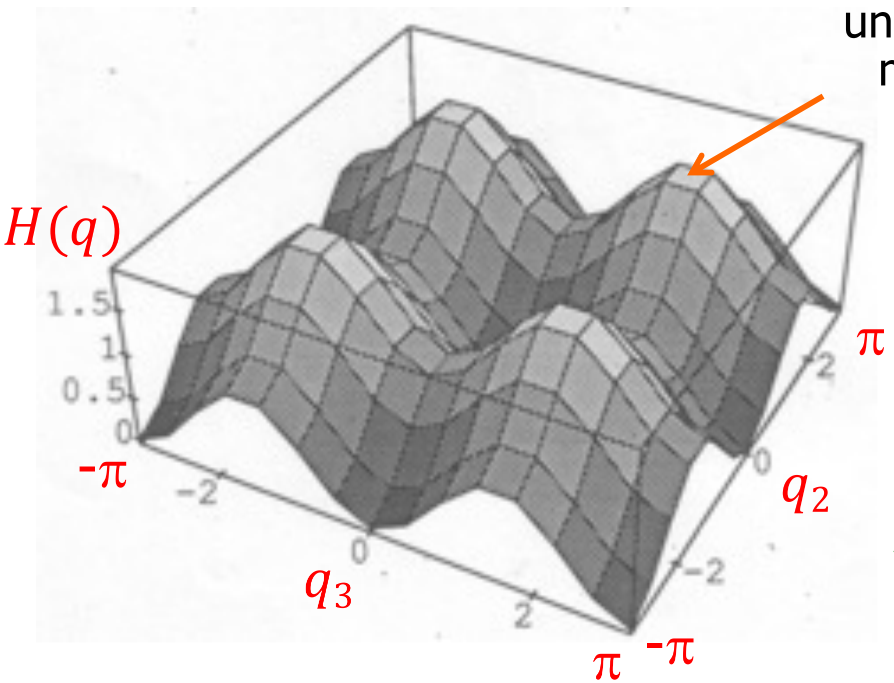
Singularities of planar 3R arm

the robot is redundant for a positioning task in the plane ($M = 2$)

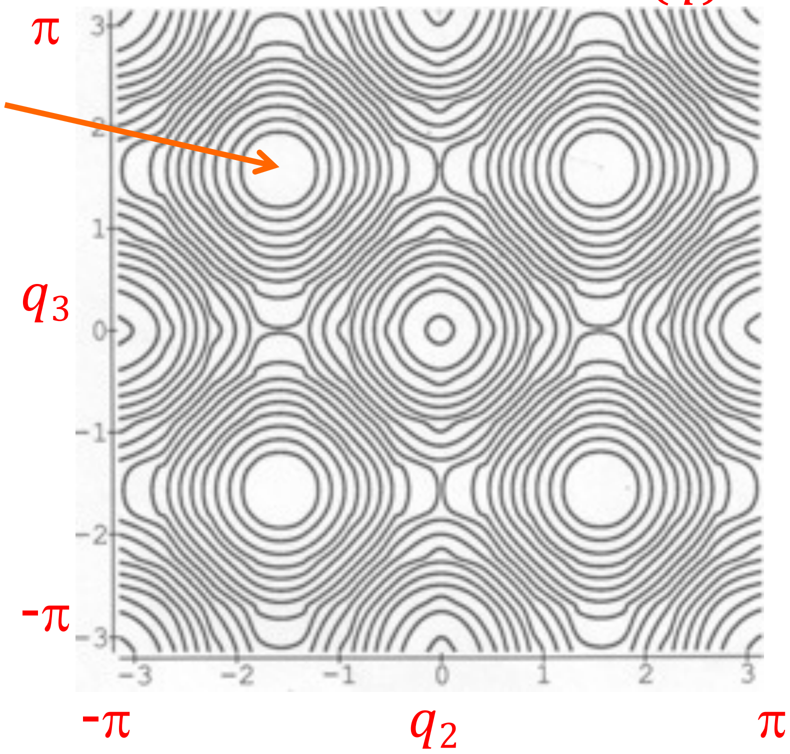
$$H(q) = \sin^2 q_2 + \sin^2 q_3$$

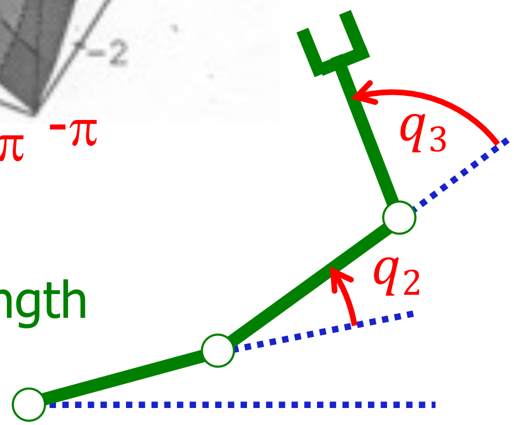
this H is **not** H_{man} but has the same minima

unconstrained maxima of $H(q)$



iso-level curves of $H(q)$





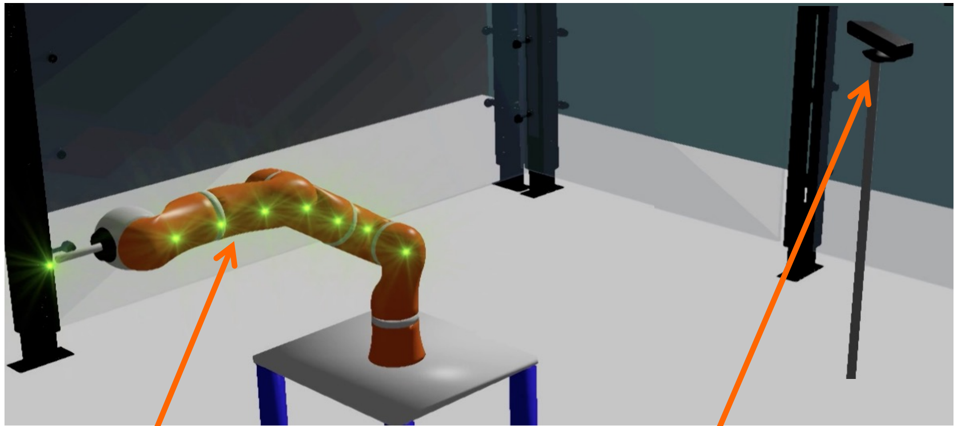
links of **equal** (unit) length

independent from q_1!

Minimum distance computation

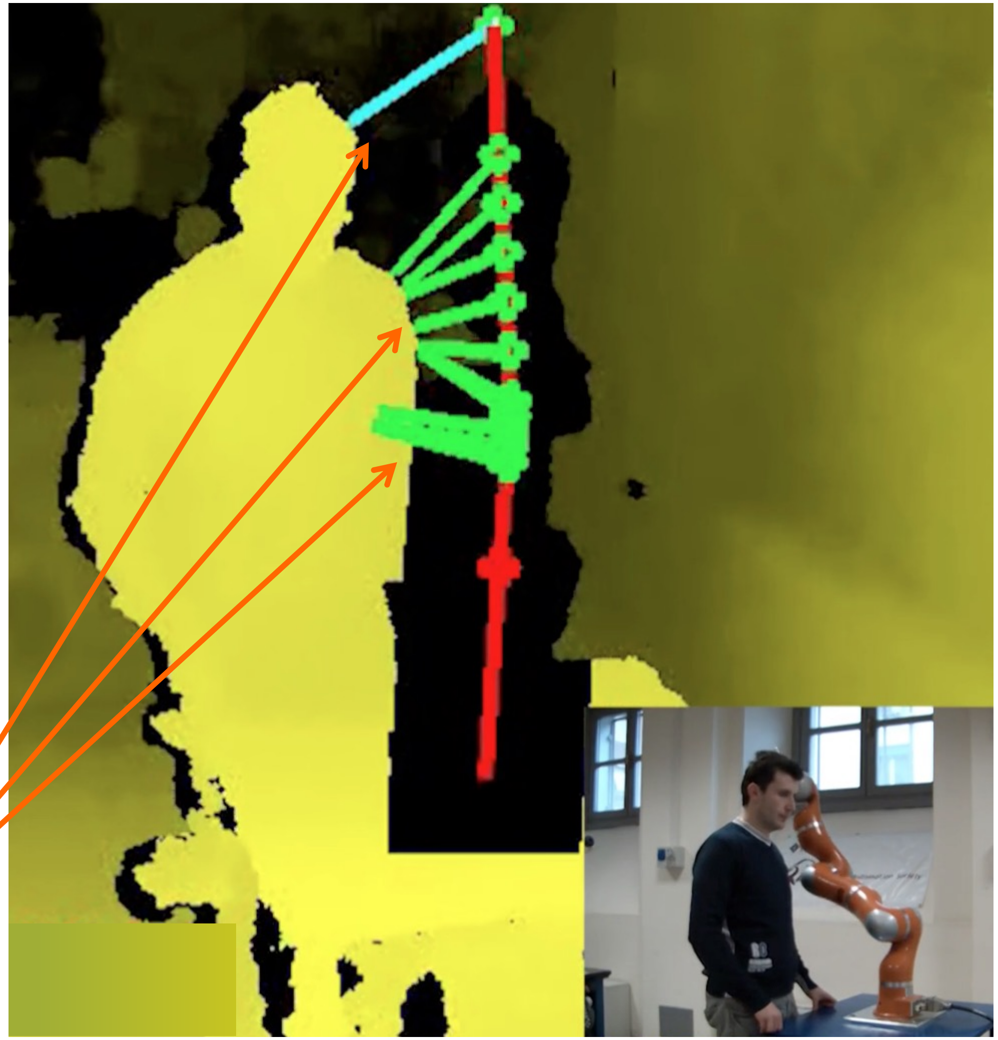
in human-robot interaction

video



LWR4 robot with a finite number of control points $a(q)$ (8, including the E-E)

a Kinect sensor monitors the workspace giving the 3D position of points b on obstacles that are fixed or moving (like humans)



distances in 3D (and then the clearance) are computed in this case as

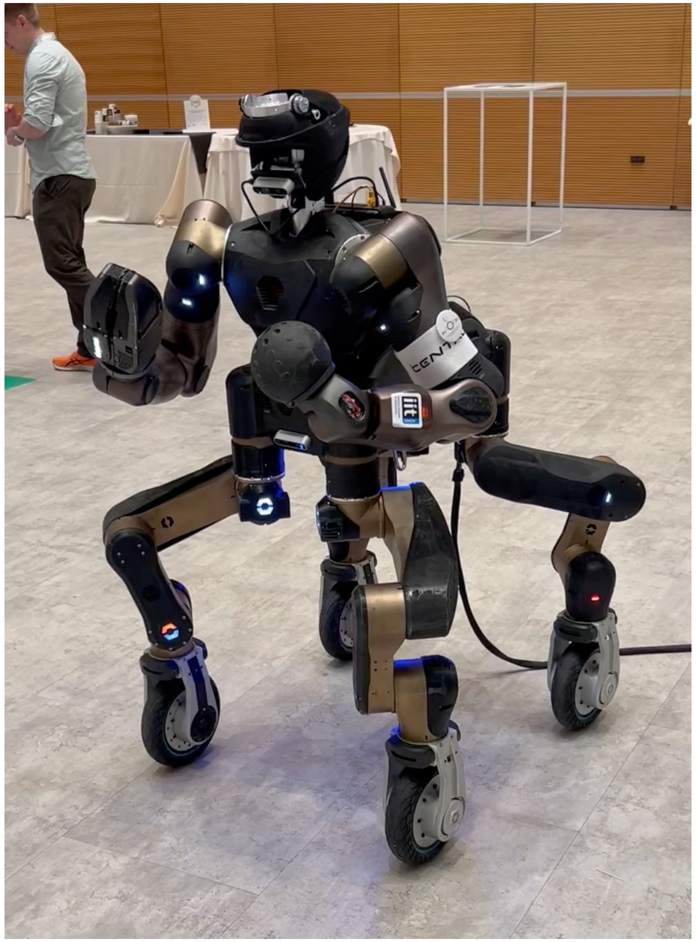
$$\min_{\substack{a \in \{\text{control points}\} \\ b \in \text{human body}}} \|a(q) - b\|^2$$

IIT Centauro robot

Execution of multiple tasks - Self-motions

https://spectrum.ieee.org/centauro-a-new-disaster-response-robot-from-iit

video



1.5 m high

93 kg

each arm
10.5 kg
(with 11 kg
payload!)

2.5 h
autonomy

cameras

RGBD sensors

laser Lidar
scanner

video

@European Robotics Forum (ERF) in Rimini, March 2024

Comments on null-space methods

- the projection matrix $(I - J^{\#}J)$ has dimension $N{\times}N$, but only rank $N - M$ (if J is full rank M), with some waste of information
- actual (efficient) evaluation of the solution

$$\boxed{\dot{q}} = J^{\#}\dot{r} + \left(I - J^{\#}J\right)\dot{q}_0 = \boxed{\dot{q}_0 + J^{\#}(\dot{r} - J\dot{q}_0)}$$

  but the pseudoinverse is needed anyway, and this is computationally intensive (SVD in the general case)
- in principle, the additional complexity of a redundancy resolution method should depend only on the redundancy degree $N - M$
- a constrained optimization method is available, which is known to be more efficient than the projected gradient (PG) —at least when the Jacobian has full rank …

Decomposition of joint space

- if $\rho\big(J(q)\big) = M$, there exists a decomposition of the set of joints (possibly, after a reordering)

$$q = \begin{pmatrix} q_a \\ q_b \end{pmatrix} \begin{matrix} \} \; M \\ \} \; N - M \end{matrix} \qquad \text{such that } J_a(q) = \overbrace{\frac{\partial f}{\partial q_a}}^{M \times M} \text{ is nonsingular}$$

- from the implicit function theorem, there exists an inverse function g

$$f(q_a, q_b) = r \quad \Rightarrow \quad q_a = g(r, q_b)$$

$$\text{with } \frac{\partial g}{\partial q_b} = -\left(\frac{\partial f}{\partial q_a}\right)^{-1} \frac{\partial f}{\partial q_b} = -J_a^{-1}(q)\, J_b(q)$$

- the $N - M$ variables q_b can be selected independently (e.g., they are used for optimizing an objective function $H(q)$, "reduced" via the use of g to a function of q_b only)
- $q_a = g(r, q_b)$ is then chosen so as to correctly execute the task

Reduced Gradient (RG)

- $H(q) = H(q_a, q_b) = H(g(r, q_b), q_b) = H'(q_b)$, with r at current value
- the Reduced Gradient (w.r.t. q_b only, but still keeping the effects of this choice into account) is

$$\nabla_{q_b} H' = \begin{bmatrix} -(J_a^{-1} J_b)^T & I_{N-M} \end{bmatrix} \nabla_q H$$

$$(\neq \nabla_{q_b} H \text{ only!!})$$

$$\boxed{\nabla_{q_b} H' = 0}$$

is a "compact" (i.e., $N - M$ dimensional) necessary condition of constrained optimality

- algorithm

$$\dot{q}_b = \nabla_{q_b} H'$$

step in the gradient direction of the reduced $(N - M)$-dim space

$$J_a \dot{q}_a + J_b \dot{q}_b = \dot{r}$$

satisfaction of the M-dim task constraints

$$\dot{q}_a = J_a^{-1}(\dot{r} - J_b \dot{q}_b)$$

Comparison between PG and RG

- Projected Gradient (PG)

$$\dot{q} = J^{\#}\dot{r} + \left(I - J^{\#}J\right)\nabla_q H$$

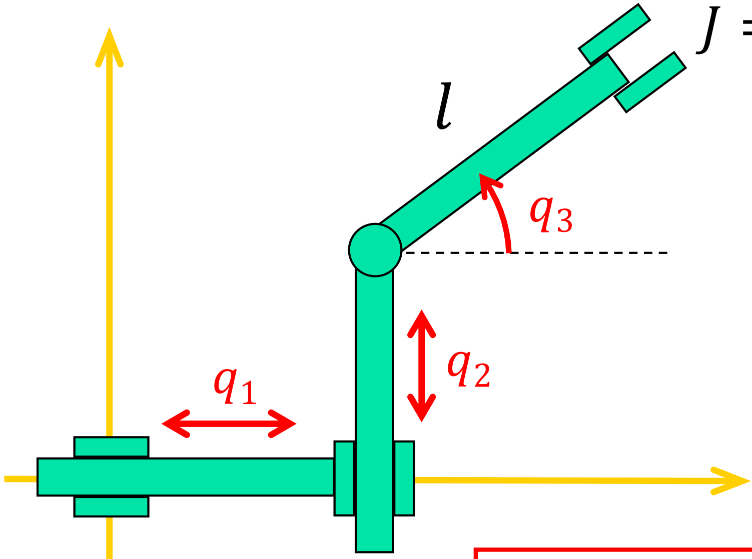
- Reduced Gradient (RG)

$$\dot{q} = \begin{pmatrix} \dot{q}_a \\ \dot{q}_b \end{pmatrix} = \begin{pmatrix} J_a^{-1} \\ 0 \end{pmatrix} \dot{r} + \begin{pmatrix} -J_a^{-1}J_b \\ I \end{pmatrix} (-(J_a^{-1}J_b)^T \quad I)\nabla_q H$$

- RG is analytically simpler and numerically faster than PG, but requires the search for a non-singular minor (J_a) of the robot Jacobian
- if r = cost & $N - M = 1$ ⇒ same (unique) direction for $\dot{q}$, but RG has automatically a larger optimization step size
- else ⇒ RG and PG methods provide always different evolutions

Analytic comparison

PPR robot



$$J = \begin{pmatrix} 1 & 0 & -ls_3 \\ 0 & 1 & lc_3 \end{pmatrix} = (\, J_a \mid J_b \,) \quad q_a = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \quad q_b = q_3$$

RG: $\dot{q} = \begin{pmatrix} J_a^{-1} \\ 0 \end{pmatrix} \dot{r} + \begin{pmatrix} -J_a^{-1}J_b \\ I \end{pmatrix} \left(-(J_a^{-1}J_b)^T \quad I\right) \nabla_q H$

$$\dot{q} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \dot{r} + \begin{pmatrix} ls_3 \\ -lc_3 \\ 1 \end{pmatrix} \left(ls_3 \quad -lc_3 \quad 1\right) \nabla_q H$$

PG: $\dot{q} = J^{\#}\dot{r} + \left(I - J^{\#}J\right)\nabla_q H$

$$J^{\#} = \frac{1}{1+l^2} \begin{pmatrix} 1 + l^2 c_3^2 & l^2 s_3 c_3 \\ l^2 s_3 c_3 & 1 + l^2 s_3^2 \\ -ls_3 & lc_3 \end{pmatrix} \qquad I - J^{\#}J = \frac{1}{1+l^2} \begin{pmatrix} l^2 s_3^2 & l^2 s_3 c_3 & ls_3 \\ l^2 s_3 c_3 & l^2 c_3^2 & -lc_3 \\ ls_3 & -lc_3 & 1 \end{pmatrix}$$

always < 1!!

Joint range limits

$$q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \theta = T\theta$$

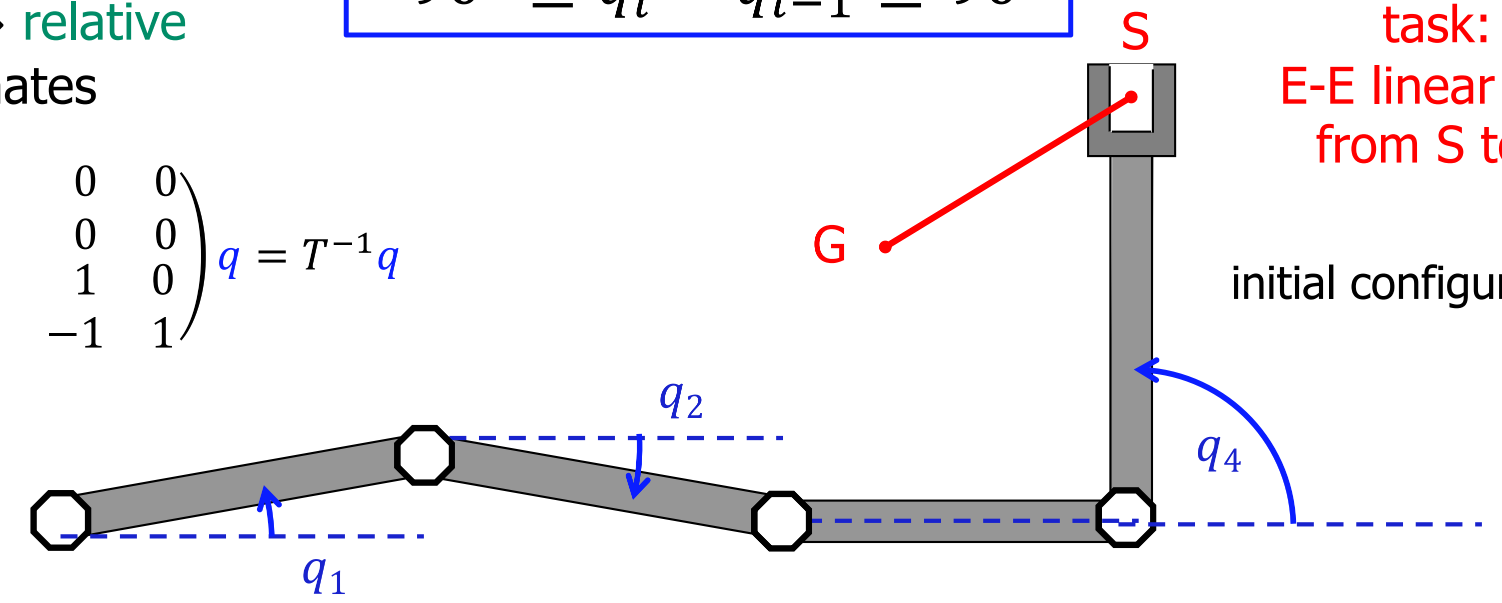
$$\boxed{-90° \leq \theta_i \leq 90°}$$

$$\Updownarrow$$

$$\boxed{-90° \leq q_i - q_{i-1} \leq 90°}$$

absolute ⇔ relative
coordinates

$$\theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} q = T^{-1} q$$



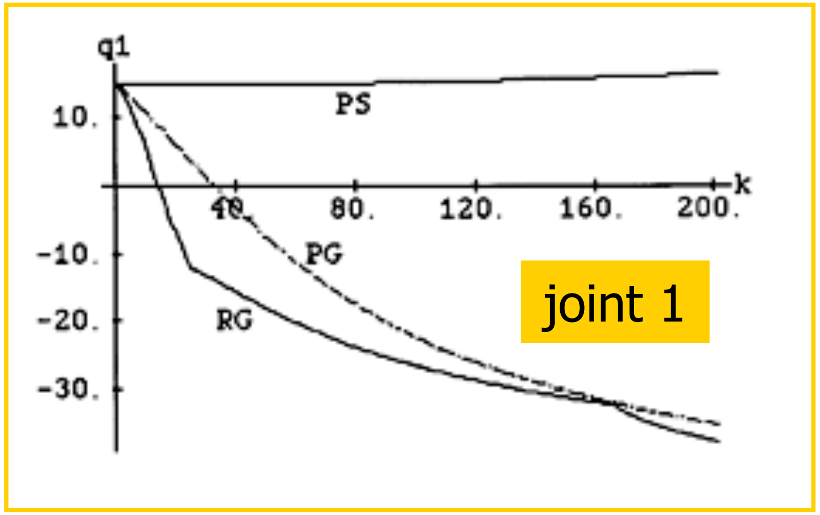
task:
E-E linear path
from S to G

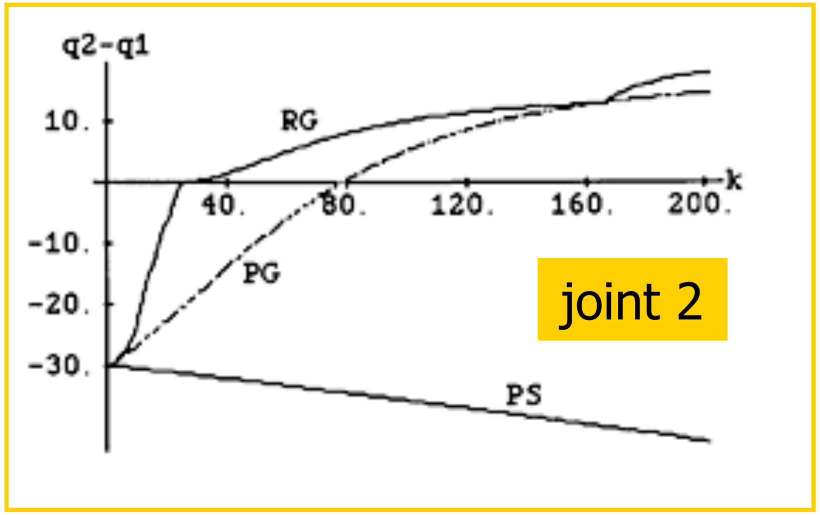
initial configuration

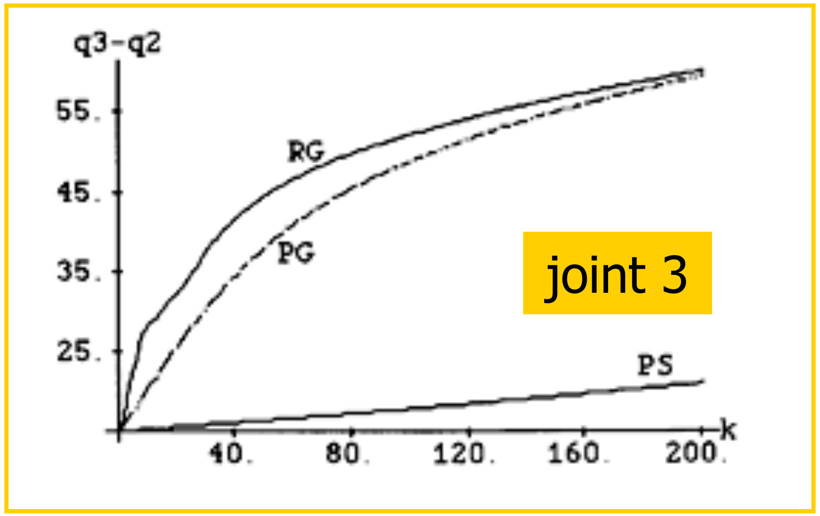
numerical comparison among pseudoinverse (PS), projected gradient (PG), and reduced gradient (RG) methods

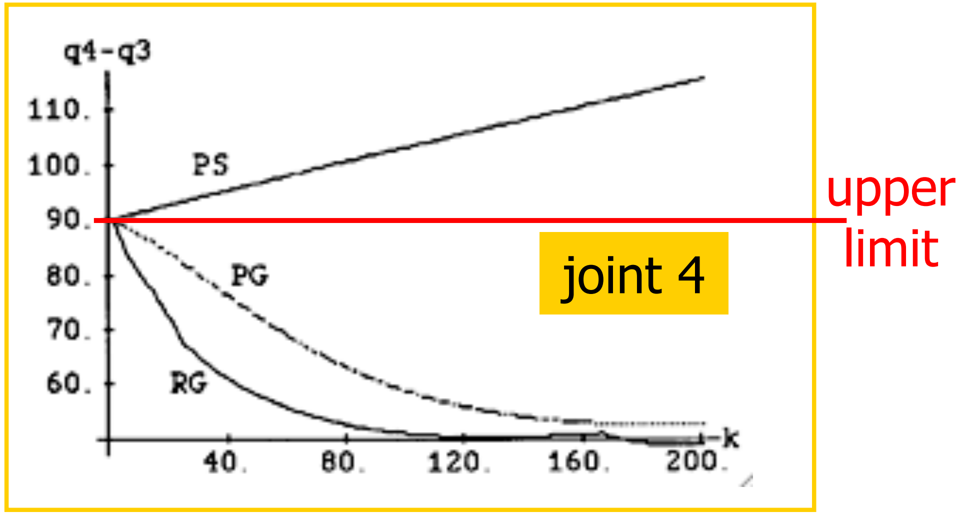
Numerical results

minimizing distance from mid joint range









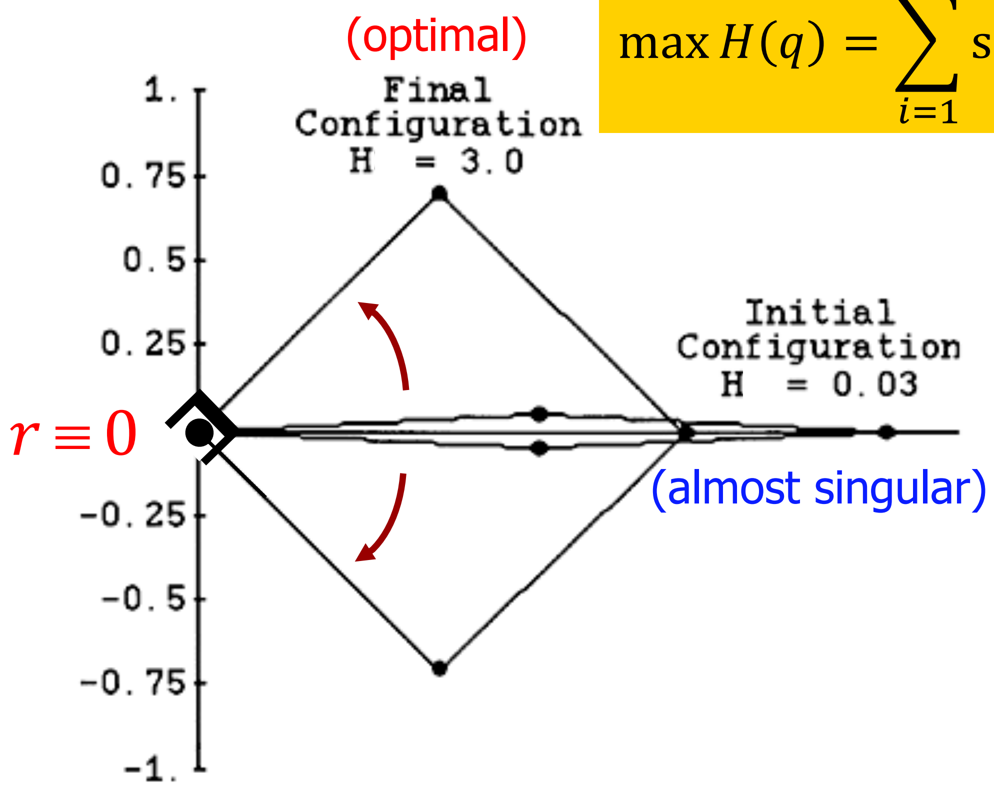
steps of numerical simulation

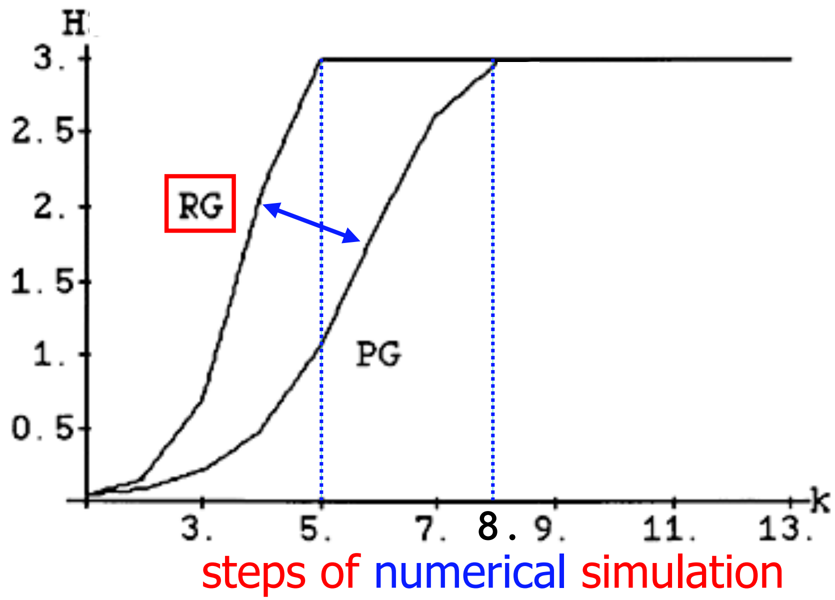
Numerical results

self-motion for escaping singularities

$$\max H(q) = \sum_{i=1}^{3} \sin^2(q_{i+1} - q_i)$$

this function is NOT the manipulability index, but has the same minima (= 0)





RG is faster than PG
(keeping the same accuracy on r)

3 Task augmentation methods

- an auxiliary task is added (task augmentation)

$$S \updownarrow \quad \boxed{f_y(q) = y} \qquad S \leq N - M$$

corresponding to some desirable feature for the solution

$$r_A = \begin{pmatrix} r \\ y \end{pmatrix} = \begin{pmatrix} f(q) \\ f_y(q) \end{pmatrix} \Rightarrow \dot{r}_A = \begin{pmatrix} J(q) \\ J_y(q) \end{pmatrix} \dot{q} = J_A(q)\dot{q} \qquad \underbrace{\boxed{J_A}}_{N} \Big\} M + S$$

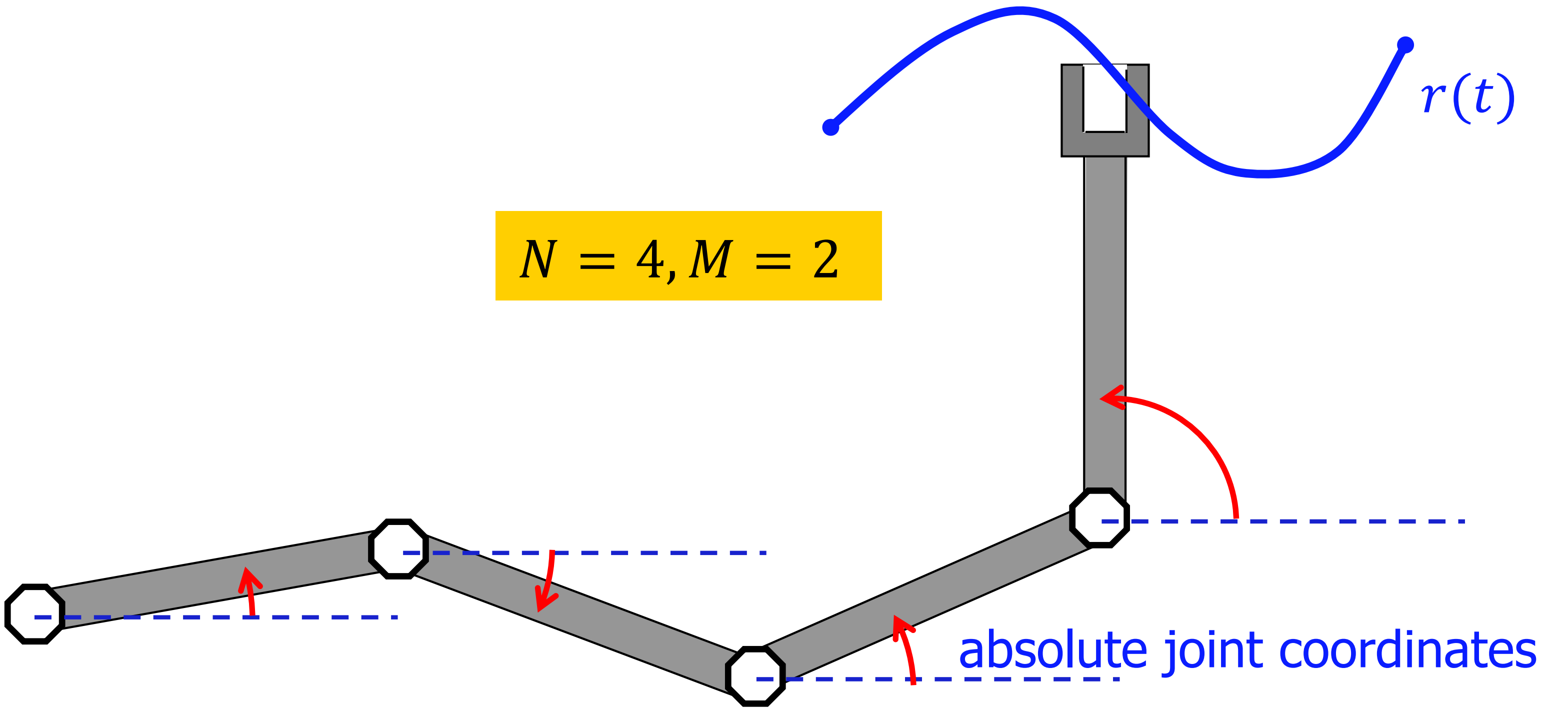
- a solution is chosen still in the form of a generalized inverse

$$\boxed{\dot{q} = K_A(q)\dot{r}_A}$$

or by adding a term in the null space of the augmented Jacobian matrix J_A

Augmented task

example



$$f_y(q) = q_4 = \pi/2 \qquad (S = 1)$$

last link is to be held vertical…

Augmenting the task …

- advantage: better shaping of the inverse kinematic solution

- disadvantage: algorithmic singularities are introduced when

$$\rho(J) = M \qquad \rho(J_y) = S \quad \text{but} \quad \rho(J_A) < M + S$$

to avoid this, it should be always $\boxed{\mathcal{R}(J^T) \cap \mathcal{R}(J_y^T) = \emptyset}$

difficult to be obtained globally!

$\Updownarrow$

rows of J AND rows of J_y
are all together linearly independent

Extended Jacobian ($S = N - M$)

- square J_A: in the absence of algorithmic singularities, we can choose

$$\dot{q} = J_A^{-1}(q)\dot{r}_A$$

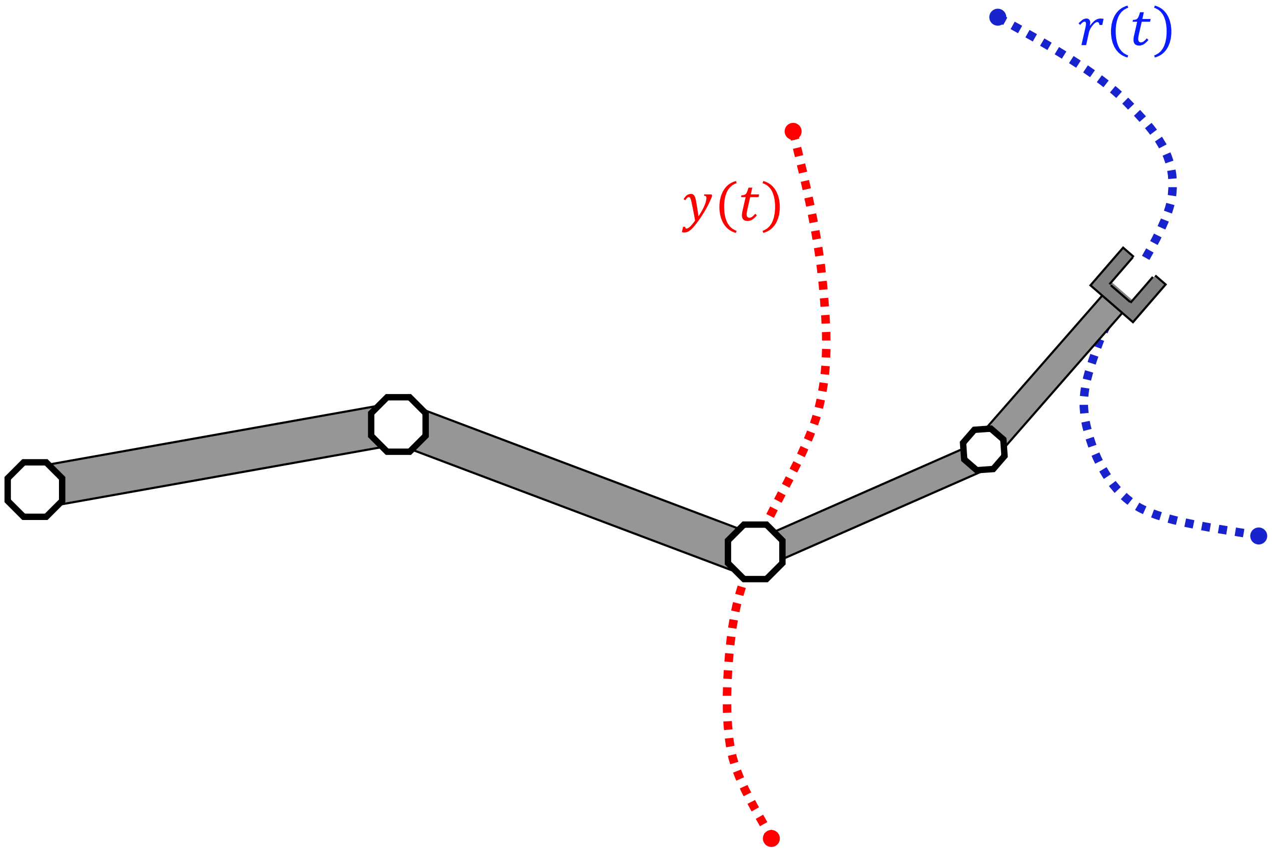
- the scheme is then repeatable
  - provided no singularities are encountered during a complete task cycle*
- when the $N - M$ conditions $f_y(q) = 0$ correspond to necessary (and sufficient) conditions for constrained optimality of a given objective function $H(q)$ (see RG method, slide #39), this scheme guarantees that constrained optimality is locally preserved during task execution
- in the vicinity of algorithmic singularities, for the simultaneous execution of the original task and the auxiliary task(s), one can use the DLS method; however, both tasks will be affected by errors

* there exists an unexpected phenomenon in some 3R manipulators having "generic" kinematics: the robot may sometimes perform a pose change after a full cycle, even if no singularity has been encountered during motion (see J. Burdick, *Mech. Mach. Theory*, 30(1), 1995)

Extended Jacobian

example

MACRO-MICRO manipulator



$N = 4, M = 2$

$$\dot{r} = J(q_1, \dots, q_4)\dot{q}$$

$$\dot{y} = J_y(q_1, q_2)\dot{q}$$

$$J_A = \left(\begin{array}{c|c} * & * \\ * & 0 \end{array}\right) \quad 4\times 4$$

Task Priority

if the original (primary) task $\dot{r}_1 = J_1(q)\dot{q}$ has higher priority than the auxiliary (secondary) task $\dot{r}_2 = J_2(q)\dot{q}$

- we first address the task with highest priority

$$\dot{q} = J_1^{\#}\dot{r}_1 + \left(I - J_1^{\#}J_1\right)v_1$$

- and then choose v_1 so as to satisfy, if possible, also the secondary (lower priority) task

$$\dot{r}_2 = J_2\dot{q} = J_2\,J_1^{\#}\dot{r}_1 + J_2\left(I - J_1^{\#}J_1\right)v_1 = J_2\,J_1^{\#}\dot{r}_1 + J_2P_1v_1$$

the general solution for v_1 takes the usual form

$$v_1 = (J_2P_1)^{\#}\left(\dot{r}_2 - J_2\,J_1^{\#}\dot{r}_1\right) + \left(I - (J_2P_1)^{\#}(J_2P_1)\right)v_2$$

v_2 is available for the execution of further tasks of lower (ordered) priorities

Task Priority (continue)

- substituting the expression of v_1 in $\dot{q}$

$$\boxed{\dot{q} = J_1^{\#}\dot{r}_1 + P_1(J_2P_1)^{\#}\left(\dot{r}_2 - J_2J_1^{\#}\dot{r}_1\right) + P_1\left(I - (J_2P_1)^{\#}(J_2P_1)\right)v_2}$$

$P(BP)^{\#} = (BP)^{\#}$
when matrix P is
idempotent and symmetric

$= (J_2P_1)^{\#}$

possibly $= 0$

- main advantage: highest priority task is ideally no longer affected by algorithmic singularities (error is restricted only to secondary task)

task 1: follow —
task 2: keep third link vertical