Robotics 1

Minimal representations of orientation
(Euler and roll-pitch-yaw angles)
Homogeneous transformations

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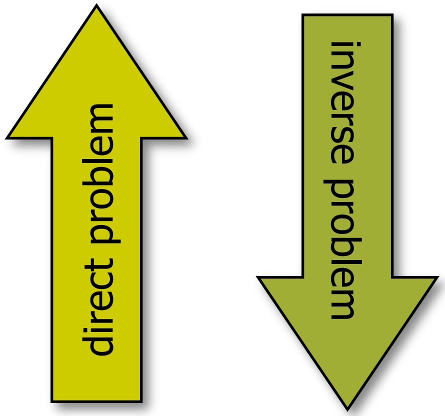
“Minimal” representations

- rotation matrices:



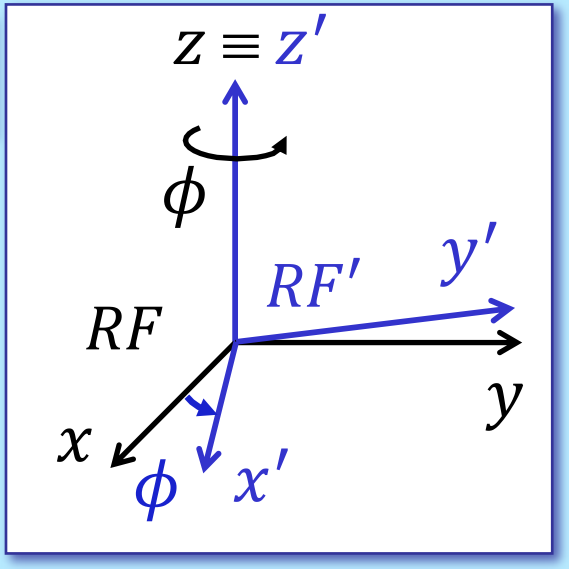
	9 elements
-	3 orthogonality relationships
-	3 unitary relationships
=	3 independent variables

- sequence of 3 rotations w.r.t. independent axes
  - by angles $\alpha_i, i = 1,2,3$, around fixed (a_i) or moving/current (a'_i) axes
    - generically called Roll-Pitch-Yaw (fixed axes) or Euler (moving axes) angles
  - 12 + 12 possible different sequences (e.g., XYX)
    - without contiguous repetitions of axes (e.g., no XXZ nor YZ’Z’)
  - actually, only 12 sequences are different since we shall see that

$$\{(a_1, \alpha_1), (a_2, \alpha_2), (a_3, \alpha_3)\} \equiv \{(a'_3, \alpha_3), (a'_2, \alpha_2), (a'_1, \alpha_1)\}$$

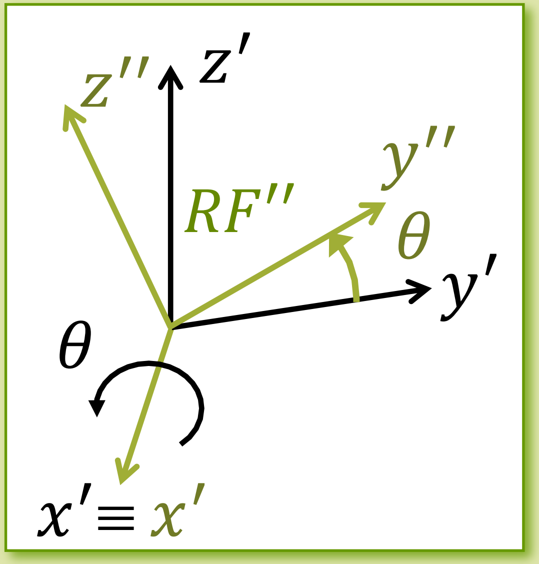
$ZX'Z''$ Euler angles

1



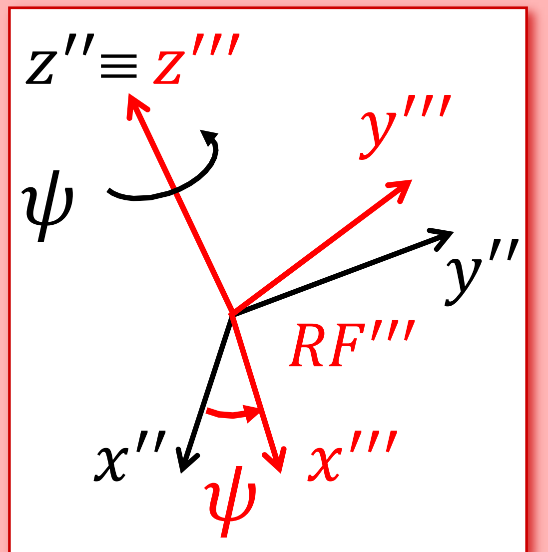
$$R_Z(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2



$$R_{X'}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

3



$$R_{Z''}(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$ZX'Z''$ Euler angles

- direct problem: given ϕ, θ, ψ, find R

$$R_{ZX'Z''}(\phi,\theta,\psi) = R_Z(\phi)R_{X'}(\theta)R_{Z''}(\psi)$$

order of definition in concatenation

$$= \begin{bmatrix} c\phi c\psi - s\phi c\theta s\psi & -c\phi s\psi - s\phi c\theta c\psi & s\phi s\theta \\ s\phi c\psi + c\phi c\theta s\psi & -s\phi s\psi + c\phi c\theta c\psi & -c\phi s\theta \\ s\theta s\psi & s\theta c\psi & c\theta \end{bmatrix}$$

- given a vector $v''' = (x''', y''', z''')$ expressed in RF''', its expression in the coordinates of RF is

$$v = R_{ZX'Z''}(\phi,\theta,\psi)v'''$$

- the orientation of RF''' is the same that would be obtained with the sequence of rotations

ψ around z, θ around x (fixed), ϕ around z (fixed)

$ZX'Z''$ Euler angles

- inverse problem: given $R = \{r_{ij}\}$, find ϕ, θ, ψ

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c\phi c\psi - s\phi c\theta s\psi & -c\phi s\psi - s\phi c\theta c\psi & s\phi s\theta \\ s\phi c\psi + c\phi c\theta s\psi & -s\phi s\psi + c\phi c\theta c\psi & -c\phi s\theta \\ s\theta s\psi & s\theta c\psi & c\theta \end{bmatrix}$$

- $r_{13}^2 + r_{23}^2 = s^2\theta, \; r_{33} = c\theta \;\Rightarrow$ $\theta = \text{atan2}\left\{\pm\sqrt{r_{13}^2 + r_{23}^2},\; r_{33}\right\}$

  two values differing just for the sign

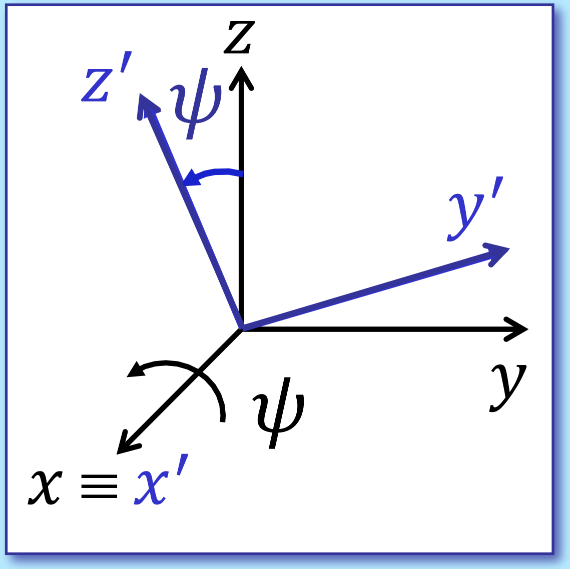
- if $r_{13}^2 + r_{23}^2 \neq 0$ (i.e., $s\theta \neq 0$)

  $r_{31}/s\theta = s\psi, \; r_{32}/s\theta = c\psi \Rightarrow$ $\psi = \text{atan2}\{r_{31}/s\theta, r_{32}/s\theta\}$

- similarly... $\phi = \text{atan2}\{r_{13}/s\theta, -r_{23}/s\theta\}$
- there is always a pair of solutions in the regular case
- there are always singularities (here $\theta = 0$ or $\pm\pi$) $\Rightarrow$ only the sum $\phi + \psi$ or the difference $\phi - \psi$ can be determined

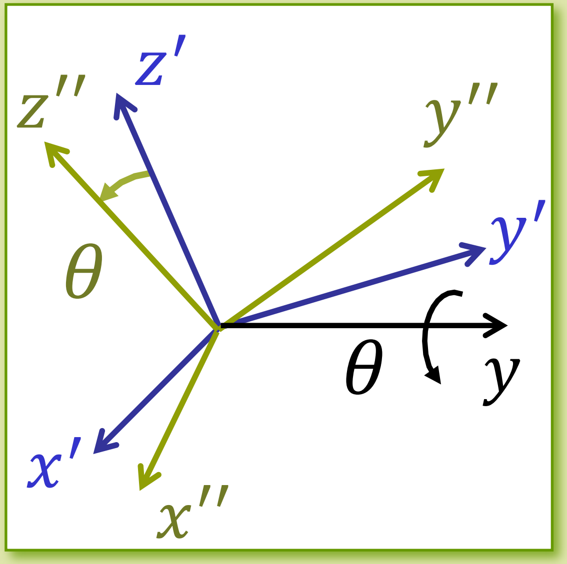
Roll-Pitch-Yaw angles (fixed XYZ)

1 **ROLL**



$$R_X(\psi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix}$$

PITCH **2**

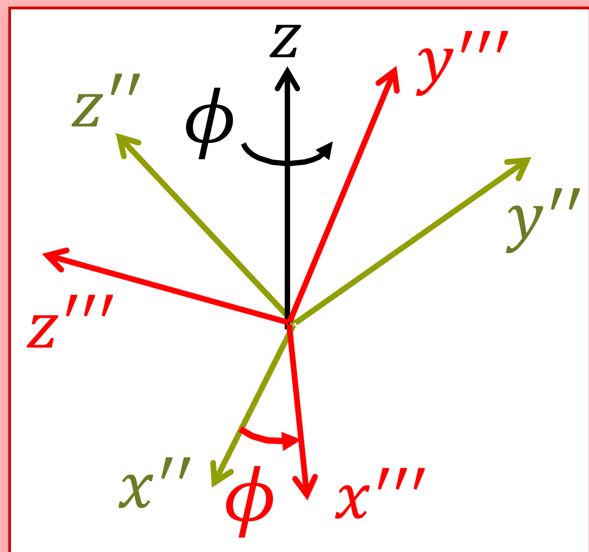


$$C_1 R_Y(\theta) C_1^T$$

with $R_Y(\theta) =$

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

3 **YAW**



$$C_2 R_Z(\phi) C_2^T$$

with $R_Z(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Roll-Pitch-Yaw angles (fixed XYZ)

- direct problem: given ψ, θ, ϕ, find R

$$R_{RPY}(\psi,\theta,\phi) = R_Z(\phi)R_Y(\theta)R_X(\psi)$$ ⇐ note the order of products!

order of definition →

$$= \begin{bmatrix} c\phi c\theta & c\phi s\theta s\psi - s\phi c\psi & c\phi s\theta c\psi + s\phi s\psi \\ s\phi c\theta & s\phi s\theta s\psi + c\phi c\psi & s\phi s\theta c\psi - c\phi s\psi \\ -s\theta & c\theta s\psi & c\theta c\psi \end{bmatrix}$$

- inverse problem: given $R = \{r_{ij}\}$, find ψ, θ, ϕ
- $r_{32}^2 + r_{33}^2 = c^2\theta,\ r_{31} = -s\theta \Rightarrow$ $$\theta = \text{atan2}\left\{-r_{31}, \pm\sqrt{r_{32}^2 + r_{33}^2}\right\}$$ for $r_{31} < 0$, two symmetric values w.r.t. $\pi/2$
- if $r_{32}^2 + r_{33}^2 \neq 0$ (i.e., $c\theta \neq 0$)

  $r_{32}/c\theta = s\psi,\ r_{33}/c\theta = c\psi \Rightarrow$ $$\psi = \text{atan2}\{r_{32}/c\theta, r_{33}/c\theta\}$$
- similarly ... $$\phi = \text{atan2}\{r_{21}/c\theta, r_{11}/c\theta\}$$
- singularities for $\theta = \pm\pi/2 \Rightarrow$ only $\phi + \psi$ or $\phi - \psi$ are defined

…why this order in the product?

$$R_{RPY}(\psi, \theta, \phi) = R_Z(\phi) R_Y(\theta) R_X(\psi)$$

order of definition →

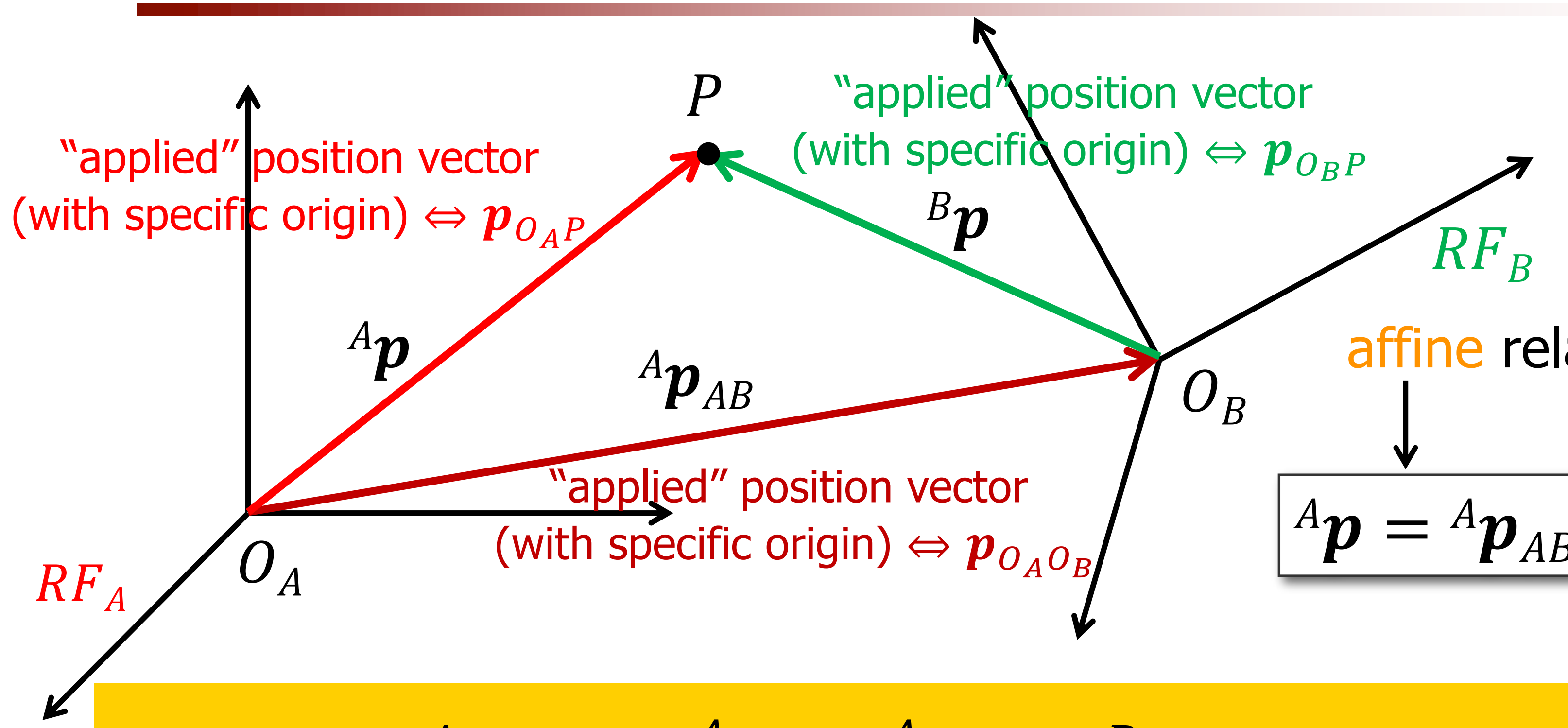
"reverse" order in the product (pre-multiplication…)

- need to refer each rotation in the sequence to one of the original fixed axes
  - use the angle/axis technique for each rotation in the sequence: $C\, R(\alpha)\, C^T$, with C being the rotation matrix reverting the previously made rotations (= "go back" to the original axes)

concatenating three rotations: [ ] [ ] [ ] (post-multiplication…)

$$\begin{aligned} R_{RPY}(\psi, \theta, \phi) = & \; [R_X(\psi)]\, [R_X^T(\psi)\, R_Y(\theta)\, R_X(\psi)] \\ & \; [R_X^T(\psi) R_Y^T(\theta)\, R_Z(\phi)\, R_Y(\theta) R_X(\psi)] \\ = & \; R_Z(\phi) R_Y(\theta) R_X(\psi) \end{aligned}$$

Homogeneous transformations



affine relationship

$${}^A\boldsymbol{p} = {}^A\boldsymbol{p}_{AB} + {}^A R_B \, {}^B\boldsymbol{p}$$

$${}^A\boldsymbol{p}_{\text{hom}} = \begin{bmatrix} {}^A\boldsymbol{p} \\ 1 \end{bmatrix} = \begin{bmatrix} {}^A R_B & {}^A\boldsymbol{p}_{AB} \\ \boldsymbol{0}^T & 1 \end{bmatrix} \begin{bmatrix} {}^B\boldsymbol{p} \\ 1 \end{bmatrix} = {}^A T_B \, {}^B\boldsymbol{p}_{\text{hom}}$$

linear relationship

vector in homogeneous coordinates

4×4 matrix of homogeneous transformation

Use of homogeneous transformation T

- describes the relation between two reference frames (relative pose = position & orientation)
- transforms the representation of a position vector (applied vector starting from the origin of the frame) from one frame to another frame
- it is a roto-translation operator on vectors in the three-dimensional space
- it is always invertible $\left({}^{A}T_{B}\right)^{-1} = {}^{B}T_{A}$
- can be composed, i.e., ${}^{A}T_{B}\,{}^{B}T_{C} = {}^{A}T_{C}$ ← note: it does not commute in general!

Affine vs linear computations

whiteboard…

$$^{1}p = {}^{1}p_{01} + {}^{1}R_0\, {}^{0}p$$

$$^{2}p = {}^{2}p_{12} + {}^{2}R_1\, {}^{1}p = {}^{2}p_{12} + {}^{2}R_1\, {}^{1}p_{01} + {}^{2}R_1\, {}^{1}R_0\, {}^{0}p$$

$$^{3}p = {}^{3}p_{23} + {}^{3}R_2\, {}^{2}p = \cdots = {}^{2}p_{23} + {}^{3}R_2\, {}^{2}p_{12} + {}^{3}R_2\, {}^{2}R_1\, {}^{1}p_{01} + {}^{3}R_2\, {}^{2}R_1\, {}^{1}R_0\, {}^{0}p$$

$$^{4}p = {}^{4}p_{34} + {}^{4}R_3\, {}^{3}p = \cdots$$

heavy on notation (and not only!)

$$^{1}T_0 = \begin{bmatrix} {}^{1}R_0 & {}^{1}p_{01} \\ 0^T & 1 \end{bmatrix} \quad \Rightarrow \quad {}^{1}p_{hom} = {}^{1}T_0\, {}^{0}p_{hom}$$

$$^{2}T_1 = \begin{bmatrix} {}^{2}R_1 & {}^{2}p_{12} \\ 0^T & 1 \end{bmatrix} \quad \Rightarrow \quad {}^{2}p_{hom} = {}^{2}T_1\, {}^{1}T_0\, {}^{0}p_{hom} = {}^{2}T_0\, {}^{0}p_{hom}$$

$$^{3}T_2 = \begin{bmatrix} {}^{3}R_1 & {}^{3}p_{23} \\ 0^T & 1 \end{bmatrix} \quad \Rightarrow \quad {}^{3}p_{hom} = {}^{3}T_2\, {}^{2}T_1\, {}^{1}T_0\, {}^{0}p_{hom} = {}^{3}T_0\, {}^{0}p_{hom}$$

$$^{4}T_3 = \begin{bmatrix} {}^{4}R_3 & {}^{4}p_{34} \\ 0^T & 1 \end{bmatrix} \quad \Rightarrow \quad {}^{4}p_{hom} = {}^{4}T_3\, {}^{3}T_2\, {}^{2}T_1\, {}^{1}T_0\, {}^{0}p_{hom} = {}^{4}T_0\, {}^{0}p_{hom}$$

Inverse of a homogeneous transformation

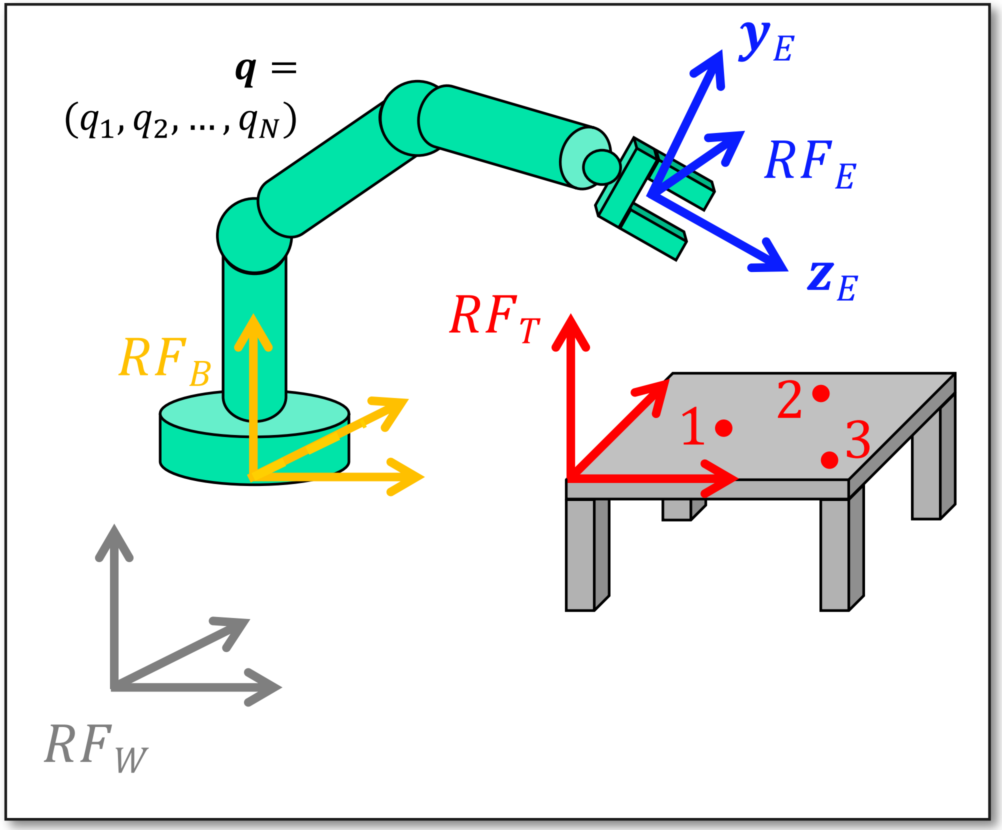
exchange $A \rightleftarrows B$ ... with the original vectors/matrices ...

$$ {}^A\mathbf{p} = {}^A\mathbf{p}_{AB} + {}^AR_B\,{}^B\mathbf{p} \qquad {}^B\mathbf{p} = {}^B\mathbf{p}_{BA} + {}^BR_A\,{}^A\mathbf{p} = -\,{}^AR_B^T\,{}^A\mathbf{p}_{AB} + {}^AR_B^T\,{}^A\mathbf{p} $$

$$ \left[\begin{array}{c|c} {}^AR_B & {}^A\mathbf{p}_{AB} \\ \hline 0\;0\;0 & 1 \end{array}\right] \qquad \left[\begin{array}{c|c} {}^BR_A & {}^B\mathbf{p}_{BA} \\ \hline 0\;0\;0 & 1 \end{array}\right] = \left[\begin{array}{c|c} {}^AR_B^T & -\,{}^AR_B^T\,{}^A\mathbf{p}_{AB} \\ \hline 0\;\;0\;\;0 & 1 \end{array}\right] $$

$$ {}^AT_B \qquad {}^BT_A \qquad ({}^AT_B)^{-1} $$

Defining a robotic task



absolute definition of task

task definition relative to the robot end-effector

$$^{W}T_{T} = {}^{W}T_{B}\, {}^{B}T_{E}\, {}^{E}T_{T}$$

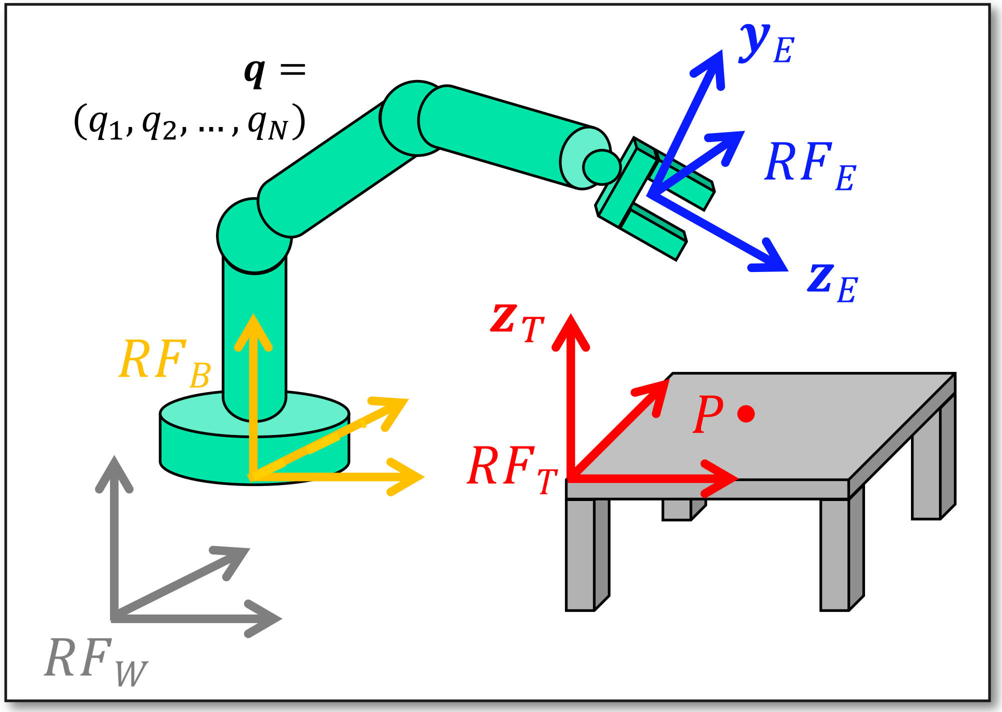
known, once the robot is placed

direct kinematics of the robot arm (function of $\boldsymbol{q}$)

solve for $\boldsymbol{q}$ (inverse kinematics)

$$^{B}T_{E}(\boldsymbol{q}) = {}^{W}T_{B}^{-1}\, {}^{W}T_{T}\, {}^{E}T_{T}^{-1} = \text{constant}$$

Example of task definition



- the robot carries a depth camera (e.g., a Kinect) on the end-effector
- the end-effector should go to a pose above the point P on the table, pointing its approach axis $\boldsymbol{z}_E$ downward and being aligned with the table sides

$${}^E R_T = \begin{bmatrix} {}^E\boldsymbol{x}_T & {}^E\boldsymbol{y}_T & {}^E\boldsymbol{z}_T \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

- point P is known in the table frame RF_T

$${}^T\boldsymbol{p} = \begin{bmatrix} p_x \\ p_y \\ 0 \end{bmatrix}$$

- the robot proceeds by centering point P in its camera image until it senses a depth h from the table (in RF_E)

$${}^E\boldsymbol{p} = \begin{bmatrix} 0 \\ 0 \\ h \end{bmatrix}$$

Q: where is the EE frame w.r.t. the table frame?

$${}^T T_E = \begin{bmatrix} {}^T R_E & {}^T\boldsymbol{p}_{TE} \\ 0^T & 1 \end{bmatrix} = {}^E T_T^{-1}$$

with ${}^T R_E = \left({}^E R_T\right)^T = {}^E R_T$

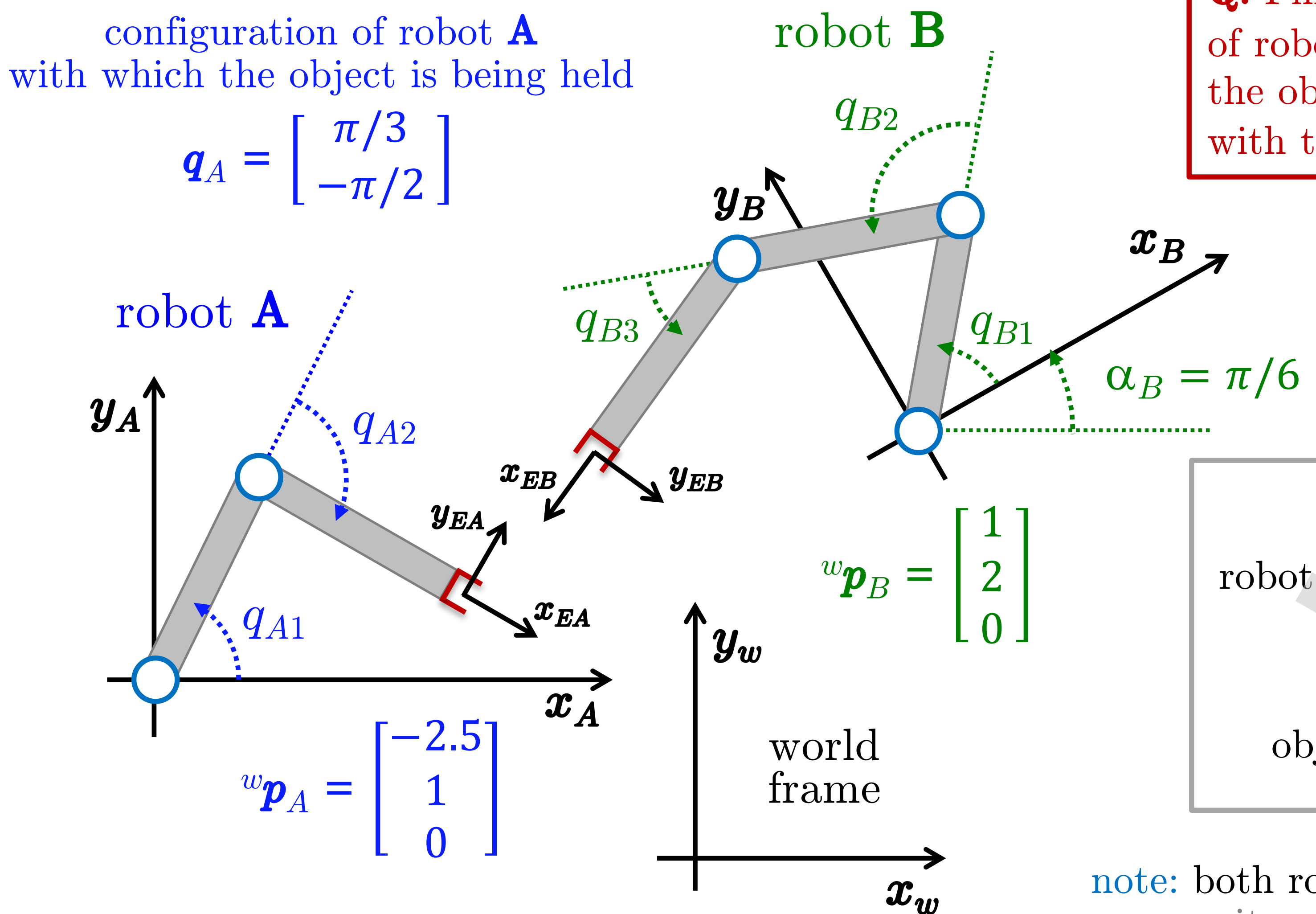
$${}^T\boldsymbol{p}_{TE} = {}^T\boldsymbol{p} - {}^T R_E \, {}^E p = \begin{bmatrix} p_x \\ p_y \\ h \end{bmatrix}$$

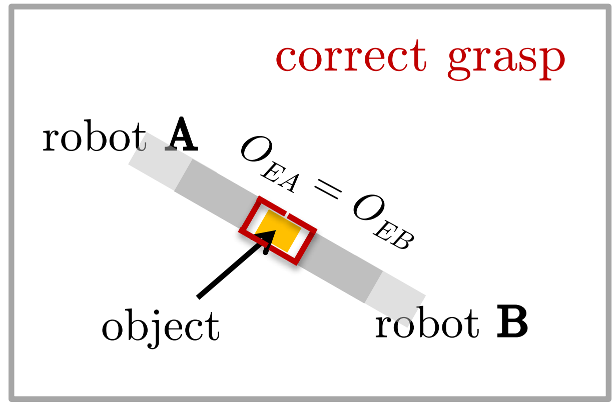
A robotic problem with T matrices

Task: 2R planar robot **A** should hand over an object at a given location to 3R planar robot **B**

Q: Find a configuration $\boldsymbol{q}_B$ of robot **B** so as to grasp the object held by robot **A** with the right orientation

Ex #3, Robotics 1
exam of Sep 11, 2020





note: both robots have unitary link lengths

Solution procedure

$$ {}^{w}\boldsymbol{T}_A = \begin{pmatrix} {}^{w}\boldsymbol{R}_A & {}^{w}\boldsymbol{p}_A \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{I}_{3\times 3} & \begin{matrix} -2.5 \\ 1 \\ 0 \end{matrix} \\ \boldsymbol{0}^T & 1 \end{pmatrix} $$

base frame of robot **A** w.r.t. world

$$ {}^{w}\boldsymbol{T}_B = \begin{pmatrix} {}^{w}\boldsymbol{R}_B & {}^{w}\boldsymbol{p}_B \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \cos\alpha_B & -\sin\alpha_B & 0 & 1 \\ \sin\alpha_B & \cos\alpha_B & 0 & 2 \\ 0 & 0 & 1 & 0 \\ & \boldsymbol{0}^T & & 1 \end{pmatrix} = \begin{pmatrix} 0.8660 & -0.5 & 0 & 1 \\ 0.5 & 0.8660 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ & \boldsymbol{0}^T & & 1 \end{pmatrix} $$

base frame of robot **B** w.r.t. world

$$ {}^{A}\boldsymbol{T}_{EA} = \begin{pmatrix} {}^{A}\boldsymbol{R}_{EA} & {}^{A}\boldsymbol{p}_{EA} \\ \boldsymbol{0}^T & 1 \end{pmatrix} $$

$$ = \begin{pmatrix} \cos(q_{A1}+q_{A2}) & -\sin(q_{A1}+q_{A2}) & 0 & \cos q_{A1} + \cos(q_{A1}+q_{A2}) \\ \sin(q_{A1}+q_{A2}) & \cos(q_{A1}+q_{A2}) & 0 & \sin q_{A1} + \sin(q_{A1}+q_{A2}) \\ 0 & 0 & 1 & 0 \\ & \boldsymbol{0}^T & & 1 \end{pmatrix} $$

$$ = \begin{pmatrix} 0.8660 & 0.5 & 0 & 1.3660 \\ -0.5 & 0.8660 & 0 & 0.3660 \\ 0 & 0 & 1 & 0 \\ & \boldsymbol{0}^T & & 1 \end{pmatrix} $$

end-effector frame of robot **A** w.r.t. its base frame (uses $\boldsymbol{q}_A$) = direct kinematics of robot **A**!

$$ {}^{EA}\boldsymbol{T}_{EB} = \begin{pmatrix} {}^{EA}\boldsymbol{R}_{EB} & {}^{EA}\boldsymbol{p}_{EB} \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ & \boldsymbol{0}^T & & 1 \end{pmatrix} $$

end-effector frame of robot **B** w.r.t. end-effector frame of robot **A** to realize the right grasp for correct handover

Solution procedure

$${}^{w}\boldsymbol{T}_{A}\,{}^{A}\boldsymbol{T}_{EA}\,{}^{EA}\boldsymbol{T}_{EB} = {}^{w}\boldsymbol{T}_{B}\,{}^{B}\boldsymbol{T}_{EB}$$

kinematic equation defining the task

end-effector frame of robot **B** w.r.t. world passing via the given configuration of robot **A**	=	end-effector frame of robot **B** w.r.t. world passing via its base frame

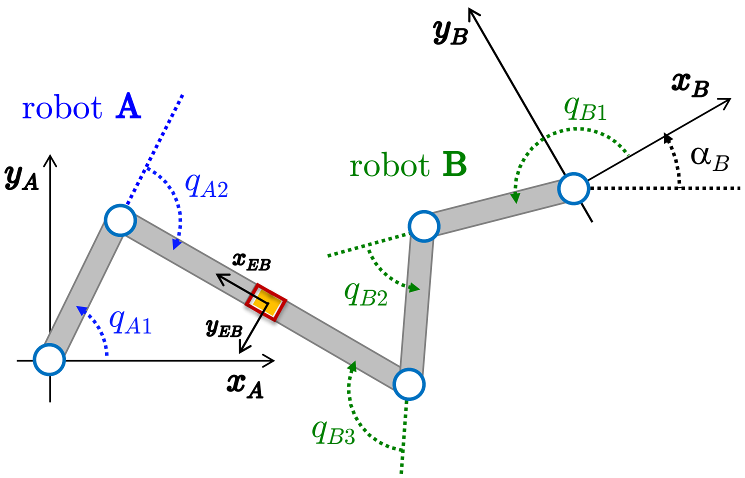
$${}^{B}\boldsymbol{T}_{EB,d} = \begin{pmatrix} {}^{B}\boldsymbol{R}_{EB,d} & {}^{B}\boldsymbol{p}_{EB,d} \\ \mathbf{0}^T & 1 \end{pmatrix} = ({}^{w}\boldsymbol{T}_{B})^{-1}\,{}^{w}\boldsymbol{T}_{A}\,{}^{A}\boldsymbol{T}_{EA}\,{}^{EA}\boldsymbol{T}_{EB}$$

$$= \begin{pmatrix} -0.5 & -0.8660 & 0 & -2.1651 \\ 0.8660 & -0.5 & 0 & 0.5179 \\ 0 & 0 & 1 & 0 \\ & \mathbf{0}^T & & 1 \end{pmatrix} =$$

desired end-effector frame of robot **B** w.r.t. its base = input for the inverse kinematics of robot **B**!

one solution $\boldsymbol{q}_B$ (out of 2) of the inverse kinematics of robot **B**

$$q_B = \begin{bmatrix} q_{B1} \\ q_{B2} \\ q_{B3} \end{bmatrix} = \begin{bmatrix} 2.7939 \\ 1.1076 \\ -1.8071 \end{bmatrix} [\text{rad}] = \begin{bmatrix} 160.08° \\ 63.46° \\ -103.54° \end{bmatrix}$$



Remarks on homogeneous matrices

- the main tool used for computing the direct kinematics of robot manipulators
- relevant in many other applications (in robotics and beyond)
  - in positioning/orienting a vision camera (matrix ${}^{b}T_{c}$ with extrinsic parameters of the camera pose)
  - in computer graphics, for the real-time visualization of 3D solid objects when changing the observation point

$$ {}^{A}T_{B} = \begin{bmatrix} & {}^{A}R_{B} & & {}^{A}p_{AB} \\ \alpha_x & \alpha_y & \alpha_z & \sigma \end{bmatrix} $$

all zero in robotics — coefficients of perspective deformation (α_x, α_y, α_z)

scaling coefficient (σ) — always unitary in robotics