Robotics 2
Midterm Test – April 15, 2026

Exercise 1

Consider a robot with n degrees of freedom executing an m-dimensional task velocity $\dot{\boldsymbol{y}}$, with $m \leq n$, in the vicinity of a singularity of the corresponding task Jacobian $\boldsymbol{J}(\boldsymbol{q})$. Provide the expression of the joint velocity command $\dot{\boldsymbol{q}}$ as solution of a Generalized Damped Least Squares (G-DLS) problem

$$\min_{\dot{\boldsymbol{q}}} H(\dot{\boldsymbol{q}}) = \frac{1}{2}\,\|\dot{\boldsymbol{q}}\|^2_{\boldsymbol{W_q}} + \frac{1}{2}\,\|\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} - \dot{\boldsymbol{y}}\|^2_{\boldsymbol{W_y}}, \tag{1}$$

being $\boldsymbol{W_q}$ and $\boldsymbol{W_y}$ two symmetric and positive definite matrices of dimensions $n \times n$ and $m \times m$, respectively. The solution should return the classical DLS joint velocity when $\boldsymbol{W_q} = \mu^2 \boldsymbol{I}_n$ and $\boldsymbol{W_y} = \boldsymbol{I}_m$. Moreover, like in the DLS case, show that the solution may involve the inversion of an $m \times m$ matrix only. *Hint. Use the 'push-through' result on product of invertible matrices of suitable dimensions:* $(\boldsymbol{I} + \boldsymbol{U}\boldsymbol{V})^{-1}\boldsymbol{U} = \boldsymbol{U}(\boldsymbol{I} + \boldsymbol{V}\boldsymbol{U})^{-1}$.

Exercise 2

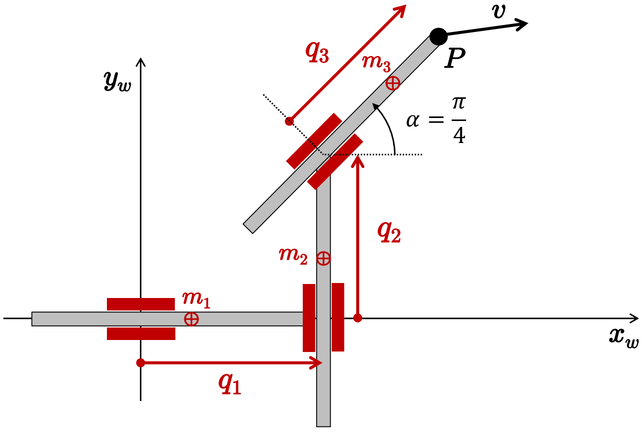

Figure 1: A 3P planar robot

Figure 1 shows a 3P planar robot with the last joint axis skewed by an angle $\alpha = \pi/4$. The robot is controlled by the joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^3$ and the task is specified by a velocity $\boldsymbol{v} \in \mathbb{R}^2$ of the end-effector point P. Determine the joint velocity $\dot{\boldsymbol{q}}_T$ that executes instantaneously the task with minimum kinetic energy T and compare it with the minimum norm velocity solution $\dot{\boldsymbol{q}}_{PS}$. Next, provide the numerical values of these two joint velocity solutions when all robot links have unitary mass and the desired end-effector velocity is first $\boldsymbol{v}_a = (1, 1)$ and then $\boldsymbol{v}_b = (-1, 1)$ [m/s]. Discuss the obtained results.

Exercise 3

An RP robot with a lateral offset at the base moves in a vertical plane as in Fig. 2, where the relevant kinematic and dynamic parameters are also shown. The robot is driven by generalized forces $\boldsymbol{\tau} \in \mathbb{R}^2$ at the joints, while dissipative terms can be neglected.

i) Derive all terms of the dynamic model in the Lagrangian form

$$\boldsymbol{M}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}. \tag{2}$$

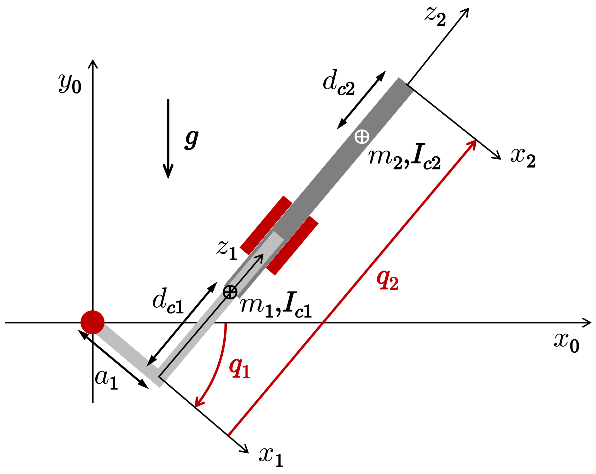

Figure 2: An RP planar robot with its DH frames and relevant kinematic and dynamic parameters

ii) Find two different factorization matrices $\boldsymbol{N}_i(\boldsymbol{q},\dot{\boldsymbol{q}})$, $i=1,2$, giving $\boldsymbol{N}_i(\boldsymbol{q},\dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}} = \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}})$ and such that matrix $\dot{\boldsymbol{M}} - 2\boldsymbol{N}$ is skew-symmetric when $\boldsymbol{N} = \boldsymbol{N}_1$ and it is not when $\boldsymbol{N} = \boldsymbol{N}_2$.

iii) Define a set of dynamic coefficients $\boldsymbol{\rho} \in \mathbb{R}^r$ so that the dynamic model (2) can be rewritten in the linearly parametrized form

$$\boldsymbol{Y}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})\,\boldsymbol{\rho} = \boldsymbol{\tau}. \tag{3}$$

Provide the symbolic expressions of the (typically unknown) dynamic coefficients $\boldsymbol{\rho}$ and of the $2\times r$ regressor matrix $\boldsymbol{Y}$ as a function of the kwown variables $\boldsymbol{q}$, $\dot{\boldsymbol{q}}$, and $\ddot{\boldsymbol{q}}$ only. With the aim of minimizing the number r of such dynamic coefficients, assume that the acceleration of gravity $g_0 = 9.81$ m/s^2 and the kinematic quantity a_1 are also known.

Exercise 4

A 2R planar robot moves on a horizontal plane in the absence of dissipative effects. The robot follows the desired joint trajectory

$$q_{d1}(t) = \frac{\pi}{4}\left(1 - 3t^2 + 2t^3\right) \qquad q_{d2}(t) = -\frac{\pi}{2}\left(2 - \cos 2\pi t\right) \qquad t \in [0,1],$$

thanks to suitable torques $\tau_{d1}(t)$ and $\tau_{d2}(t)$ applied in the same time interval by the motors. What will be the variation of the total robot energy E at the end of the entire motion? In a second motion repetition, two external torques act as disturbances on the desired motion as follows:

$$u_{e1}(t) = -2\,[\text{Nm}] \quad \text{for } t \in [0, 0.5], \qquad u_{e2}(t) = 1\,[\text{Nm}] \quad \text{for } t \in [0.5, 1].$$

Assume that the two motors are controlled to deliver suitable additional torques so that the desired motion remains unaffected. Which is the energy change ΔE due to $\boldsymbol{u}_e = (u_{e1}, u_{e2})$ only? What is the variation of the total robot energy E in this case?

[240 minutes (4 hours); open books]

Solution

April 15, 2026

Exercise 1

Rewrite more explicitly the function $H(\dot{\boldsymbol{q}})$ in (1) to be minimized as

$$H(\dot{\boldsymbol{q}}) = \frac{1}{2}\,\dot{\boldsymbol{q}}^T \boldsymbol{W_q}\,\dot{\boldsymbol{q}} + \frac{1}{2}\left(\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} - \dot{\boldsymbol{y}}\right)^T \boldsymbol{W_y}\left(\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} - \dot{\boldsymbol{y}}\right).$$

Since

$$\frac{\partial H}{\partial \dot{\boldsymbol{q}}} = \dot{\boldsymbol{q}}^T \boldsymbol{W_q} + \left(\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} - \dot{\boldsymbol{y}}\right)^T \boldsymbol{W_y}\,\boldsymbol{J}(\boldsymbol{q}),$$

the stationarity condition is

$$\nabla_{\dot{\boldsymbol{q}}} H = \left(\frac{\partial H}{\partial \dot{\boldsymbol{q}}}\right)^T = \boldsymbol{W_q}\,\dot{\boldsymbol{q}} + \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{W_y}\left(\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} - \dot{\boldsymbol{y}}\right) = \boldsymbol{0}. \tag{4}$$

Being

$$\nabla^2_{\dot{\boldsymbol{q}}} H = \boldsymbol{W_q} + \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{W_y}\,\boldsymbol{J}(\boldsymbol{q}) > 0, \tag{5}$$

condition (4) is necessary and sufficient for a minimum of H. Solving it for $\dot{\boldsymbol{q}}$ gives

$$\dot{\boldsymbol{q}} = \left(\boldsymbol{W_q} + \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{W_y}\,\boldsymbol{J}(\boldsymbol{q})\right)^{-1}\boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{W_y}\,\dot{\boldsymbol{y}}. \tag{6}$$

When $\boldsymbol{W_q} = \mu^2\boldsymbol{I}_n$ and $\boldsymbol{W_y} = \boldsymbol{I}_m$, equation (6) boils down to

$$\dot{\boldsymbol{q}} = \left(\mu^2\boldsymbol{I}_n + \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q})\right)^{-1}\boldsymbol{J}^T(\boldsymbol{q})\dot{\boldsymbol{y}} = \boldsymbol{J}^T(\boldsymbol{q})\left(\mu^2\boldsymbol{I}_m + \boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\right)^{-1}\dot{\boldsymbol{y}},$$

which is the classical DLS solution, written either with the inversion of an $n \times n$ matrix or with the inversion of an $m \times m$ matrix, the latter being clearly more convenient in the redundant case. To obtain the same result for the general case, starting from (6), the following identities are used:

$$\begin{aligned}
\dot{\boldsymbol{q}} &= \left(\boldsymbol{W_q}\big(\boldsymbol{I}_n + \boldsymbol{W_q}^{-1}\boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{W_y}\,\boldsymbol{J}(\boldsymbol{q})\big)\right)^{-1}\boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{W_y}\,\dot{\boldsymbol{y}}\\
&= \left(\boldsymbol{I}_n + \boldsymbol{W_q}^{-1}\boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{W_y}\,\boldsymbol{J}(\boldsymbol{q})\right)^{-1}\boldsymbol{W_q}^{-1}\boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{W_y}\,\dot{\boldsymbol{y}}\\
&= \boldsymbol{W_q}^{-1}\boldsymbol{J}^T(\boldsymbol{q})\left(\boldsymbol{I}_m + \boldsymbol{W_y}\,\boldsymbol{J}(\boldsymbol{q})\boldsymbol{W_q}^{-1}\boldsymbol{J}^T(\boldsymbol{q})\right)^{-1}\boldsymbol{W_y}\,\dot{\boldsymbol{y}} \quad \text{... or also ...}\\
&= \boldsymbol{W_q}^{-1}\boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{W_y}\left(\boldsymbol{I}_m + \boldsymbol{J}(\boldsymbol{q})\boldsymbol{W_q}^{-1}\boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{W_y}\right)^{-1}\dot{\boldsymbol{y}} \quad \text{... as well as ...}\\
&= \boldsymbol{W_q}^{-1}\boldsymbol{J}^T(\boldsymbol{q})\left(\boldsymbol{W_y}^{-1} + \boldsymbol{J}(\boldsymbol{q})\boldsymbol{W_q}^{-1}\boldsymbol{J}^T(\boldsymbol{q})\right)^{-1}\dot{\boldsymbol{y}}
\end{aligned}$$

where the mentioned 'push-through' identity has been used twice, first with $\boldsymbol{U} = \boldsymbol{W_q}^{-1}\boldsymbol{J}^T$ and $\boldsymbol{V} = \boldsymbol{W_y}\,\boldsymbol{J}$ and then with $\boldsymbol{U} = \boldsymbol{W_y}$ and $\boldsymbol{V} = \boldsymbol{J}\boldsymbol{W_q}^{-1}\boldsymbol{J}^T$. In the last passage, we set $\boldsymbol{I}_m = \boldsymbol{W_y}^{-1}\boldsymbol{W_y}$, factored $\boldsymbol{W_y}$ on the right, and used the inverse of a product of matrices.

Exercise 2

We need the task Jacobian $\boldsymbol{J}$ and the inertia matrix $\boldsymbol{M}$ of this robot. Since the position of point P is

$$\boldsymbol{p} = \begin{pmatrix} q_1 + q_3\cos\alpha \\ q_2 + q_3\sin\alpha \end{pmatrix} = \begin{pmatrix} q_1 + \frac{\sqrt{2}}{2}\,q_3 \\ q_2 + \frac{\sqrt{2}}{2}\,q_3 \end{pmatrix} = \boldsymbol{k}(\boldsymbol{q}),$$

the Jacobian is constant and equal to

$$\boldsymbol{J}=\frac{\partial \boldsymbol{k}}{\partial \boldsymbol{q}}=\begin{pmatrix} 1 & 0 & \cos\alpha \\ 0 & 1 & \sin\alpha \end{pmatrix}=\begin{pmatrix} 1 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & \frac{\sqrt{2}}{2} \end{pmatrix}. \tag{7}$$

Next, we compute the kinetic energy T of the robot. One has

$$T_1=\frac{1}{2}m_1\dot{q}_1^2 \qquad\qquad T_2=\frac{1}{2}m_2\left(\dot{q}_1^2+\dot{q}_2^2\right)$$

and

$$\boldsymbol{v}_{c3}=\begin{pmatrix} \dot{q}_1+\frac{\sqrt{2}}{2}\dot{q}_3 \\ \dot{q}_2+\frac{\sqrt{2}}{2}\dot{q}_3 \end{pmatrix} \qquad \Rightarrow \qquad T_3=\frac{1}{2}m_3\left(\dot{q}_1^2+\dot{q}_2^2+\dot{q}_3^2+\sqrt{2}\,\dot{q}_3\left(\dot{q}_1+\dot{q}_2\right)\right).$$

As a result

$$T=T_1+T_2+T_3=\frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{M}\dot{\boldsymbol{q}},$$

where the robot inertia matrix is constant and given by

$$\boldsymbol{M}=\begin{pmatrix} m_1+m_2+m_3 & 0 & \frac{\sqrt{2}}{2}m_3 \\ 0 & m_2+m_3 & \frac{\sqrt{2}}{2}m_3 \\ \frac{\sqrt{2}}{2}m_3 & \frac{\sqrt{2}}{2}m_3 & m_3 \end{pmatrix}. \tag{8}$$

The minimum norm joint velocity that realizes the desired end-effector velocity $\boldsymbol{v}$ is given by the pseudoinverse of the Jacobian (7). Since there are no singularities, we have

$$\dot{\boldsymbol{q}}_{PS}=\boldsymbol{J}^{\#}\boldsymbol{v}=\boldsymbol{J}^T\left(\boldsymbol{J}\boldsymbol{J}^T\right)^{-1}\boldsymbol{v}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{pmatrix}^{-1}\boldsymbol{v}=\begin{pmatrix} 0.75 & -0.25 \\ -0.25 & 0.75 \\ \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \end{pmatrix}\boldsymbol{v}.$$

In the two numerical cases, one obtains

$$\dot{\boldsymbol{q}}_{PS,a}=\boldsymbol{J}^{\#}\boldsymbol{v}_a=\begin{pmatrix} 0.5 \\ 0.5 \\ \frac{\sqrt{2}}{2} \end{pmatrix} \qquad \text{and} \qquad \dot{\boldsymbol{q}}_{PS,b}=\boldsymbol{J}^{\#}\boldsymbol{v}_b=\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad [\text{m/s}]. \tag{9}$$

In the first case, $\boldsymbol{v}_a=(1,1)$ has the same (constant) direction of the third link and the pseudoinverse solution equally distributes the velocity effort among the pairs of joints that contribute to the single Cartesian components of the task velocity $\boldsymbol{v}$, i.e., joints 1 and 3 for v_x, and joints 2 and 3 for v_y. Moreover, the contribution of joint 3 is weighted by its orientation. In the second case, $\boldsymbol{v}_b=(-1,1)$ is orthogonal to the third link and no motion contribution can be obtained from joint 3. Thus, the pseudoinverse solution assigns each component of the task velocity $\boldsymbol{v}$ to the corresponding joint (v_x to joint 1 for and v_y to joint 2).

The joint velocity that realizes the desired end-effector velocity $\boldsymbol{v}$ while minimizing the kinetic energy is given by the weighted pseudoinverse of the Jacobian (7), with the robot inertia (8) as weighting matrix:

$$\dot{\boldsymbol{q}}_T=\boldsymbol{J}^{\#}_{\boldsymbol{M}}\boldsymbol{v}=\boldsymbol{M}^{-1}\boldsymbol{J}^T\left(\boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{J}^T\right)^{-1}\boldsymbol{v}. \tag{10}$$

The following general expression of the solution (10) is obtained quite simply when using, e.g., the symbolic toolbox of MATLAB:

$$\dot{\boldsymbol{q}}_T = \frac{1}{m_1 + 2m_2} \begin{pmatrix} m_2 & -m_2 \\ -(m_1+m_2) & m_1+m_2 \\ \sqrt{2}\,(m_1+m_2) & \sqrt{2}\,m_2 \end{pmatrix} \boldsymbol{v}.$$

However, in the lack of symbolic manipulation tools, we will proceed next with the numerical solution. Replacing the unitary masses of the robot links in (8), one has

$$\boldsymbol{M} = \begin{pmatrix} 3 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 2 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 1 \end{pmatrix} \qquad \Rightarrow \qquad \det \boldsymbol{M} = 3.5 = \frac{7}{2}.$$

The inverse of the inertia matrix is

$$\boldsymbol{M}^{-1} = \frac{1}{7} \begin{pmatrix} 3 & 1 & -2\sqrt{2} \\ 1 & 5 & -3\sqrt{2} \\ -2\sqrt{2} & -3\sqrt{2} & 12 \end{pmatrix}.$$

Thus

$$\boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{J}^T = \frac{1}{7}\begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix} \qquad \Rightarrow \qquad \left(\boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{J}^T\right)^{-1} = \frac{1}{3}\begin{pmatrix} 5 & -2 \\ -2 & 5 \end{pmatrix}$$

and finally

$$\dot{\boldsymbol{q}}_T = \boldsymbol{M}^{-1}\boldsymbol{J}^T\left(\boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{J}^T\right)^{-1}\boldsymbol{v} = \frac{1}{3}\begin{pmatrix} 1 & -1 \\ -2 & 2 \\ 2\sqrt{2} & \sqrt{2} \end{pmatrix}\boldsymbol{v}.$$

In the two numerical cases, one obtains

$$\dot{\boldsymbol{q}}_{T,a} = \boldsymbol{J}^{\#}_{\boldsymbol{M}}\,\boldsymbol{v}_a = \begin{pmatrix} 0 \\ 0 \\ \sqrt{2} \end{pmatrix} \qquad \text{and} \qquad \dot{\boldsymbol{q}}_{T,b} = \boldsymbol{J}^{\#}_{\boldsymbol{M}}\,\boldsymbol{v}_b = \frac{1}{3}\begin{pmatrix} -2 \\ 4 \\ -\sqrt{2} \end{pmatrix} \quad [\text{m/s}]. \tag{11}$$

The solution in the first case has a very intuitive explanation. Roughly speaking, minimization of kinetic energy involves the motion of the least possible amount of robot masses. In fact, the task velocity $\boldsymbol{v}_a = (1, 1)$ can be fully realized by moving only the last link and none of the other two. On the other hand, the mass of the third link has to be carried anyway if any of the first two joints would be involved. In the second case, for $\boldsymbol{v}_b = (-1, 1)$, the solution found in (11) is physically motivated by the need of reducing the load on the joints q_1 and q_2, which move respectively three and two link masses, by using the contribution coming from the less loaded joint q_3.

Exercise 3

This exercise can be completed without using any symbolic toolbox or recursive computation method, taking advantage of the simple kinematic structure of this planar RP robot. Nonetheless, note that frames and joint variables in Fig. 2 are defined according to the standard Denavit–Hartenberg (DH) convention. Thus, the moving frames algorithm for computing the robot kinetic energy would be easily applicable. Hereafter, we will follow a more direct derivation in which, to simplify computations, we take advantage of the first DH rotation matrix

$$ {}^0\boldsymbol{R}_1(q_1) = \begin{pmatrix} c_1 & 0 & -s_1 \\ s_1 & 0 & c_1 \\ 0 & -1 & 0 \end{pmatrix} \qquad\qquad (\text{being } \alpha_1 = -\pi/2). $$

Morever, since ${}^1\boldsymbol{R}_2 = \boldsymbol{I}$, then ${}^0\boldsymbol{R}_2(q_1) = {}^0\boldsymbol{R}_1(q_1)$.

Since robot motion occurs in the plane (x_0, y_0) and the second joint is prismatic, the angular velocities of the two links are

$$ {}^0\boldsymbol{\omega}_1 = \begin{pmatrix} 0 \\ 0 \\ \dot{q}_1 \end{pmatrix} \quad \Rightarrow \quad {}^1\boldsymbol{\omega}_1 = {}^0\boldsymbol{R}_1^T(q_1)\, {}^0\boldsymbol{\omega}_1 = \begin{pmatrix} 0 \\ -\dot{q}_1 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{S}({}^1\boldsymbol{\omega}_1) = \begin{pmatrix} 0 & 0 & -\dot{q}_1 \\ 0 & 0 & 0 \\ \dot{q}_1 & 0 & 0 \end{pmatrix} $$

and similarly

$$ {}^0\boldsymbol{\omega}_2 = \begin{pmatrix} 0 \\ 0 \\ \dot{q}_1 \end{pmatrix} \quad \Rightarrow \quad {}^2\boldsymbol{\omega}_2 = {}^0\boldsymbol{R}_2^T(q_1)\, {}^0\boldsymbol{\omega}_2 = \begin{pmatrix} 0 \\ -\dot{q}_1 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{S}({}^2\boldsymbol{\omega}_2) = \begin{pmatrix} 0 & 0 & -\dot{q}_1 \\ 0 & 0 & 0 \\ \dot{q}_1 & 0 & 0 \end{pmatrix}. $$

The positions of the center of mass (CoM) of the two links are

$$ \boldsymbol{p}_{c1} = \begin{pmatrix} a_1 c_1 - d_{c1} s_1 \\ a_1 s_1 + d_{c1} c_1 \\ 0 \end{pmatrix} = {}^0\boldsymbol{R}_1(q_1) \begin{pmatrix} a_1 \\ 0 \\ d_{c1} \end{pmatrix} $$

and

$$ \boldsymbol{p}_{c2} = \begin{pmatrix} a_1 c_1 - (q_2 - d_{c2})\, s_1 \\ a_1 s_1 + (q_2 - d_{c2})\, c_1 \\ 0 \end{pmatrix} = {}^0\boldsymbol{R}_2(q_1) \begin{pmatrix} a_1 \\ 0 \\ q_2 - d_{c2} \end{pmatrix}. $$

The corresponding velocities are computed as

$$ \boldsymbol{v}_{c1} = \dot{\boldsymbol{p}}_{c1} = {}^0\dot{\boldsymbol{R}}_1(q_1) \begin{pmatrix} a_1 \\ 0 \\ d_{c1} \end{pmatrix} = {}^0\boldsymbol{R}_1(q_1)\boldsymbol{S}({}^1\boldsymbol{\omega}_1) \begin{pmatrix} a_1 \\ 0 \\ d_{c1} \end{pmatrix} = {}^0\boldsymbol{R}_1(q_1) \begin{pmatrix} -d_{c1}\dot{q}_1 \\ 0 \\ a_1\dot{q}_1 \end{pmatrix} $$

and

$$ \begin{aligned} \boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2} &= {}^0\boldsymbol{R}_2(q_1) \begin{pmatrix} 0 \\ 0 \\ \dot{q}_2 \end{pmatrix} + {}^0\dot{\boldsymbol{R}}_2(q_1) \begin{pmatrix} a_1 \\ 0 \\ q_2 - d_{c2} \end{pmatrix} \\ &= {}^0\boldsymbol{R}_2(q_1)\Bigg(\begin{pmatrix} 0 \\ 0 \\ \dot{q}_2 \end{pmatrix} + \boldsymbol{S}({}^2\boldsymbol{\omega}_2) \begin{pmatrix} a_1 \\ 0 \\ q_2 - d_{c2} \end{pmatrix} \Bigg) = {}^0\boldsymbol{R}_2(q_1) \begin{pmatrix} -(q_2 - d_{c2})\,\dot{q}_1 \\ 0 \\ \dot{q}_2 + a_1\dot{q}_1 \end{pmatrix}. \end{aligned} $$

The kinetic energy of the first link is then given by

$$ T_1 = \frac{1}{2} m_1 \|\boldsymbol{v}_{c1}\|^2 + \frac{1}{2}\, {}^1\boldsymbol{\omega}_1^T\, {}^1\boldsymbol{I}_{c1}\, {}^1\boldsymbol{\omega}_1 = \frac{1}{2}\left(I_{c1,yy} + m_1\left(a_1^2 + d_{c1}^2\right)\right)\dot{q}_1^2. $$

For the second link, the kinetic energy is

$$\begin{aligned}
T_2 &= \frac{1}{2}\, m_2 \|\boldsymbol{v}_{c2}\|^2 + \frac{1}{2}\, {}^2\boldsymbol{\omega}_2^T\, {}^2\boldsymbol{I}_{c2}\, {}^1\boldsymbol{\omega}_2 \\
&= \frac{1}{2}\, m_2 \left(\left(a_1^2 + (q_2 - d_{c2})^2 \right) \dot{q}_1^2 + \dot{q}_2^2 + 2 a_1 \dot{q}_1 \dot{q}_2 \right) + \frac{1}{2}\, I_{c2,yy}\, \dot{q}_1^2.
\end{aligned}$$

Note that the barycentric inertia matrices $\boldsymbol{I}_{c1}$ and $\boldsymbol{I}_{c2}$ of the two links need not to be diagonal: in fact, the only element that is involved in robot motion are the diagonal elements along the local y axes ($I_{ci,yy}$, for $i = 1, 2$).
Therefore,

$$T = T_1 + T_2 = \frac{1}{2}\, \dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\, \dot{\boldsymbol{q}},$$

with

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} I_{c1,yy} + m_1 \left(a_1^2 + d_{c1}^2 \right) + I_{c2,yy} + m_2 \left(a_1^2 + (q_2 - d_{c2})^2 \right) & m_2 a_1 \\ m_2 a_1 & m_2 \end{pmatrix}.$$

Defining the three dynamic coefficients

$$\begin{aligned}
\rho_1 &= I_{c1,yy} + m_1 d_{c1}^2 + I_{c2,yy} + m_2 d_{c2}^2 + (m_1 + m_2)\, a_1^2 \\
\rho_2 &= m_2 d_{c2} \\
\rho_3 &= m_2,
\end{aligned} \tag{12}$$

the inertia matrix can be rewritten more compactly as

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} \rho_1 - 2\rho_2\, q_2 + \rho_3\, q_2^2 & \rho_3\, a_1 \\ \rho_3\, a_1 & \rho_3 \end{pmatrix}.$$

The velocity terms are computed from the Christoffel matrices as

$$c_i(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_i(\boldsymbol{q}) \dot{\boldsymbol{q}} \qquad \boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2} \left(\frac{\partial \boldsymbol{m}_i}{\partial \boldsymbol{q}} + \left(\frac{\partial \boldsymbol{m}_i}{\partial \boldsymbol{q}} \right)^T - \frac{\partial \boldsymbol{M}}{\partial q_i} \right) \qquad i = 1, 2,$$

where $\boldsymbol{m}_i(\boldsymbol{q})$ is the i-th column of the inertia matrix $\boldsymbol{M}(\boldsymbol{q})$. We have

$$\boldsymbol{C}_1(\boldsymbol{q}) = \begin{pmatrix} 0 & -\left(\rho_2 - \rho_3\, q_2\right) \\ -\left(\rho_2 - \rho_3\, q_2\right) & 0 \end{pmatrix} \qquad \Rightarrow \qquad c_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -2\left(\rho_2 - \rho_3\, q_2\right) \dot{q}_1 \dot{q}_2,$$

while

$$\boldsymbol{C}_2(\boldsymbol{q}) = \begin{pmatrix} \rho_2 - \rho_3\, q_2 & 0 \\ 0 & 0 \end{pmatrix} \qquad \Rightarrow \qquad c_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \left(\rho_2 - \rho_3\, q_2\right) \dot{q}_1^2,$$

Thus,

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} -2\left(\rho_2 - \rho_3\, q_2\right) \dot{q}_1 \dot{q}_2 \\ \left(\rho_2 - \rho_3\, q_2\right) \dot{q}_1^2 \end{pmatrix}.$$

A factorization $\boldsymbol{N}(\boldsymbol{q}, \dot{\boldsymbol{q}})\, \dot{\boldsymbol{q}}$ of $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ that satisfies the skew-symmetric property of $\dot{\boldsymbol{M}} - 2\boldsymbol{N}$ is

$$\boldsymbol{N}_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} -\left(\rho_2 - \rho_3\, q_2\right) \dot{q}_2 & -\left(\rho_2 - \rho_3\, q_2\right) \dot{q}_1 \\ \left(\rho_2 - \rho_3\, q_2\right) \dot{q}_1 & 0 \end{pmatrix} = \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})$$

This matrix is obtained directly from the Christoffel matrices, with the first row being $\dot{\boldsymbol{q}}^T \boldsymbol{C}_1(\boldsymbol{q})$ and the second row being $\dot{\boldsymbol{q}}^T \boldsymbol{C}_2(\boldsymbol{q})$. Since

$$\dot{\boldsymbol{M}}(\boldsymbol{q}) = \begin{pmatrix} -2\,(\rho_2 - \rho_3\,q_2)\,\dot{q}_2 & 0 \\ 0 & 0 \end{pmatrix},$$

the above factorization leads as expected to the skew-symmetric matrix

$$\boldsymbol{S} = \dot{\boldsymbol{M}} - 2\boldsymbol{C} = \begin{pmatrix} 0 & 2\,(\rho_2 - \rho_3\,q_2)\,\dot{q}_1 \\ -2\,(\rho_2 - \rho_3\,q_2)\,\dot{q}_1 & 0 \end{pmatrix}.$$

On the other hand, the factorization $\boldsymbol{N}_2(\boldsymbol{q}, \dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}}$ of vector $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ with

$$N_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} 0 & -2\,(\rho_2 - \rho_3\,q_2)\,\dot{q}_1 \\ (\rho_2 - \rho_3\,q_2)\,\dot{q}_1 & 0 \end{pmatrix}$$

leads to the matrix

$$\dot{\boldsymbol{M}} - 2\boldsymbol{N}_2 = \begin{pmatrix} -2\,(\rho_2 - \rho_3\,q_2)\,\dot{q}_2 & 4\,(\rho_2 + \rho_3\,q_2)\,\dot{q}_1 \\ -2\,(\rho_2 - \rho_3\,q_2)\,\dot{q}_1 & 0 \end{pmatrix}$$

which is clearly not skew-symmetric. Still, one can easily check that $\dot{\boldsymbol{q}}^T \left(\dot{\boldsymbol{M}} - 2\boldsymbol{N}_2\right) \dot{\boldsymbol{q}} = 0$ for any $\dot{\boldsymbol{q}} \in \mathbb{R}^2$ (a property that follows from conservation of energy).

Being the gravity acceleration $\boldsymbol{g} = \begin{pmatrix} 0 & -g_0 & 0 \end{pmatrix}^T$ when expressed in frame 0, the potential energy of the two links is given by

$$\begin{aligned} U_1 &= -m_1\,\boldsymbol{g}^T \boldsymbol{p}_{0,c1} = m_1 g_0\,(a_1 s_1 + d_{c1} c_1) + U_{10} \\ U_2 &= -m_2\,\boldsymbol{g}^T \boldsymbol{p}_{0,c2} = m_2 g_0\,(a_1 s_1 + (q_2 - d_{c2})\,c_1) + U_{20} \end{aligned}$$

Therefore, from $U = U_1 + U_2$ one has

$$\begin{aligned} \boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U}{\partial \boldsymbol{q}}\right)^T &= \begin{pmatrix} (m_1 + m_2)\,a_1 g_0\,c_1 + (m_2 d_{c2} - m_1 d_{c1})\,g_0\,s_1 - m_2\,g_0\,q_2 s_1 \\ m_2\,g_0\,c_1 \end{pmatrix} \\ &= g_0 \begin{pmatrix} \rho_4\,a_1\,c_1 + \rho_5\,s_1 - \rho_3\,q_2 s_1 \\ \rho_3\,c_1 \end{pmatrix}, \end{aligned} \tag{13}$$

with the introduction of the two additional dynamic coefficients

$$\begin{aligned} \rho_4 &= m_1 + m_2 \\ \rho_5 &= m_2 d_{c2} - m_1 d_{c1}. \end{aligned} \tag{14}$$

With the total of $r = 5$ dynamic coefficients in (12) and (14), the dynamic model (2) can be rewritten in the linearly parametrized form (3)

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}.\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})\,\boldsymbol{\rho} = \boldsymbol{\tau},$$

where

$$\boldsymbol{Y} = \begin{pmatrix} \ddot{q}_1 & -2\,(q_2\,\ddot{q}_1 + \dot{q}_1\dot{q}_2) & q_2^2\,\ddot{q}_1 + a_1\ddot{q}_2 + 2q_2\,\dot{q}_1\dot{q}_2 - g_0\,q_2 s_1 & a_1 g_0\,c_1 & g_0\,s_1 \\ 0 & \dot{q}_1^2 & a_1\ddot{q}_1 + \ddot{q}_2 - q_2\,\dot{q}_1^2 + g_0\,c_1 & 0 & 0 \end{pmatrix} \quad \boldsymbol{\rho} = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \\ \rho_5 \end{pmatrix}.$$

Exercise 4

The nominal joint trajectory for $t \in [0,1]$ is

$$\boldsymbol{q}_d(t) = \begin{pmatrix} \dfrac{\pi}{4}\left(1 - 3t^2 + 2t^3\right) \\ -\dfrac{\pi}{2}\left(2 - \cos 2\pi t\right) \end{pmatrix},$$

with its first and second derivatives

$$\dot{\boldsymbol{q}}_d(t) = \begin{pmatrix} \dfrac{3\pi}{2}\left(t^2 - t\right) \\ -\pi^2 \sin 2\pi t \end{pmatrix} \qquad \ddot{\boldsymbol{q}}_d(t) = \begin{pmatrix} \dfrac{3\pi}{2}\left(2t - 1\right) \\ -4\pi^3 \cos 2\pi t \end{pmatrix}.$$

In the absence of external disturbances and on a horizontal plane, this trajectory is executed with the inverse dynamics joint torque

$$\boldsymbol{\tau}_d(t) = \boldsymbol{M}(\boldsymbol{q}_d(t))\,\ddot{\boldsymbol{q}}_d(t) + \boldsymbol{c}(\boldsymbol{q}_d(t), \dot{\boldsymbol{q}}_d(t)),$$

starting in $\boldsymbol{q}_d(0) = (\pi/4, -\pi/2)$ at time $t = 0$ and ending in $\boldsymbol{q}_d(1) = (0, -\pi/2)$ at time $t = 1$. Since $\dot{\boldsymbol{q}}_d(0) = \dot{\boldsymbol{q}}_d(1) = \boldsymbol{0}$, this is a rest-to-rest motion. With the constant potential energy set to zero, the total energy E of the robot is given by the kinetic energy T only. Therefore, being the initial and final joint velocities zero, $E(0) = E(1) = 0$.

Consider now the presence of disturbances, expressed as the two external torques $u_{e1}(t)$ and $u_{e2}(t)$ being applied to the robot for some time intervals. To keep the same original desired motion, the two actuators should them deliver the control torques

$$\tau_1(t) = \tau_{d1}(t) - u_{e1}(t) \qquad \tau_2(t) = \tau_{d2}(t) - u_{e2}(t), \tag{15}$$

so that both disturbances will be perfectly cancelled. As a result, the nominal trajectory will remain the same realizing once more a rest-to-rest motion. Thus, we obtain again a final $E(1) = 0$. However, there is an energy change when considering the disturbance $\boldsymbol{u}_e(t)$ only. This is computed as

$$\Delta E_e = E(1) - E(0) = \int_0^1 \dot{E}(t)dt = \int_0^1 \dot{\boldsymbol{q}}_d^T(t)\boldsymbol{u}_e(t)dt = \int_0^1 \left(\dot{q}_{d1}(t)u_{e1}(t) + \dot{q}_{d2}(t)u_{e2}(t)\right)dt. \tag{16}$$

Expliciting the two addends in the integral, one obtains

$$\int_0^1 \dot{q}_{d1}(t)u_{e1}(t)dt = -2\int_0^{0.5} \dot{q}_{d1}(t)dt = -3\pi\int_0^{0.5}\left(t^2 - t\right)dt = -3\pi\left(\left[\frac{t^3}{3} - \frac{t^2}{2}\right]_{t=0}^{t=0.5}\right) = \frac{\pi}{4}$$

and

$$\int_0^1 \dot{q}_{d2}(t)u_{e2}(t)dt = \int_{0.5}^1 \dot{q}_{d2}(t)dt = -\pi^2\int_{0.5}^1 \sin 2\pi t\,dt = \frac{\pi}{2}\Big[\cos 2\pi t\Big]_{t=0.5}^{t=1} = \pi.$$

Thus, from (16)

$$\Delta E_e = \frac{\pi}{4} + \pi = \frac{5\pi}{4} \simeq 3.927 \text{ J}.$$

Indeed, this increase $\Delta E_e > 0$ in energy due to the effect of the disturbances is perfectly balanced by the energy decrease $\Delta E_c = -\Delta E_e < 0$ induced by the extra control torques in (15) needed to cancel the two disturbances.

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