Robotics 2

July 17, 2026

Exercise #1

Consider a rigid robot link of length $L = 1$ m with square cross-section of side $D = 0.2$ m. A force/torque (F/T) sensor is mounted at the proximal end of the link. The sensor frame is chosen such that the z-axis coincides with the longitudinal axis of the link and the x- and y-axes are aligned with the sides of the square section. Assume that a contact force $\boldsymbol{f}_c \in \mathbb{R}^3$ acts at a contact point $\boldsymbol{p}_c \in \mathbb{R}^3$ on the upper face of the link. The F/T sensor measures a force and a moment

$$\boldsymbol{f}_s = \begin{pmatrix} 1 & 2 & 1 \end{pmatrix}^T \text{ [N]} \qquad\qquad \boldsymbol{\mu}_s = \begin{pmatrix} -1.1 & 0.55 & 0 \end{pmatrix}^T \text{ [Nm]}.$$

Determine the contact force $\boldsymbol{f}_c$ and the contact point $\boldsymbol{p}_c$.

Exercise #2

a) Consider a rigid robot with n joints and dynamics

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}, \tag{1}$$

where the matrix $\boldsymbol{C}$ is computed using the Christoffel symbols. Define a control torque $\boldsymbol{\tau}$ such that the following behavior is assigned to the robot kinetic energy $\mathcal{T}(\boldsymbol{q}, \dot{\boldsymbol{q}})$:

$$\dot{\mathcal{T}} = -\alpha\, \mathcal{T} \qquad \alpha > 0. \tag{2}$$

b) At a given state $(\boldsymbol{q}, \dot{\boldsymbol{q}})$, the robot should now realize a desired primary task acceleration $\ddot{\boldsymbol{y}}_d \in \mathbb{R}^m$, with $m < n$, where $\boldsymbol{y} = \boldsymbol{k}(\boldsymbol{q})$. Define a control torque $\boldsymbol{\tau}$ that executes this task out of singularities while trying to achieve at best the behavior specified in **a)**. What if some position and/or velocity task error occurs during the execution of the primary task?

c) Consider now the dynamics of a robot with elastic joints

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{K}\,(\boldsymbol{q} - \boldsymbol{\theta}) = \boldsymbol{0} \tag{3}$$

$$\boldsymbol{M}_m\ddot{\boldsymbol{\theta}} + \boldsymbol{K}\,(\boldsymbol{\theta} - \boldsymbol{q}) = \boldsymbol{\tau}, \tag{4}$$

with diagonal matrices $\boldsymbol{M}_m > 0$ and $\boldsymbol{K} > 0$. At a given state $(\boldsymbol{q}, \boldsymbol{\theta}, \dot{\boldsymbol{q}}, \dot{\boldsymbol{\theta}})$, define a control torque $\boldsymbol{\tau}$ such that, as long as $\dot{\boldsymbol{\theta}} \neq \boldsymbol{0}$, the following behavior is assigned to the total robot kinetic energy $\mathcal{T}_{\rm tot}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \dot{\boldsymbol{\theta}}) = \mathcal{T}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \mathcal{T}_{\rm m}(\dot{\boldsymbol{\theta}})$:

$$\dot{\mathcal{T}}_{\rm tot} = -\beta\, \mathcal{T}_{\rm tot} \qquad \beta > 0. \tag{5}$$

Exercise #3

A 2R polar robot, with the classical Denavit-Hartenberg (DH) parameters given in Tab. 1 and a vertical first joint axis, has a perfectly known dynamics expressed as in (1). An unknown payload is added to the robot tip, having mass m_p, center of mass placed at the origin of the DH frame of the second link, and barycentric inertia ${}^2\boldsymbol{I}_p = \text{diag}\{I_{px}, I_{py}, I_{pz}\}$. Design an adaptive control law for $\boldsymbol{\tau}$ that achieves global asymptotic tracking of a desired smooth trajectory $\boldsymbol{q}_d(t)$ and has a minimum number of controller states. *Hint: The derivation of explicit expressions for the terms in the dynamic model of the robot* without *payload is not strictly required!*

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	d_1	q_1
2	0	a_2	0	q_2

Table 1: DH parameters of a 2R polar robot.

[180 minutes (3 hours); open books]

Solution

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Exercise #1

With reference to the situation sketched in Fig. 1, the transformation of forces and moments on a rigid body, with all vectors expressed in the same sensor frame, provides

$$\boldsymbol{f}_s = \boldsymbol{f}_c \tag{6}$$

$$\boldsymbol{\mu}_s = \boldsymbol{r}_c \times \boldsymbol{f}_c = \boldsymbol{S}(\boldsymbol{r}_c)\boldsymbol{f}_c = -\boldsymbol{S}(\boldsymbol{f}_c)\,\boldsymbol{r}_c, \tag{7}$$

where $\boldsymbol{r}_c \in \mathbb{R}^3$ is the unknown distance vector from the origin of the sensor frame to the line of action of the contact force $\boldsymbol{f}_c$. Using (6) in (7) and plugging in the data of the problem leads to a linear system of 3 equations in the 3 unknown components r_x, r_y, r_z of $\boldsymbol{r}_c$

$$\boldsymbol{S}(\boldsymbol{f}_s)\,\boldsymbol{r}_c = -\boldsymbol{\mu}_s \quad \Rightarrow \quad \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix} \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} = \begin{pmatrix} 1.1 \\ -0.55 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} -r_y + 2r_z &= 1.1 \\ r_x - r_z &= -0.55 \\ -2r_x + r_y &= 0. \end{aligned} \tag{8}$$

Note that the 3×3 skew-symmetric matrix $\boldsymbol{S}$ is never of full rank. In this case, the third equation is the sum of the first and of the second multiplied by 2. Thus, matrix $\boldsymbol{S}$ has rank 2 and there is an infinite number of solutions to (8), namely all the points along the *line of action* of the force $\boldsymbol{f}_c$. In fact, all vectors $\boldsymbol{p}$ to points on this line return the same vector value in the cross product with $\boldsymbol{f}_c$, i.e.,

$$\boldsymbol{p} \times \boldsymbol{f}_c = \boldsymbol{r}_c \times \boldsymbol{f}_c = \boldsymbol{\mu}_s. \tag{9}$$

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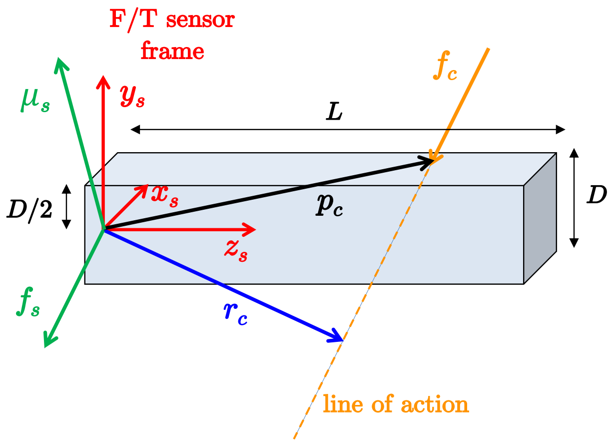

Figure 1: Contact point and contact force reconstruction

The minimum norm solution to (8) obtained using the pseudoinverse of $\boldsymbol{S}$ provides the vector $\boldsymbol{r}_c$ in Fig. 1. It can be easily checked that, for any $\boldsymbol{v} \in \mathbb{R}^3$, it is

$$\boldsymbol{S}^2(\boldsymbol{v}) = \boldsymbol{v}\boldsymbol{v}^T - \|\boldsymbol{v}\|^2 \boldsymbol{I} \qquad\qquad \boldsymbol{S}^{\#}(\boldsymbol{v}) = -\frac{1}{\|\boldsymbol{v}\|^2}\,\boldsymbol{S}(\boldsymbol{v}).$$

Thus, we obtain

$$\boldsymbol{r}_c = -\boldsymbol{S}^{\#}(\boldsymbol{f}_s)\,\boldsymbol{\mu}_s = \frac{1}{\|\boldsymbol{f}_s\|^2}\,\boldsymbol{S}(\boldsymbol{f}_s)\,\boldsymbol{\mu}_s = \frac{1}{6}\begin{pmatrix} -0.55 \\ -1.1 \\ 2.75 \end{pmatrix} = \begin{pmatrix} -0.0917 \\ -0.1833 \\ 0.4583 \end{pmatrix} \text{ [m]}. \tag{10}$$

The entire line of action passing through the point with position $\boldsymbol{r}_c$ can be described by the parametric equation

$$\boldsymbol{p}(\lambda) = \boldsymbol{r}_c + \lambda \boldsymbol{f}_c = \begin{pmatrix} -0.0917 \\ -0.1833 \\ 0.4583 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \qquad \forall \lambda \in \mathbb{R}. \tag{11}$$

Intersecting the line (11) with the upper side of the link, namely with the plane $p_y = D/2 = 0.1$, gives

$$p_y(\lambda) = -0.1833 + 2\lambda = 0.1 \quad \Rightarrow \quad \lambda_c = 0.1417.$$

Thus, the contact force associated with the measured wrench $(\boldsymbol{f}_s, \boldsymbol{\mu}_s)$ is

$$\boxed{\boldsymbol{f}_c = \boldsymbol{f}_s = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \text{ [N]},} \tag{12}$$

with the contact point

$$\boxed{\boldsymbol{p}_c = \boldsymbol{r}_c + \lambda_c \boldsymbol{f}_c = \begin{pmatrix} 0.05 \\ 0.1 \\ 0.6 \end{pmatrix} \text{ [m]}.} \tag{13}$$

This derivation neatly separates the wrench inversion step (using the pseudoinverse of $\boldsymbol{S}(\boldsymbol{f}_s) = \boldsymbol{S}(\boldsymbol{f}_c)$) from the geometric constraint step (imposing the intersection of the line of action with a plane). The fact that when $p_{c,y} = D/2 = 0.1$ also the other two components of $\boldsymbol{p}_c$ are within the dimensions of the upper side of the link, i.e., $p_{c,z} \in [0, 1]$ and $|p_{c,x}| \leq 0.1$, means that the sensor data $\boldsymbol{f}_s$ and $\boldsymbol{\mu}_s$ given in the problem are coherent with the physics (in practice, some added noise in the data could perturb the outcome and a geometric tolerance should be used).

We note finally that the same result could have been obtained by imposing directly the geometric constraint for the contact point, i.e., $\boldsymbol{p}_c$ should be on the upper side of the link (or $p_{c,y} = D/2 = 0.1$ [m]), by using (9) with $\boldsymbol{p} = \boldsymbol{p}_c$ and $\boldsymbol{f}_c = \boldsymbol{f}_s = \begin{pmatrix} 1 & 2 & 1 \end{pmatrix}^T$:

$$\boldsymbol{p}_c \times \boldsymbol{f}_s = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ p_{c,x} & 0.1 & p_{c,z} \\ 1 & 2 & 1 \end{vmatrix} = \begin{pmatrix} 0.1 \cdot 1 - 2p_{c,z} \\ p_{c,z} - p_{c,x} \\ 2p_{c,x} - 0.1 \cdot 1 \end{pmatrix} = \begin{pmatrix} -1.1 \\ 0.55 \\ 0 \end{pmatrix} = \boldsymbol{\mu}_s \quad \Rightarrow \quad \boldsymbol{p}_c = \begin{pmatrix} 0.05 \\ 0.1 \\ 0.6 \end{pmatrix} \text{ [m]}.$$

Exercise #2

Part a) The time evolution of the kinetic energy for a rigid robot with dynamics (1) is

$$\dot{\mathcal{T}} = \frac{d}{dt}\left(\frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}\right) = \dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \frac{1}{2}\dot{\boldsymbol{q}}^T \dot{\boldsymbol{M}}(\boldsymbol{q})\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}^T \left(\boldsymbol{\tau} - \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{g}(\boldsymbol{q})\right) + \frac{1}{2}\dot{\boldsymbol{q}}^T \dot{\boldsymbol{M}}(\boldsymbol{q})\dot{\boldsymbol{q}}. \tag{14}$$

Since matrix $\boldsymbol{C}$ satisfies the skew-symmetric property

$$\dot{\boldsymbol{q}}^T \left(\dot{\boldsymbol{M}}(\boldsymbol{q}) - 2\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\right)\dot{\boldsymbol{q}} = 0 \qquad \forall (\boldsymbol{q},\dot{\boldsymbol{q}}), \tag{15}$$

we can simplify (14) as

$$\dot{\mathcal{T}} = \dot{\boldsymbol{q}}^T \left(\boldsymbol{\tau} - \boldsymbol{g}(\boldsymbol{q})\right). \tag{16}$$

Imposing to (16) the desired behavior (2)

$$\dot{\mathcal{T}} = -\alpha\,\mathcal{T} = -\frac{\alpha}{2}\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

leads to the torque solution

$$\boxed{\boldsymbol{\tau} = \boldsymbol{g}(\boldsymbol{q}) - \frac{\alpha}{2}\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}} \tag{17}$$

Note that the first action performed by the torque (17) is to remove gravity from the picture.

Part b) We have

$$\boldsymbol{y} = \boldsymbol{k}(\boldsymbol{q}) \qquad \Rightarrow \qquad \dot{\boldsymbol{y}} = \frac{\partial \boldsymbol{k}}{\partial \boldsymbol{q}}\,\dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}},$$

with a non-square $m \times n$ Jacobian matrix $\boldsymbol{J}(\boldsymbol{q})$, which is supposed to be of full row rank m. Furthermore

$$\ddot{\boldsymbol{y}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}. \tag{18}$$

At the current robot state $(\boldsymbol{q}, \dot{\boldsymbol{q}})$, we would like to find a control torque $\boldsymbol{\tau}$ that enforces $\ddot{\boldsymbol{y}} = \ddot{\boldsymbol{y}}_d$ while staying as close as possible to the energy-shaping torque previously found in (17) and labeled now for convenience as

$$\boldsymbol{\tau}_N = \boldsymbol{g}(\boldsymbol{q}) - \frac{\alpha}{2}\,\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}. \tag{19}$$

One can obtain different (infinite) possible solutions, depending on how the norm of $\boldsymbol{\tau} - \boldsymbol{\tau}_N$ is being weighted in a linear-quadratic (LQ) optimization problem. For instance, one can solve (18) for all joint acceleration $\ddot{\boldsymbol{q}}$ that realize $\ddot{\boldsymbol{y}} = \ddot{\boldsymbol{y}}_d$

$$\ddot{\boldsymbol{q}} = \boldsymbol{J}^{\#}(\boldsymbol{q})\left(\ddot{\boldsymbol{y}}_d - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}\right) + \left(\boldsymbol{I} - \boldsymbol{J}^{\#}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q})\right)\ddot{\boldsymbol{q}}_N \qquad \forall \ddot{\boldsymbol{q}}_N \in \mathbb{R}^n. \tag{20}$$

Substituting (20) in (1), where we set for compactness $\boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q})$, one obtains

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})\boldsymbol{J}^{\#}(\boldsymbol{q})\left(\ddot{\boldsymbol{y}}_d - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}\right) + \boldsymbol{M}(\boldsymbol{q})\left(\boldsymbol{I} - \boldsymbol{J}^{\#}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q})\right)\ddot{\boldsymbol{q}}_N + \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}). \tag{21}$$

In a state $(\boldsymbol{q}, \dot{\boldsymbol{q}})$, the unique acceleration associated to the desired torque $\boldsymbol{\tau}_N$ in (19) is

$$\ddot{\boldsymbol{q}}_N = \boldsymbol{M}^{-1}(\boldsymbol{q})\left(\boldsymbol{\tau}_N - \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}})\right) = -\boldsymbol{M}^{-1}(\boldsymbol{q})\left(\frac{\alpha}{2}\,\boldsymbol{M}(\boldsymbol{q}) + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\right)\dot{\boldsymbol{q}} = -\frac{\alpha}{2}\,\dot{\boldsymbol{q}} - \boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}. \tag{22}$$

Substituting (22) in (21) and rearranging terms yields finally

$$\boxed{\boldsymbol{\tau} = \boldsymbol{\tau}_N + \boldsymbol{M}(\boldsymbol{q})\boldsymbol{J}^{\#}(\boldsymbol{q})\left(\ddot{\boldsymbol{y}}_d - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} + \boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})(\boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{\tau}_N)\right)} \tag{23}$$

or

$$\boldsymbol{\tau} = \boldsymbol{g}(\boldsymbol{q}) - \frac{\alpha}{2}\,\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}} + \boldsymbol{M}(\boldsymbol{q})\boldsymbol{J}^{\#}(\boldsymbol{q})\Big(\ddot{\boldsymbol{y}}_d - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} + \boldsymbol{J}(\boldsymbol{q})\big(\boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \frac{\alpha}{2}\,\boldsymbol{I}\big)\dot{\boldsymbol{q}}\Big). \tag{24}$$

This solution minimizes the weighted norm[1]

$$H_2 = \frac{1}{2}\,\|\boldsymbol{\tau} - \boldsymbol{\tau}_N\|^2_{\boldsymbol{M}^{-2}} = \frac{1}{2}\,\left\|\boldsymbol{\tau} - \boldsymbol{g}(\boldsymbol{q}) + \frac{\alpha}{2}\,\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}\right\|^2_{\boldsymbol{M}^{-2}}.$$

An alternative solution is obtained by substituting the acceleration $\ddot{\boldsymbol{q}}$ from the dynamic model (1) into (18) with $\ddot{\boldsymbol{y}} = \ddot{\boldsymbol{y}}_d$. This gives

$$\boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\,\boldsymbol{\tau} = \ddot{\boldsymbol{y}}_d - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} + \boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}), \tag{25}$$

which is in the linear form $\boldsymbol{A}\,\boldsymbol{\tau} = \boldsymbol{b}$, with $\boldsymbol{A} = \boldsymbol{J}\boldsymbol{M}^{-1}$ and $\boldsymbol{b} = \ddot{\boldsymbol{y}}_d - \dot{\boldsymbol{J}}\dot{\boldsymbol{q}} + \boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{n}$. When minimizing the norm

$$H_1 = \frac{1}{2}\,\|\boldsymbol{\tau} - \boldsymbol{\tau}_N\|^2 = \frac{1}{2}\,\left\|\boldsymbol{\tau} - \boldsymbol{g}(\boldsymbol{q}) + \frac{\alpha}{2}\,\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}\right\|^2,$$

we obtain the torque

$$\boxed{\boldsymbol{\tau} = \boldsymbol{\tau}_N + \left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\right)^{\#}\left(\ddot{\boldsymbol{y}}_d - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} + \boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})(\boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{\tau}_N)\right)} \tag{26}$$

Similarly, if we proceed by using the dynamically consistent decomposition of the joint torque as

$$\boldsymbol{\tau} = \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{F} + \left(\boldsymbol{I} - \boldsymbol{J}^T(\boldsymbol{q})\left(\boldsymbol{J}^{\#}_{\boldsymbol{M}}(\boldsymbol{q})\right)^T\right)\boldsymbol{\tau}_N, \tag{27}$$

[1] See also Sect. 6.5.3 in the new textbook or the LQ solutions in slide #11 of the block 02_KinematicRedundancy_2, being here $\boldsymbol{n} - \boldsymbol{\tau}_N = \left(\boldsymbol{C} + (\alpha/2)\boldsymbol{M}\right)\dot{\boldsymbol{q}}$.

where $\boldsymbol{F} \in \mathbb{R}^m$ is a generalized force performing work on $\boldsymbol{y}$ and $\boldsymbol{J}_M^{\#} = \boldsymbol{M}^{-1}\boldsymbol{J}^T(\boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{J}^T)^{-1}$, it can be easily shown that we obtain the torque

$$\boxed{\boldsymbol{\tau} = \boldsymbol{\tau}_N + \boldsymbol{J}^T(\boldsymbol{q})\big(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\big)^{-1}\big(\ddot{\boldsymbol{y}}_d - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} + \boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})(\boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}}) - \boldsymbol{\tau}_N)\big)} \tag{28}$$

which minimizes the norm of the torque difference when weighted by the inverse inertia matrix, i.e.,

$$H_3 = \frac{1}{2}\left\|\boldsymbol{\tau} - \boldsymbol{\tau}_N\right\|^2_{\boldsymbol{M}^{-1}} = \frac{1}{2}\left\|\boldsymbol{\tau} - \boldsymbol{g}(\boldsymbol{q}) + \frac{\alpha}{2}\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}\right\|^2_{\boldsymbol{M}^{-1}}.$$

If position and/or velocity errors occur during the execution of the primary task, a modified controller can recover these errors at a tunable exponential rate. Just substitute $\ddot{\boldsymbol{y}}_d$ in (23), (26), or (28), with

$$\ddot{\boldsymbol{y}}_d + \boldsymbol{K}_D(\dot{\boldsymbol{y}}_d - \dot{\boldsymbol{y}}) + \boldsymbol{K}_P(\boldsymbol{y}_d - \boldsymbol{y}),$$

where $\boldsymbol{K}_P$ and $\boldsymbol{K}_D$ are positive definite, symmetric (typically, diagonal) gain matrices. The task error $\boldsymbol{e} = \boldsymbol{y}_d - \boldsymbol{y}$ will satisfy then the linear, second-order differential equation

$$\ddot{\boldsymbol{e}} + \boldsymbol{K}_D\dot{\boldsymbol{e}} + \boldsymbol{K}_P\boldsymbol{e} = \boldsymbol{0},$$

whose solution $\boldsymbol{e}(t)$ exponentially converges to zero with a desired transient behavior (convergence at a fast rate and non-oscillatory) governed by suitable choices of the gain matrices $\boldsymbol{K}_P$ and $\boldsymbol{K}_D$.

Part c) We proceed similarly to the rigid case in **a)**. The total kinetic energy of a robot with elastic joints having the dynamics (3)–(4) is

$$\mathcal{T}_{\rm tot} = \mathcal{T} + \mathcal{T}_{\rm m} = \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}} + \frac{1}{2}\dot{\boldsymbol{\theta}}^T\boldsymbol{M}_m\dot{\boldsymbol{\theta}}.$$

Thus, the time evolution of the total kinetic energy is

$$\begin{aligned}\dot{\mathcal{T}}_{\rm tot} &= \dot{\boldsymbol{q}}^T\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \frac{1}{2}\dot{\boldsymbol{q}}^T\dot{\boldsymbol{M}}(\boldsymbol{q})\dot{\boldsymbol{q}} + \dot{\boldsymbol{\theta}}^T\boldsymbol{M}_m\ddot{\boldsymbol{\theta}} \\ &= \dot{\boldsymbol{q}}^T\left(\boldsymbol{\tau}_J - \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{g}(\boldsymbol{q})\right) + \frac{1}{2}\dot{\boldsymbol{q}}^T\dot{\boldsymbol{M}}(\boldsymbol{q})\dot{\boldsymbol{q}} + \dot{\boldsymbol{\theta}}^T\left(\boldsymbol{\tau} - \boldsymbol{\tau}_J\right),\end{aligned} \tag{29}$$

where we introduced for compactness the elastic joint torque

$$\boldsymbol{\tau}_J = \boldsymbol{K}\left(\boldsymbol{\theta} - \boldsymbol{q}\right). \tag{30}$$

Using again the property (15) of skew-symmetry, one obtains

$$\dot{\mathcal{T}}_{\rm tot} = -(\dot{\boldsymbol{\theta}} - \dot{\boldsymbol{q}})^T\boldsymbol{K}(\boldsymbol{\theta} - \boldsymbol{q}) - \dot{\boldsymbol{q}}^T\boldsymbol{g}(\boldsymbol{q}) + \dot{\boldsymbol{\theta}}^T\boldsymbol{\tau} = \dot{\boldsymbol{q}}^T\left(\boldsymbol{\tau}_J - \boldsymbol{g}(\boldsymbol{q})\right) + \dot{\boldsymbol{\theta}}^T\left(\boldsymbol{\tau} - \boldsymbol{\tau}_J\right). \tag{31}$$

Imposing to (31) the desired behavior (5)

$$\dot{\mathcal{T}}_{\rm tot} = -\beta\,\mathcal{T}_{\rm tot} = -\frac{\beta}{2}\left(\dot{\boldsymbol{q}}^T\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}} + \dot{\boldsymbol{\theta}}^T\boldsymbol{M}_m\dot{\boldsymbol{\theta}}\right), \tag{32}$$

leads to the torque solution

$$\boxed{\boldsymbol{\tau} = \boldsymbol{\tau}_J - \frac{\beta}{2}\boldsymbol{M}_m\dot{\boldsymbol{\theta}} - \frac{\dot{\boldsymbol{\theta}}}{\|\dot{\boldsymbol{\theta}}\|^2}\,\dot{\boldsymbol{q}}^T\left(\boldsymbol{\tau}_J - \boldsymbol{g}(\boldsymbol{q}) + \frac{\beta}{2}\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}\right)} \tag{33}$$

provided that $\|\dot{\boldsymbol{\theta}}\| \neq 0$. In fact, one can check that using (33) we have

$$\dot{\boldsymbol{\theta}}^T\left(\boldsymbol{\tau} - \boldsymbol{\tau}_J\right) = -\frac{\beta}{2}\dot{\boldsymbol{\theta}}^T\boldsymbol{M}_m\dot{\boldsymbol{\theta}} - \dot{\boldsymbol{q}}^T\left(\boldsymbol{\tau}_J - \boldsymbol{g}(\boldsymbol{q})\right) - \frac{\beta}{2}\dot{\boldsymbol{q}}^T\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}},$$

so that from (31) one obtains $\dot{\mathcal{T}}_{\rm tot} = -\beta\,\mathcal{T}_{\rm tot}$ as desired.

We note that the control expression (33) requires $\dot{\boldsymbol{\theta}} \neq \mathbf{0}$ only for the cancellation of dynamic terms related to the link equation (3). Indeed, this is only one of the infinitely many possible torque solutions of the scalar equation (31), with $\dot{\mathcal{T}}_{\rm tot}$ given by (32). For instance, in place of (33) one can choose the torque

$$\boldsymbol{\tau} = \frac{\dot{\boldsymbol{\theta}}}{\|\dot{\boldsymbol{\theta}}\|^2} \left((\dot{\boldsymbol{\theta}} - \dot{\boldsymbol{q}})^T \boldsymbol{\tau}_J + \dot{\boldsymbol{q}}^T \boldsymbol{g}(\boldsymbol{q}) - \beta \mathcal{T}_{\rm tot} \right), \tag{34}$$

which is the solution with minimum norm of $\boldsymbol{\tau}$. In fact, the motor velocity normed factor in (34) is just the pseudoinverse of the row vector $\dot{\boldsymbol{\theta}}^T$. More in general, we can write this factor as

$$(\dot{\boldsymbol{\theta}}^T)^\# = \begin{cases} \dfrac{\dot{\boldsymbol{\theta}}}{\|\dot{\boldsymbol{\theta}}\|^2} & \dot{\boldsymbol{\theta}} \neq \mathbf{0} \\ \mathbf{0} & \dot{\boldsymbol{\theta}} = \mathbf{0}, \end{cases}$$

which is defined also for $\dot{\boldsymbol{\theta}} = \mathbf{0}$. However, the discontinuity when approaching zero motor velocity would not be avoided since

$$\lim_{\|\dot{\boldsymbol{\theta}}\| \to \mathbf{0}} \left\| (\dot{\boldsymbol{\theta}}^T)^\# \right\| = \infty.$$

A damped least squares (DLS) solution may be adopted in that case by using

$$(\dot{\boldsymbol{\theta}}^T)^{DLS} = \frac{\dot{\boldsymbol{\theta}}}{\|\dot{\boldsymbol{\theta}}\|^2 + \mu^2}$$

for a suitable small scalar $\mu > 0$, possibly activated only for very small motor velocity.

Exercise #3

In the presence of an unknown payload, let the robot dynamics (1) be rewritten as

$$\boldsymbol{M}_0(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}_0(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}_0(\boldsymbol{q}) + \boldsymbol{M}_p(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}_p(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}_p(\boldsymbol{q}) = \boldsymbol{\tau} \tag{35}$$

where the terms with subscript 0 contain only known parameters (as related to the 2R polar robot without payload), while the terms with subscript p are the additional unknown dynamic contributions due to the payload. Thus, we need to derive only the kinetic and potential energy of the payload.

From the DH parameters in Tab. 1 and with reference to Fig. 2, we have

$${}^0\boldsymbol{R}_1(q_1) = \begin{pmatrix} \cos q_1 & 0 & \sin q_1 \\ \sin q_1 & 0 & -\cos q_1 \\ 0 & 1 & 0 \end{pmatrix} \qquad {}^1\boldsymbol{R}_2(q_2) = \begin{pmatrix} \cos q_2 & -\sin q_2 & 0 \\ \sin q_2 & \cos q_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and thus

$${}^0\boldsymbol{R}_2(\boldsymbol{q}) = \begin{pmatrix} \cos q_1 \cos q_2 & -\cos q_1 \sin q_2 & \sin q_1 \\ \sin q_1 \cos q_2 & -\sin q_1 \sin q_2 & -\cos q_1 \\ \sin q_2 & \cos q_2 & 0 \end{pmatrix}.$$

The position of the payload center of mass coincides with the origin of the DH frame 2, i.e.,

$${}^0\boldsymbol{p}_p = {}^0\boldsymbol{p}_2 = \begin{pmatrix} a_2 \cos q_2 \cos q_1 \\ a_2 \cos q_2 \sin q_1 \\ d_1 + a_2 \sin q_2 \end{pmatrix}. \tag{36}$$

Thus, its velocity is

$${}^0\boldsymbol{v}_p = {}^0\dot{\boldsymbol{p}}_p = \begin{pmatrix} -a_2 \left(\sin q_1 \cos q_2 \, \dot{q}_1 + \cos q_1 \sin q_2 \, \dot{q}_2 \right) \\ a_2 \left(\cos q_1 \cos q_2 \, \dot{q}_1 - \sin q_1 \sin q_2 \, \dot{q}_2 \right) \\ a_2 \cos q_2 \, \dot{q}_2 \end{pmatrix}.$$

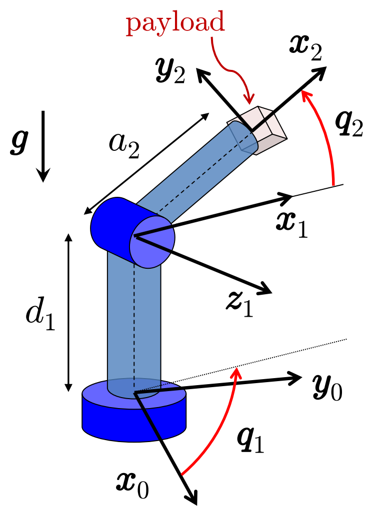

Figure 2: DH frames for the 2R polar robot

However, it is convenient to express this velocity in the local DH frame 2,

$$ {}^2\boldsymbol{v}_p = {}^0\boldsymbol{R}_2^T(\boldsymbol{q})\,{}^0\boldsymbol{v}_p = \begin{pmatrix} 0 \\ a_2\,\dot{q}_2 \\ -a_2\cos q_2\,\dot{q}_1 \end{pmatrix}, $$

so that the linear kinetic energy of the payload is

$$ \mathcal{T}_{p,l} = \frac{1}{2}\,m_p\,\|{}^2\boldsymbol{v}_p\|^2 = \frac{1}{2}\,m_p\,a_2^2\left(\cos^2 q_2\,\dot{q}_1^2 + \dot{q}_2^2\right). \tag{37} $$

The angular velocity of the payload is

$$ {}^0\omega_p = {}^0\boldsymbol{z}_0\,\dot{q}_1 + {}^0\boldsymbol{z}_1\,\dot{q}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\dot{q}_1 + \begin{pmatrix} \sin q_1 \\ -\cos q_1 \\ 0 \end{pmatrix}\dot{q}_2. $$

Expressing this angular velocity in the local DH frame 2, one obtains

$$ {}^2\omega_p = {}^0\boldsymbol{R}_2^T(\boldsymbol{q})\,{}^0\omega_p = \begin{pmatrix} \sin q_2\,\dot{q}_1 \\ \cos q_2\,\dot{q}_1 \\ \dot{q}_2 \end{pmatrix}. $$

The angular kinetic energy of the payload is then

$$ \mathcal{T}_{p,a} = \frac{1}{2}\,{}^2\omega_p^T\,{}^2\boldsymbol{I}_p\,{}^2\omega_p = \frac{1}{2}\left(\left(I_{px}\sin^2 q_2 + I_{py}\cos^2 q_2\right)\dot{q}_1^2 + I_{pz}\,\dot{q}_2^2\right). \tag{38} $$

As a result, using $\sin^2 q_2 = 1 - \cos^2 q_2$ for compactness,[2] the kinetic energy of the payload is

$$ \mathcal{T}_p = \mathcal{T}_{p,l} + \mathcal{T}_{p,a} = \frac{1}{2}\left(\left(I_{px} + \left(I_{py} + m_p\,a_2^2 - I_{px}\right)\cos^2 q_2\right)\dot{q}_1^2 + \left(I_{pz} + m_p\,a_2^2\right)\dot{q}_2^2\right) = \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{M}_p(\boldsymbol{q})\dot{\boldsymbol{q}}, \tag{39} $$

[2] This substitution is not strictly necessary. The final minimal number of dynamic coefficients remains the same, i.e., 4. As discussed below, there are multiple different minimal parametrizations, while the minimal dimension of all possible parametrizations is unique.

with

$$\boxed{\boldsymbol{M}_p(\boldsymbol{q}) = \begin{pmatrix} \rho_1 + \rho_2 \cos^2 q_2 & 0 \\ 0 & \rho_3 \end{pmatrix}} \tag{40}$$

having defined the dynamic coefficients

$$\begin{aligned} \rho_1 &= I_{px} \\ \rho_2 &= I_{py} + m_p\, a_2^2 - I_{px} \\ \rho_3 &= I_{pz} + m_p\, a_2^2. \end{aligned} \tag{41}$$

Using the Christoffel symbols, we obtain also the Coriolis and centrifugal terms

$$\boxed{\boldsymbol{c}_p(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} -\rho_2 \sin(2q_2)\, \dot{q}_1 \dot{q}_2 \\ \dfrac{\rho_2}{2} \sin(2q_2)\, \dot{q}_1^2 \end{pmatrix} = \begin{pmatrix} -\rho_2 \sin q_2 \cos q_2\, \dot{q}_2 & -\rho_2 \sin q_2 \cos q_2\, \dot{q}_1 \\ \rho_2 \sin q_2 \cos q_2\, \dot{q}_1 & 0 \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} = \boldsymbol{C}_p(\boldsymbol{q}, \dot{\boldsymbol{q}})\, \dot{\boldsymbol{q}}} \tag{42}$$

with the factorization matrix $\boldsymbol{C}_p$ satisfying the skew-symmetry of $\dot{\boldsymbol{M}}_p - 2\boldsymbol{C}_p$.

As for the potential energy of the payload due to gravity, being ${}^0\boldsymbol{g}^T = \begin{pmatrix} 0 & 0 & -g \end{pmatrix}$, with $g = 9.81$ m/s^2, and using (36), we have

$$\mathcal{U}_p = -m_p\, {}^0\boldsymbol{g}^{T\,0}\boldsymbol{p}_p = m_p g \left(d_1 + a_2 \sin q_2 \right),$$

and so

$$\boxed{\boldsymbol{g}_p(\boldsymbol{q}) = \begin{pmatrix} 0 \\ m_p\, g a_2 \cos q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \rho_4\, g a_2 \cos q_2 \end{pmatrix}} \tag{43}$$

having introduced a fourth (and last) dynamic coefficient $\rho_4 = m_p$.

Therefore, we can rewrite the dynamic terms related to the payload in (35) in a linearly parameterized form as

$$\boldsymbol{M}_p(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}_p(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}_p(\boldsymbol{q}) = \boldsymbol{Y}_p(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})\, \boldsymbol{\rho}_p$$

with the regressor

$$\boxed{\boldsymbol{Y}_p = \begin{pmatrix} \ddot{q}_1 & \cos^2 q_2\, \ddot{q}_1 - \sin(2q_2)\, \dot{q}_1 \dot{q}_2 & 0 & 0 \\ 0 & \sin q_2 \cos q_2\, \dot{q}_1^2 & \ddot{q}_2 & a_2 g\, \cos q_2 \end{pmatrix}} \tag{44}$$

and the *minimal* set of $r = 4$ dynamic coefficients

$$\boxed{\boldsymbol{\rho}_p = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \end{pmatrix} = \begin{pmatrix} I_{px} \\ I_{py} + m_p\, a_2^2 - I_{px} \\ I_{pz} + m_p\, a_2^2 \\ m_p \end{pmatrix}} \tag{45}$$

Observing the minimal set (45) of dynamic coefficients, one can immediately see that there is a one-to-one correspondence with the 4 relevant dynamic parameters of the payload, namely m_p, I_{px}, I_{py}, and I_{pz} (being the CoM of the payload in a known position). We have

$$\boldsymbol{\rho}_p = \begin{pmatrix} I_{px} \\ I_{py} + m_p\, a_2^2 - I_{px} \\ I_{pz} + m_p\, a_2^2 \\ m_p \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & a_2^2 \\ 0 & 0 & 1 & a_2^2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} I_{px} \\ I_{py} \\ I_{pz} \\ m_p \end{pmatrix} = \boldsymbol{T}\, \boldsymbol{\pi}_p, \tag{46}$$

with $\boldsymbol{T}$ being a constant, invertible and known matrix. When the dynamic model is linearly parametrized with $\boldsymbol{\pi}_p \in \mathbb{R}^4$, the associated regressor is then

$$\begin{aligned} \boldsymbol{Y}_\pi &= \boldsymbol{Y}_p \boldsymbol{T} \\ &= \begin{pmatrix} \ddot{q}_1 - \cos^2 q_2\, \ddot{q}_1 + \sin(2q_2)\, \dot{q}_1 & \cos^2 q_2\, \ddot{q}_1 - \sin(2q_2)\, \dot{q}_1 \dot{q}_2 & 0 & a_2^2 \left(\cos^2 q_2\, \ddot{q}_1 - \sin(2q_2)\, \dot{q}_1 \dot{q}_2 \right) \\ -\sin q_2 \cos q_2\, \dot{q}_1^2 & \sin q_2 \cos q_2\, \dot{q}_1^2 & \ddot{q}_2 & a_2 g\, \cos q_2 + a_2^2 \left(\ddot{q}_2 + \sin q_2 \cos q_2\, \dot{q}_1^2 \right) \end{pmatrix}. \end{aligned} \tag{47}$$

Although the result is the same, the amount of real-time computations for an adaptive control law using the regressor $\boldsymbol{Y}_\pi$ in (47) is clearly larger than when using $\boldsymbol{Y}_p$ in (44).

An adaptive control law for asymptotic tracking of a desired trajectory $\boldsymbol{q}_d(t)$ law that uses the knowledge already available on the dynamics of the robot without payload takes then the following form

$$\begin{aligned}\boldsymbol{\tau} &= \boldsymbol{M}_0(\boldsymbol{q})\ddot{\boldsymbol{q}}_r + \boldsymbol{C}_0(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_r + \boldsymbol{g}_0(\boldsymbol{q}) + \boldsymbol{Y}_p(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r)\,\widehat{\boldsymbol{\rho}}_p + \boldsymbol{K}_P\boldsymbol{e} + \boldsymbol{K}_D\dot{\boldsymbol{e}} & (48)\\ \dot{\widehat{\boldsymbol{\rho}}}_p &= \boldsymbol{\Gamma}\,\boldsymbol{Y}_p^T(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r)\,(\dot{\boldsymbol{q}}_r - \dot{\boldsymbol{q}})\,, & (49)\end{aligned}$$

with

$$\boldsymbol{e} = \boldsymbol{q}_d - \boldsymbol{q} \quad \dot{\boldsymbol{e}} = \dot{\boldsymbol{q}}_d - \dot{\boldsymbol{q}} \quad \text{diagonal } \boldsymbol{K}_P > 0, \boldsymbol{K}_D > 0, \boldsymbol{\Gamma} > 0 \quad \dot{\boldsymbol{q}}_r = \dot{\boldsymbol{q}}_d + \boldsymbol{\Lambda}\boldsymbol{e} \quad \boldsymbol{\Lambda} = \boldsymbol{K}_D^{-1}\boldsymbol{K}_P$$

and

$$\begin{aligned}\boldsymbol{Y}_p(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r) &= \boldsymbol{M}_p(\boldsymbol{q})\ddot{\boldsymbol{q}}_r + \boldsymbol{C}_p(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_r + \boldsymbol{g}_p(\boldsymbol{q})\\ &= \begin{pmatrix} \ddot{q}_{r1} & \cos^2 q_2\,\ddot{q}_{r1} - \sin q_2\cos q_2\,(\dot{q}_2\dot{q}_{r1} + \dot{q}_1\dot{q}_{r2}) & 0 & 0\\ 0 & \sin q_2\cos q_2\,\dot{q}_1\dot{q}_{r1} & \ddot{q}_{r2} & a_2 g\,\cos q_2\end{pmatrix},\end{aligned} \qquad (50)$$

where all the dynamic terms related to the payload have been defined. The known terms in (48), those with the subscript 0, need not to be described. Note also that $\boldsymbol{K}_D\,(\dot{\boldsymbol{q}}_r - \dot{\boldsymbol{q}}) = \boldsymbol{K}_D\,(\dot{\boldsymbol{q}}_d + \boldsymbol{\Lambda}\boldsymbol{e} - \dot{\boldsymbol{q}}) = \boldsymbol{K}_D\dot{\boldsymbol{e}} + \boldsymbol{K}_P\boldsymbol{e}$.

Being the set of dynamic coefficients $\boldsymbol{\rho}_p$ in (45) minimal, also the number of states $\widehat{\boldsymbol{\rho}}_p$ of the dynamic control law (48)–(49) will be the minimum possible. Note that if the trajectory is persistently exciting (in the nonlinear setting), we may also have convergence of the estimates $\widehat{\boldsymbol{\rho}}_p$ to the true value. Moreover, using the inverse of matrix $\boldsymbol{T}$ in (46), from the estimate of the four dynamic coefficients $\boldsymbol{\rho}_p$ one can extract also the estimated values of the four dynamic parameters $\boldsymbol{\pi}_p$ of the payload, i.e.,

$$\widehat{\boldsymbol{\pi}}_p = \begin{pmatrix}\widehat{I}_{px}\\ \widehat{I}_{py}\\ \widehat{I}_{pz}\\ \widehat{m}_p\end{pmatrix} = \boldsymbol{T}^{-1}\,\widehat{\boldsymbol{\rho}}_p = \begin{pmatrix}1 & 0 & 0 & 0\\ 1 & 1 & 0 & -a_2^2\\ 0 & 0 & 1 & -a_2^2\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\widehat{\rho}_1\\ \widehat{\rho}_2\\ \widehat{\rho}_3\\ \widehat{\rho}_4\end{pmatrix} = \begin{pmatrix}\widehat{\rho}_1\\ \widehat{\rho}_1 + \widehat{\rho}_2 - a_2^2\,\widehat{\rho}_4\\ \widehat{\rho}_3 - a_2^2\,\widehat{\rho}_4\\ \widehat{\rho}_4\end{pmatrix}.$$

The stability proof of the adaptive tracking controller (48)–(49) is a small variant of the classical one, when all dynamic parameters/coefficients are unknown. Consider in fact the same positive definite Lyapunov candidate

$$V = \frac{1}{2}\,\boldsymbol{s}^T\boldsymbol{M}(\boldsymbol{q})\boldsymbol{s} + \boldsymbol{e}^T\boldsymbol{K}_P\boldsymbol{e} + \frac{1}{2}\,\widetilde{\boldsymbol{\rho}}_p^T\,\boldsymbol{\Gamma}^{-1}\widetilde{\boldsymbol{\rho}}_p \geq 0, \qquad (51)$$

where $\boldsymbol{s} = \dot{\boldsymbol{q}}_r - \dot{\boldsymbol{q}} = \dot{\boldsymbol{e}} + \boldsymbol{\Lambda}\boldsymbol{e}$, $\widetilde{\boldsymbol{\rho}}_p = \boldsymbol{\rho}_p - \widehat{\boldsymbol{\rho}}_p$. Note that only the unknown dynamic coefficients and their estimates that need to be updated online appear in the Lyapunov candidate. The time derivative of (51) is

$$\dot{V} = \frac{1}{2}\,\boldsymbol{s}^T\dot{\boldsymbol{M}}(\boldsymbol{q})\boldsymbol{s} + \boldsymbol{s}^T\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{s}} + 2\,\boldsymbol{e}^T\boldsymbol{K}_P\dot{\boldsymbol{e}} - \widetilde{\boldsymbol{\rho}}_p^T\,\boldsymbol{\Gamma}^{-1}\dot{\widehat{\boldsymbol{\rho}}}_p. \qquad (52)$$

The closed-loop dynamics is

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{Y}_0(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r)\,\boldsymbol{\rho}_0 + \boldsymbol{Y}_p(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r)\,\widehat{\boldsymbol{\rho}}_p + \boldsymbol{K}_P\boldsymbol{e} + \boldsymbol{K}_D\dot{\boldsymbol{e}} \qquad (53)$$

Subtracting both sides of eq. (53) from the identity

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}}_r + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_r + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{Y}_0(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r)\,\boldsymbol{\rho}_0 + \boldsymbol{Y}_p(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r)\,\boldsymbol{\rho}_p$$

yields

$$\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{s}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\boldsymbol{s} = \boldsymbol{Y}_p(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r)\,\widetilde{\boldsymbol{\rho}}_p - \boldsymbol{K}_P\boldsymbol{e} - \boldsymbol{K}_D\dot{\boldsymbol{e}}. \qquad (54)$$

Substituting $\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{s}}$ from (54) into (52) and using the update law $\dot{\widehat{\boldsymbol{\rho}}}_p = \boldsymbol{\Gamma}\,\boldsymbol{Y}_p^T(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r)\boldsymbol{s}$ gives

$$\begin{aligned}\dot{V} &= \frac{1}{2}\,\boldsymbol{s}^T\big(\dot{\boldsymbol{M}}(\boldsymbol{q}) - 2\,\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\big)\boldsymbol{s} + \boldsymbol{s}^T\boldsymbol{Y}_p(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r)\,\widetilde{\boldsymbol{\rho}}_p - \boldsymbol{s}^T(\boldsymbol{K}_P\boldsymbol{e} + \boldsymbol{K}_D\dot{\boldsymbol{e}})\\ &\quad + 2\,\boldsymbol{e}^T\boldsymbol{K}_P\dot{\boldsymbol{e}} - \widetilde{\boldsymbol{\rho}}_p^T\,\boldsymbol{Y}_p^T(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r)\boldsymbol{s} = -\big(\dot{\boldsymbol{e}} + \boldsymbol{K}_D^{-1}\boldsymbol{K}_P\,\boldsymbol{e}\big)^T(\boldsymbol{K}_P\boldsymbol{e} + \boldsymbol{K}_D\dot{\boldsymbol{e}}) + 2\,\boldsymbol{e}^T\boldsymbol{K}_P\dot{\boldsymbol{e}}\\ &= -\dot{\boldsymbol{e}}^T\boldsymbol{K}_D\,\dot{\boldsymbol{e}} - \boldsymbol{e}^T\boldsymbol{K}_P\boldsymbol{K}_D^{-1}\boldsymbol{K}_P\,\boldsymbol{e} \leq 0,\end{aligned}$$

where we used the skew-symmetry of $\dot{\boldsymbol{M}} - 2\,\boldsymbol{C}$ and the diagonality of the gain matrices. The proof is completed by Barbalat lemma and LaSalle theorem, showing convergence to $\boldsymbol{e} = \dot{\boldsymbol{e}} = \boldsymbol{0}$ and boundedness of the estimation error $\widetilde{\boldsymbol{\rho}}_p$.

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