Robotics 2
June 12, 2026

Exercise 1 [students with midterm should skip this]

Consider the RRP planar robot in Fig. 1, where the Denavit-Hartenberg (DH) frames of the base and the end-effector are also shown. Assume that the center of mass of each link is placed along one of the DH axes attached to the link.

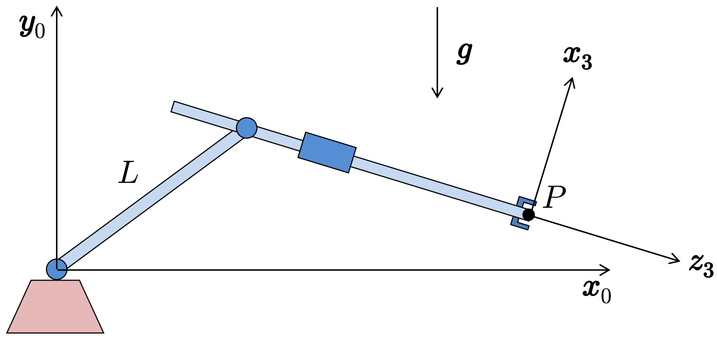

Figure 1: A RRP robot moving in a vertical plane

a) Using the moving frames algorithm (based on DH frames), compute the robot inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ in symbolic terms.

b) Determine all free equilibrium configurations $\boldsymbol{q}_e$ of this robot. Are these configurations maxima or minima of the potential energy $\mathcal{U}(\boldsymbol{q})$?

c) Provide a linear factorization of the gravity vector in the model in terms of dynamic coefficients as

$$\boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{Y}_g(\boldsymbol{q})\boldsymbol{\rho}_g. \tag{1}$$

d) Let the three robot links have masses $m_1 \gg \max\{m_2, m_3\}$. Assume that point P is fixed at $\boldsymbol{p}_0 = (p_x, 0)$, for some $p_x \gg L > 0$. In this condition, find a configuration $\boldsymbol{q}_m$ that provides the constrained minimum of the potential energy $\mathcal{U}(\boldsymbol{q})$. How would you command the joint velocity $\dot{\boldsymbol{q}}$ to get to $\boldsymbol{q}_m$ with a self-motion that keeps the position of P fixed at $\boldsymbol{p}_0$?

Exercise 2

The dynamic equation of an actuated single link moving in the vertical plane is given by

$$I\ddot{q} + md_c g \sin q = \tau. \tag{2}$$

The standard dynamic parameters I and md_c (the only two that appear in the model) are given or have been previously identified. Then, an unknown payload is added, with mass m_p, center of mass at the link tip, and barycentric inertia I_p. Assume that the link length l and the gravity acceleration g are known.

a) Discuss whether or not the two payload parameters m_p and I_p can be separately identified: if so, describe how; if not, explain why and what can be identified instead in a reliable way.

b) Repeat the analysis of the identification problem in a) when the link is moving on a horizontal plane.

c) Suppose now that m, d_c and I are unknown and that the link moves again under gravity with the unknown tip payload attached. Repeat the complete analysis of the identification problem.

Exercise 3

The end-effector of the RP robot in Fig. 2 is constrained to move on a physical guide described by the Cartesian line $x + y = L$, with $L > 0$.

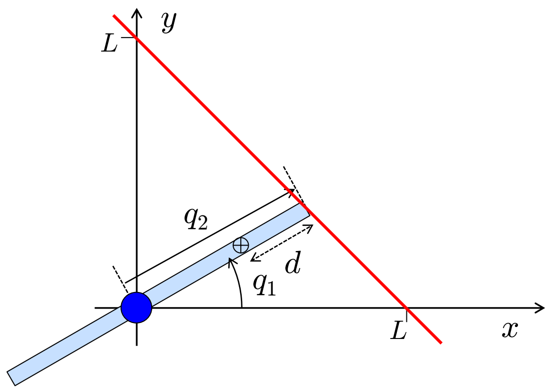

Figure 2: An RP robot moving on a horizontal plane, with its end-effector constrained to a linear guide

a) Derive the expression of a globally valid *reduced* robot dynamics that automatically satisfies the constraint. Design then a nominal joint torque command $\boldsymbol{\tau}_d \in \mathbb{R}^2$ so that the tangent velocity to the constraint keeps a constant value $u_d \neq 0$ while the normal force is being zeroed, i.e., $\lambda_d = 0$.

b) Suppose next that the initial velocity of the robot is $\dot{\boldsymbol{q}}(0) = \mathbf{0}$, that a normal force $\lambda(0) = \lambda_0 > 0$ is being applied to the guide, and that $\ddot{\boldsymbol{q}}(0) = \mathbf{0}$ is being imposed. In this condition, which is the symbolic expression of the joint torque $\boldsymbol{\tau}(0) = \boldsymbol{\tau}_0$ being applied at $t = 0$?

c) How would you modify the joint torque command $\boldsymbol{\tau}$ by using feedback so to recover any initial error on u and λ? Explain if and how a force/torque sensor mounted at the tip is needed for this task.

[full exam (all Exercises): 270 minutes (4,5 hours); open books]

[students with midterm (only Ex #2 and #3): 150 minutes (2,5 hours); open books]

Solution

June 12, 2026

Exercise 1

Part a) Figure 3 shows a possible assignment of DH frames, together with the definition of all relevant dynamic parameters of the considered RRP planar robot. Note that $d_{ci} \geq 0$, for all $i = 1, 2, 3$. The DH parameters are given in Tab. 1.

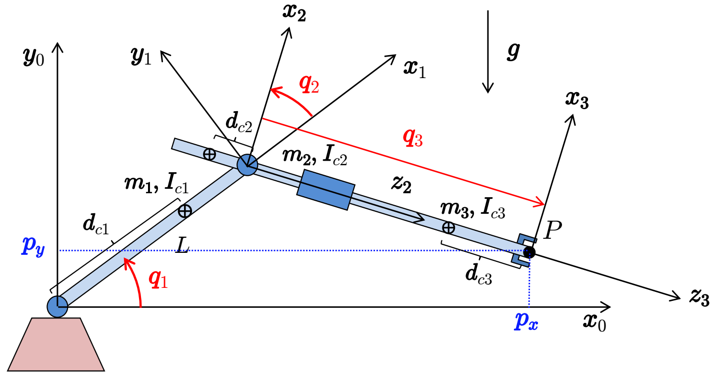

Figure 3: A possible set of DH frames and definition of dynamic parameters for the RRP robot of Fig. 1

i	α_i	a_i	d_i	θ_i
1	0	L	0	q_1
2	$\pi/2$	0	0	q_2
3	0	0	q_3	0

Table 1: DH parameters for the frame assignment in Fig. 3

To use the moving frame algorithm for computing the robot kinetic energy $\mathcal{T}$, and thus $\boldsymbol{M}(\boldsymbol{q})$, we need the three DH rotation matrices

$$ {}^0\boldsymbol{R}_1(q_1) = \begin{pmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad {}^1\boldsymbol{R}_2(q_2) = \begin{pmatrix} c_2 & 0 & s_2 \\ s_2 & 0 & -c_2 \\ 0 & 1 & 0 \end{pmatrix} \qquad {}^2\boldsymbol{R}_3 = \boldsymbol{I}_{3\times 3}, $$

as well as the quantities ${}^{i-1}\boldsymbol{r}_{i-1,i}$ (relative position between O_{i-1} and O_i, expressed in frame $i-1$) and ${}^i\boldsymbol{r}_{ci}$ (position of the center of mass, constant when expressed in frame i), for $i = 1, 2, 3$:

$$ {}^0\boldsymbol{r}_{0,1} = \begin{pmatrix} L\,c_1 \\ L\,s_1 \\ 0 \end{pmatrix} \qquad {}^1\boldsymbol{r}_{1,2} = \boldsymbol{0} \qquad {}^2\boldsymbol{r}_{2,3} = \begin{pmatrix} 0 \\ 0 \\ q_3 \end{pmatrix} $$

$$ {}^1\boldsymbol{r}_{c1} = \begin{pmatrix} -L + d_{c1} \\ 0 \\ 0 \end{pmatrix} \qquad {}^2\boldsymbol{r}_{c2} = \begin{pmatrix} 0 \\ 0 \\ -d_{c2} \end{pmatrix} \qquad {}^3\boldsymbol{r}_{c3} = \begin{pmatrix} 0 \\ 0 \\ -d_{c3} \end{pmatrix}. $$

With the basic equations

$$
\begin{aligned}
{}^{i}\boldsymbol{\omega}_i &= {}^{i-1}\boldsymbol{R}_i^T(q_i)\Big({}^{i-1}\boldsymbol{\omega}_{i-1} + (1-\sigma_i)\dot{q}_i\,{}^{i-1}\boldsymbol{z}_{i-1}\Big)\\
{}^{i}\boldsymbol{v}_i &= {}^{i-1}\boldsymbol{R}_i^T(q_i)\Big({}^{i-1}\boldsymbol{v}_{i-1} + \sigma_i\dot{q}_i\,{}^{i-1}\boldsymbol{z}_{i-1} + {}^{i-1}\boldsymbol{\omega}_i \times {}^{i-1}\boldsymbol{r}_{i-1,i}\Big)\\
{}^{i}\boldsymbol{v}_{ci} &= {}^{i}\boldsymbol{v}_i + {}^{i}\boldsymbol{\omega}_i \times {}^{i}\boldsymbol{r}_{ci}\\
\mathcal{T}_i &= \frac{1}{2}\,m_i\,{}^{i}\boldsymbol{v}_{ci}^T\,{}^{i}\boldsymbol{v}_{ci} + \frac{1}{2}\,{}^{i}\boldsymbol{\omega}_i^T\,{}^{i}\boldsymbol{I}_i\,\boldsymbol{\omega}_i,
\end{aligned}
$$

where ${}^{i}\boldsymbol{z}_i = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$ and ${}^{i}\boldsymbol{I}_i$ is the constant (symmetric) inertia matrix of link i, for any i, the algorithm proceeds as follows, starting with zero velocity and angular velocity of the base.

Link 1 (joint 1 revolute, $\sigma_1 = 0$)

$$
\begin{aligned}
{}^{1}\boldsymbol{\omega}_1 &= {}^{0}\boldsymbol{R}_1^T(q_1)\left({}^{0}\boldsymbol{\omega}_0 + \dot{q}_1\begin{pmatrix}0\\0\\1\end{pmatrix}\right) = \begin{pmatrix}0\\0\\\dot{q}_1\end{pmatrix}\\
{}^{1}\boldsymbol{v}_1 &= {}^{0}\boldsymbol{R}_1^T(q_1)\left({}^{0}\boldsymbol{v}_0 + \begin{pmatrix}0\\0\\\dot{q}_1\end{pmatrix}\times\begin{pmatrix}L\,c_1\\L\,s_1\\0\end{pmatrix}\right) = \begin{pmatrix}0\\L\,\dot{q}_1\\0\end{pmatrix}\\
{}^{1}\boldsymbol{v}_{c1} &= \begin{pmatrix}0\\L\,\dot{q}_1\\0\end{pmatrix} + \begin{pmatrix}0\\0\\\dot{q}_1\end{pmatrix}\times\begin{pmatrix}-L+d_{c1}\\0\\0\end{pmatrix} = \begin{pmatrix}0\\d_{c1}\dot{q}_1\\0\end{pmatrix}\\
\mathcal{T}_1 &= \frac{1}{2}\left(I_{c1} + m_1 d_{c1}^2\right)\dot{q}_1^2 \qquad\qquad (\ldots\text{as expected!}),
\end{aligned}
$$

being $I_{c1} = {}^{1}I_{1,zz}$.

Link 2 (joint 2 revolute, $\sigma_2 = 0$)

$$
\begin{aligned}
{}^{2}\boldsymbol{\omega}_2 &= {}^{1}\boldsymbol{R}_2^T(q_2)\left({}^{1}\boldsymbol{\omega}_1 + \dot{q}_2\begin{pmatrix}0\\0\\1\end{pmatrix}\right) = \begin{pmatrix}0\\\dot{q}_1+\dot{q}_2\\0\end{pmatrix}\\
{}^{2}\boldsymbol{v}_2 &= {}^{1}\boldsymbol{R}_2^T(q_2)\left({}^{1}\boldsymbol{v}_1 + \begin{pmatrix}0\\0\\\dot{q}_1+\dot{q}_2\end{pmatrix}\times\boldsymbol{0}\right) = \begin{pmatrix}L\,s_2\,\dot{q}_1\\0\\-L\,c_2\,\dot{q}_1\end{pmatrix}\\
{}^{2}\boldsymbol{v}_{c2} &= \begin{pmatrix}L\,s_2\,\dot{q}_1\\0\\-L\,c_2\,\dot{q}_1\end{pmatrix} + \begin{pmatrix}0\\\dot{q}_1+\dot{q}_2\\0\end{pmatrix}\times\begin{pmatrix}0\\0\\-d_{c2}\end{pmatrix} = \begin{pmatrix}L\,s_2\,\dot{q}_1 - d_{c2}\,(\dot{q}_1+\dot{q}_2)\\0\\-L\,c_2\,\dot{q}_1\end{pmatrix}\\
\mathcal{T}_2 &= \frac{1}{2}\left(I_{c2} + m_2 d_{c2}^2\right)(\dot{q}_1+\dot{q}_2)^2 + \frac{1}{2}\,m_2\left(L^2\dot{q}_1^2 \;-2\,L\,d_{c2}\,s_2\,\dot{q}_1\,(\dot{q}_1+\dot{q}_2)\right),
\end{aligned}
$$

being $I_{c2} = {}^{2}I_{2,yy}$.

Link 3 (joint 3 prismatic, $\sigma_3 = 1$)

$$
\begin{aligned}
{}^3\boldsymbol{\omega}_3 &= {}^2\boldsymbol{R}_3^T \begin{pmatrix} 0 \\ \dot{q}_1 + \dot{q}_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \dot{q}_1 + \dot{q}_2 \\ 0 \end{pmatrix} \\
{}^3\boldsymbol{v}_3 &= {}^2\boldsymbol{R}_3^T \left({}^2\boldsymbol{v}_2 + \dot{q}_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ \dot{q}_1 + \dot{q}_2 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ q_3 \end{pmatrix} \right) = \begin{pmatrix} L\, s_2\, \dot{q}_1 + q_3\,(\dot{q}_1 + \dot{q}_2) \\ 0 \\ -L\, c_2\, \dot{q}_1 + \dot{q}_3 \end{pmatrix} \\
{}^3\boldsymbol{v}_{c3} &= \begin{pmatrix} L\, s_2\, \dot{q}_1 + q_3\,(\dot{q}_1 + \dot{q}_2) \\ 0 \\ -L\, c_2\, \dot{q}_1 + \dot{q}_3 \end{pmatrix} + \begin{pmatrix} 0 \\ \dot{q}_1 + \dot{q}_2 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ -d_{c3} \end{pmatrix} = \begin{pmatrix} L\, s_2\, \dot{q}_1 + (q_3 - d_{c3})\,(\dot{q}_1 + \dot{q}_2) \\ 0 \\ -L\, c_2\, \dot{q}_1 + \dot{q}_3 \end{pmatrix} \\
\mathcal{T}_3 &= \frac{1}{2} \left(I_{c3} + m_3\,(q_3 - d_{c3})^2 \right) (\dot{q}_1 + \dot{q}_2)^2 + \frac{1}{2} m_3 \left(L^2 \dot{q}_1^2 + \dot{q}_3^2 - 2\, L\, c_2\, \dot{q}_1 \dot{q}_3 + 2\, L\,(q_3 - d_{c3})\, s_2\, \dot{q}_1\,(\dot{q}_1 + \dot{q}_2) \right),
\end{aligned}
$$

being $I_{c3} = {}^3I_{3,yy}$.

Summarizing, we have

$$
\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 = \frac{1}{2}\, \dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q}) \dot{\boldsymbol{q}}
$$

with

$$
\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix}
\begin{array}{c} I_{c1} + m_1 d_{c1}^2 + I_{c2} + m_2 d_{c2}^2 + (m_2 + m_3)\, L^2 \\ -\, 2\, m_2\, L\, d_{c2}\, s_2 + I_{c3} + m_3\,(q_3 - d_{c3})^2 \\ +\, 2\, m_3\, L\,(q_3 - d_{c3})\, s_2 \end{array} & symm & symm \\
\begin{array}{c} I_{c2} + m_2 d_{c2}^2 - m_2\, L\, d_{c2}\, s_2 \\ +\, I_{c3} + m_3\,(q_3 - d_{c3})^2 + m_3\, L\,(q_3 - d_{c3})\, s_2 \end{array} & I_{c2} + m_2 d_{c2}^2 + I_{c3} + m_3\,(q_3 - d_{c3})^2 & symm \\
-m_3\, L\, c_2 & 0 & m_3
\end{pmatrix}
$$

Part b) First, we need to compute the robot potential energy due to gravity. For the i-th link, we have

$$
\mathcal{U}_i = -m_i\, {}^0\boldsymbol{g}^T\, {}^0\boldsymbol{r}_{ci},
$$

where

$$
{}^0\boldsymbol{g} = \begin{pmatrix} 0 \\ -g \\ 0 \end{pmatrix} \quad \text{with } g = 9.81\,\text{m/s}^2 \qquad {}^0\boldsymbol{r}_{ci,h} = \begin{pmatrix} {}^0\boldsymbol{r}_{ci} \\ 1 \end{pmatrix} = \left(\prod_{j=1}^{i} {}^{j-1}\boldsymbol{A}_j(q_j) \right) \begin{pmatrix} {}^i\boldsymbol{r}_{ci} \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_i(\boldsymbol{q})\, {}^i\boldsymbol{r}_{ci,h}.
$$

Thus, one obtains:

$$
\begin{aligned}
\mathcal{U}_1 &= m_1 g\, d_{c1} s_1 \\
\mathcal{U}_2 &= m_2 g \big(L\, s_1 + d_{c2} \sin(q_1 + q_2 + \tfrac{\pi}{2}) \big) = m_2 g\,(L\, s_1 + d_{c2}\, c_{12}) \\
\mathcal{U}_3 &= m_3 g \big(L\, s_1 + (q_3 - d_{c3}) \sin(q_1 + q_2 - \tfrac{\pi}{2}) \big) = m_3 g\,(L\, s_1 - (q_3 - d_{c3})\, c_{12}).
\end{aligned}
$$

From $\mathcal{U} = \mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3$, it follows then

$$
\boldsymbol{g}(\boldsymbol{q}) = \nabla_{\boldsymbol{q}} \mathcal{U}(\boldsymbol{q}) = \left(\frac{\partial \mathcal{U}}{\partial \boldsymbol{q}} \right)^T = g \begin{pmatrix} (m_1 d_{c1} + (m_2 + m_3) L)\, c_1 + (m_3(q_3 - d_{c3}) - m_2 d_{c2})\, s_{12} \\ (m_3(q_3 - d_{c3}) - m_2 d_{c2})\, s_{12} \\ -m_3 c_{12} \end{pmatrix}. \tag{3}
$$

At an equilibrium configuration $\boldsymbol{q}_e$, it should be $\boldsymbol{g}(\boldsymbol{q}_e) = \boldsymbol{0}$. Working backwards through the components, by setting $g_i(\boldsymbol{q}_e) = 0$, $i = 3, 2, 1$, we obtain the conditions

$$
c_{12} = 0 \quad \Rightarrow \quad s_{12} = \pm 1 \quad \Rightarrow \quad q_3 = d_{c3} + \frac{m_2}{m_3}\, d_{c2} \quad \Rightarrow \quad c_1 = 0,
$$

and thus the four equilibria

$$
\boldsymbol{q}_{e,1} = \begin{pmatrix} \pi/2 \\ 0 \\ d_{c3} + \frac{m_2}{m_3}\, d_{c2} \end{pmatrix} \quad \boldsymbol{q}_{e,2} = \begin{pmatrix} \pi/2 \\ \pi \\ d_{c3} + \frac{m_2}{m_3}\, d_{c2} \end{pmatrix} \quad \boldsymbol{q}_{e,3} = \begin{pmatrix} -\pi/2 \\ 0 \\ d_{c3} + \frac{m_2}{m_3}\, d_{c2} \end{pmatrix} \quad \boldsymbol{q}_{e,4} = \begin{pmatrix} -\pi/2 \\ \pi \\ d_{c3} + \frac{m_2}{m_3}\, d_{c2} \end{pmatrix}.
$$

All these equilibria are stationary points for $\mathcal{U}(\boldsymbol{q})$, since

$$
\nabla_{\boldsymbol{q}} \mathcal{U}(\boldsymbol{q}_e) = \left(\frac{\partial \mathcal{U}}{\partial \boldsymbol{q}} \right)^T \Bigg|_{\boldsymbol{q}=\boldsymbol{q}_e} = \boldsymbol{g}(\boldsymbol{q}_e) = \boldsymbol{0}.
$$

However, the Hessian of $\mathcal{U}(\boldsymbol{q})$ at any $\boldsymbol{q}_e$

$$
\begin{aligned}
\nabla^2_{\boldsymbol{q}} \mathcal{U}(\boldsymbol{q}_e) &= \frac{\partial^2 \mathcal{U}}{\partial \boldsymbol{q}^2} \Bigg|_{\boldsymbol{q}=\boldsymbol{q}_e} \\
&= g \left. \begin{pmatrix} -\left(m_1 d_{c1} + (m_2+m_3)L\right) s_1 + \left(m_3(q_3 - d_{c3}) - m_2 d_{c2}\right) c_{12} & symm & symm \\ \left(m_3(q_3 - d_{c3}) - m_2 d_{c2}\right) c_{12} & \left(m_3(q_3 - d_{c3}) - m_2 d_{c2}\right) c_{12} & symm \\ m_3 s_{12} & m_3 s_{12} & 0 \end{pmatrix} \right|_{\boldsymbol{q}=\boldsymbol{q}_e} \\
&= g \begin{pmatrix} \mp \left(m_1 d_{c1} + (m_2+m_3)L\right) & 0 & \pm m_3 \\ 0 & 0 & \pm m_3 \\ \pm m_3 & \pm m_3 & 0 \end{pmatrix}
\end{aligned}
$$

is an indefinite matrix. Thus, these stationary configurations are neither minima nor maxima of $\mathcal{U}(\boldsymbol{q})$, but rather saddle points. In fact, one can easily see that, by increasing indefinitely (in either direction) the variable q_3 of the prismatic joint when $q_1 = q_2 = 0$, the value of $\mathcal{U}(\boldsymbol{q})$ (in particular, of $\mathcal{U}_3$) becomes unbounded.

Part c) Define the vector of dynamic coefficients $\boldsymbol{\rho}_g \in \mathbb{R}^3$ in (3) as

$$
\begin{aligned}
\rho_{g1} &= m_3 \\
\rho_{g2} &= m_2 d_{c2} + m_3 d_{c3} \\
\rho_{g3} &= m_1 d_{c1} + (m_2 + m_3) L.
\end{aligned}
$$

Then, one can write the desired linear parametrization (4) of the gravity vector as

$$
\boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{Y}_g(\boldsymbol{q}) \boldsymbol{\rho}_g = g \begin{pmatrix} q_3 s_{12} & -s_{12} & c_1 \\ q_3 s_{12} & -s_{12} & 0 \\ -c_{12} & 0 & 0 \end{pmatrix} \begin{pmatrix} \rho_{g1} \\ \rho_{g2} \\ \rho_{g3} \end{pmatrix}. \tag{4}
$$

Part d) The configuration $\boldsymbol{q}_m$ that achieves the minimum of $\mathcal{U}(\boldsymbol{q})$ when the end-effector is constrained to be at some far position $\boldsymbol{p}_0$ on the positive $\boldsymbol{x}_0$ axis can be found by inspection, see Fig. 4. In this configuration, the center of mass of the heaviest (first) link is in its lowest position. Thus, by simple trigonometric analysis,

$$
\boldsymbol{q}_m = \begin{pmatrix} -\dfrac{\pi}{2} \\ -\dfrac{\pi}{2} - \arctan \dfrac{p_x}{L} \\ \sqrt{L^2 + p_x^2} \end{pmatrix}.
$$

When starting from a different constrained configuration, the robot can approach $\boldsymbol{q}_m$ with a self-motion joint velocity command of the projected (anti)gradient type

$$
\dot{\boldsymbol{q}} = -\alpha \left(\boldsymbol{I} - \boldsymbol{J}^{\#}(\boldsymbol{q}) \boldsymbol{J}(\boldsymbol{q}) \right) \nabla_{\boldsymbol{q}} \mathcal{U}(\boldsymbol{q}) = -\alpha \left(\boldsymbol{I} - \boldsymbol{J}^{\#}(\boldsymbol{q}) \boldsymbol{J}(\boldsymbol{q}) \right) \boldsymbol{g}(\boldsymbol{q}), \tag{5}
$$

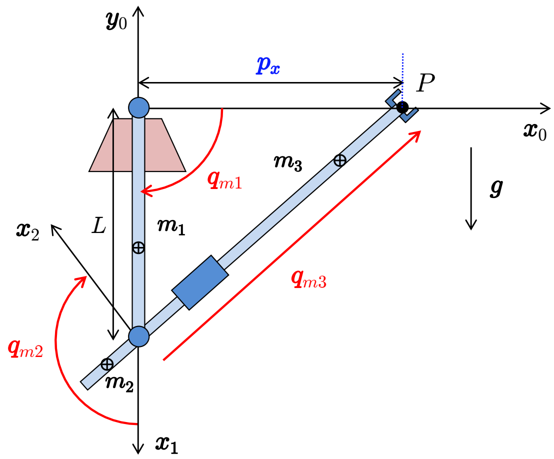

Figure 4: The configuration of the RRP robot yielding the constrained minimum value of $\mathcal{U}(\boldsymbol{q})$

for a sufficiently small $\alpha > 0$. Note that $\boldsymbol{g}(\boldsymbol{q})$, which is given in (3), will never vanish during such self-motion, since no $\boldsymbol{q}_e$ is in the backimage of any Cartesian point of the form $\boldsymbol{p}_0 = (p_x, 0)$. Moreover, being the end-effector position

$$\boldsymbol{p} = \boldsymbol{k}(\boldsymbol{q}) = \begin{pmatrix} Lc_1 + q_3 s_{12} \\ Ls_1 - q_3 c_{12} \end{pmatrix},$$

the Jacobian $\boldsymbol{J}(\boldsymbol{q})$ in (5) is the 2×3 matrix

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{k}}{\partial \boldsymbol{q}} = \begin{pmatrix} -Ls_1 + q_3 c_{12} & q_3 c_{12} & s_{12} \\ Lc_1 + q_3 s_{12} & q_3 s_{12} & -c_{12} \end{pmatrix}. \tag{6}$$

Exercise 2

Part a) In the presence of a payload placed at the tip of a link of length l, we have the additional kinetic and potential energies

$$\mathcal{T}_p = \frac{1}{2}\left(I_p + m_p l^2\right)\dot{q}^2 \qquad \mathcal{U}_p = -m_p l g \cos q.$$

Therefore, the dynamic model (2) becomes

$$\left(I + I_p + m_p l^2\right)\ddot{q} + \left(m d_c + m_p l\right) g \sin q = \tau. \tag{7}$$

Collecting already known terms in a modified torque τ_p on the right, eq. (7) can be rewritten in a linearly parametrized form as

$$\begin{pmatrix} l^2\ddot{q} + lg\sin q & \ddot{q} \end{pmatrix} \begin{pmatrix} m_p \\ I_p \end{pmatrix} = \boldsymbol{Y}_p(q, \ddot{q})\,\boldsymbol{\rho}_p = \tau_p(\tau, q, \ddot{q}) = \tau - I\ddot{q} - m d_c g \sin q. \tag{8}$$

By collecting enough sampled data $(q_k, \ddot{q}_k, \tau_k)$, for $k = 1, \ldots, s$, with the acceleration $\ddot{q}_k$ estimated offline, and defining

$$\boldsymbol{Y}_{p,k} = \begin{pmatrix} l^2\ddot{q}_k + lg\sin q_k & \ddot{q}_k \end{pmatrix} \qquad \tau_{p,k} = \tau_p(\tau_k, q_k, \ddot{q}_k) = \tau_k - I\ddot{q}_k - m d_c g \sin q_k,$$

we can stack these in a matrix/vector format as

$$\overline{\boldsymbol{Y}}_p = \begin{pmatrix} \boldsymbol{Y}_{p,1} \\ \boldsymbol{Y}_{p,2} \\ \vdots \\ \boldsymbol{Y}_{p,m} \end{pmatrix} \qquad \overline{\boldsymbol{\tau}}_p = \begin{pmatrix} \tau_{p,1} \\ \tau_{p,2} \\ \vdots \\ \tau_{p,m} \end{pmatrix}.$$

Therefore, we can solve the s-dimensional overconstrained linear system in the unknowns (I_p, m_p)

$$\overline{\boldsymbol{Y}}_p \begin{pmatrix} m_p \\ I_p \end{pmatrix} = \overline{\boldsymbol{\tau}}_p$$

in a least squares error sense using pseudoinversion as

$$\begin{pmatrix} \widehat{m}_p \\ \widehat{I}_p \end{pmatrix} = \overline{\boldsymbol{Y}}_p^{\#} \, \overline{\boldsymbol{\tau}}_p = \left(\overline{\boldsymbol{Y}}_p^T \overline{\boldsymbol{Y}}_p \right)^{-1} \overline{\boldsymbol{Y}}_p^T \overline{\boldsymbol{\tau}}_p.$$

Note that it has been assumed that $\overline{\boldsymbol{Y}}_p$ has full column rank (= 2), which can be certainly enforced by suitable sampling of the link motion. As a result, we can estimate both dynamic parameters of the payload.

Part b) If the link is moving on a horizontal plane ($g = 0$), then the dynamics (7) becomes now a linear second-order differential equation:

$$\left(I + I_p + m_p l^2 \right) \ddot{q} = \tau \tag{9}$$

Even if the link inertia I (referred to the joint axis) is known in advance, it should be rather intuitive that we cannot identify separately the two dynamic parameters of the payload. In fact, if we attempt to write

$$\begin{pmatrix} l^2 \ddot{q} & \ddot{q} \end{pmatrix} \begin{pmatrix} m_p \\ I_p \end{pmatrix} = \boldsymbol{Y}_p^0(q, \ddot{q}) \, \boldsymbol{\rho}_p = \tau_p^0(\tau, \ddot{q}) = \tau - I \ddot{q}$$

then the resulting stacked regressor $\overline{\boldsymbol{Y}}_p^0$ will always have rank 1 (the first column will always be l times the second one). Thus, pseudoinversion (or also any weighted pseudoinversion) will *arbitrarily* distribute the collected data among the two estimated values $\widehat{m}_p$ and $\widehat{I}_p$. As a matter of fact, we can only estimate the dynamic coefficient $I_p + m_p l^2$ of the payload using the basic equation

$$\begin{pmatrix} \ddot{q} \end{pmatrix} \begin{pmatrix} I_p + m_p l^2 \end{pmatrix} = Y_p^0(\ddot{q}) \, \rho_p^0 = \tau_p^0(\tau, \ddot{q}) = \tau - I \ddot{q} \tag{10}$$

in place of (8), and proceed as before.

Part c) If everything is unknown from the start and the link carries a tip payload, a correct parametrization of the link dynamics (7) is

$$\begin{pmatrix} g \sin q & \ddot{q} \end{pmatrix} \begin{pmatrix} m d_c + m_p l \\ I + I_p + m_p l^2 \end{pmatrix} = \boldsymbol{Y}(q, \ddot{q}) \, \boldsymbol{\rho} = \tau. \tag{11}$$

From here, one can proceed as before by identifying the two dynamic coefficients ρ_1 and ρ_2 as combinations of the link and payload parameters. Note that any attempt to estimate individual parameters will fail. For instance, if one tries to extract m_p and writes

$$\begin{pmatrix} l^2 \ddot{q} + l g \sin q & g \sin q & \ddot{q} \end{pmatrix} \begin{pmatrix} m_p \\ m d_c \\ I + I_p \end{pmatrix} = \boldsymbol{Y}'(q, \ddot{q}) \, \boldsymbol{\rho}' = \tau,$$

it is easy to see that the first element in $\boldsymbol{Y}'$ is a linear combination of the second two. Thus, the stacked regressor matrix $\overline{\boldsymbol{Y}}'$ will never have full column rank.

Exercise 3

Part a) Following the Lagrangian approach, for a robot with n generalized coordinates $\boldsymbol{q}$ and multipliers $\boldsymbol{\lambda}$ weighting the k-dimensional holonomic constraints $\boldsymbol{h}(\boldsymbol{q}) = \boldsymbol{0}$, the dynamic equations (in the absence of gravity) take the form

$$\boldsymbol{M}(\boldsymbol{q}) \ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{\tau} + \boldsymbol{A}^T(\boldsymbol{q}) \boldsymbol{\lambda} \qquad s.t. \quad \boldsymbol{h}(\boldsymbol{q}) = \boldsymbol{0}, \tag{12}$$

with $\boldsymbol{A}(\boldsymbol{q}) = \partial \boldsymbol{h}(\boldsymbol{q}) / \partial \boldsymbol{q}$. A reduced dynamic model is obtained by restricting motion to an r-dimensional configuration space, with $r = n - k$, that is automatically compatible with the constraints $\boldsymbol{h}(\boldsymbol{q}) = \boldsymbol{0}$. The constraints are thus discarded from the formulation. At the same time, it is also possible to eliminate the appearance of the multipliers (i.e., of the generalized forces that arise when attempting to violate the constraints) from the resulting dynamic equations.

In the present case, it is $n = 2$, $k = 1$, and thus $r = 1$. We provide first the terms that appear in (12), namely the robot inertia matrix $\boldsymbol{M}$, the Coriolis and centrifugal vector $\boldsymbol{c}$, and the matrix $\boldsymbol{A}$. The kinetic energy[1] is

$$T = T_1 + T_2 = \frac{1}{2} I_1 \dot{q}_1^2 + \frac{1}{2} \left(I_2 \dot{q}_1^2 + m_2 \boldsymbol{v}_{c2}^T \boldsymbol{v}_{c2} \right).$$

Working with two-dimensional quantities in the plane, since

$$\boldsymbol{p}_{c2} = \begin{pmatrix} (q_2 - d)\cos q_1 \\ (q_2 - d)\sin q_1 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2} = \begin{pmatrix} -(q_2 - d)\sin q_1 \dot{q}_1 + \dot{q}_2 \cos q_1 \\ (q_2 - d)\cos q_1 \dot{q}_1 + \dot{q}_2 \sin q_1 \end{pmatrix} = \boldsymbol{R}(q_1) \begin{pmatrix} \dot{q}_2 \\ q_2 - d \end{pmatrix},$$

it follows that

$$\begin{aligned} T = \frac{1}{2} \left(I_1 + I_2 + m_2 (q_2 - d)^2 \right) \dot{q}_1^2 + \frac{1}{2} m_2 \dot{q}_2^2 &= \frac{1}{2} \dot{\boldsymbol{q}}^T \begin{pmatrix} I_1 + I_2 + m_2 (q_2 - d)^2 & 0 \\ 0 & m_2 \end{pmatrix} \dot{\boldsymbol{q}} \\ &= \frac{1}{2} \dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q}) \dot{\boldsymbol{q}}. \end{aligned}$$

From the elements of the inertia matrix, using the Christoffel symbols, we obtain

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} 2m_2 (q_2 - d)\, \dot{q}_1 \dot{q}_2 \\ -m_2 (q_2 - d)\, \dot{q}_1^2 \end{pmatrix}.$$

The (scalar) Cartesian constraint on the end-effector is

$$h(\boldsymbol{q}) = q_2 \cos q_1 + q_2 \sin q_1 - L = 0.$$

Thus,

$$\boldsymbol{A}(\boldsymbol{q}) = \frac{\partial h(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} q_2 (\cos q_1 - \sin q_1) & \sin q_1 + \cos q_1 \end{pmatrix}.$$

Define the matrix $\boldsymbol{D}(\boldsymbol{q})$ (a row in our case) such that

$$\begin{pmatrix} \boldsymbol{A}(\boldsymbol{q}) \\ \boldsymbol{D}(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} q_2 (\cos q_1 - \sin q_1) & \sin q_1 + \cos q_1 \\ d_1(\boldsymbol{q}) & d_2(\boldsymbol{q}) \end{pmatrix} \tag{13}$$

is nonsingular. A good choice is

$$\boldsymbol{D}(\boldsymbol{q}) = \begin{pmatrix} -\frac{q_2}{2} (\sin q_1 + \cos q_1) & \frac{1}{2} (\cos q_1 - \sin q_1) \end{pmatrix} \quad \Rightarrow \quad \det \begin{pmatrix} \boldsymbol{A}(\boldsymbol{q}) \\ \boldsymbol{D}(\boldsymbol{q}) \end{pmatrix} = q_2.$$

Since the end-effector is constrained to live on $q_2 (\sin q_1 + \cos q_1) = L > 0$, it will always be $q_2 \neq 0$. Thus, the requested nonsingularity of the matrix holds globally as long as the constraint is enforced. This implies that the following derivations will lead to a globally defined reduced model.

Define then the reduced velocity u (a scalar) and its derivative $\dot{u}$ (the reduced acceleration) as

$$u = \boldsymbol{D}(\boldsymbol{q}) \dot{\boldsymbol{q}} = -\frac{q_2}{2} (\sin q_1 + \cos q_1)\, \dot{q}_1 + \frac{1}{2} (\cos q_1 - \sin q_1)\, \dot{q}_2 \tag{14}$$

$$\begin{aligned} \dot{u} = \boldsymbol{D}(\boldsymbol{q}) \ddot{\boldsymbol{q}} + \dot{\boldsymbol{D}}(\boldsymbol{q}) \dot{\boldsymbol{q}} &= -\frac{q_2}{2} (\sin q_1 + \cos q_1)\, \ddot{q}_1 + \frac{1}{2} (\cos q_1 - \sin q_1)\, \ddot{q}_2 \\ &\quad - \frac{q_2}{2} (\cos q_1 - \sin q_1)\, \dot{q}_1^2 - (\sin q_1 + \cos q_1)\, \dot{q}_1 \dot{q}_2. \end{aligned} \tag{15}$$

[1] For simplicity, it is assumed that the first link has its center of mass on the axis of the first joint. Otherwise, if the center of mass is at a distance d_{c1}, simply replace I_1 by $I_1 + m_1 d_{c1}^2$ in the following.

To invert these relations, define

$$\begin{pmatrix} \boldsymbol{E}(\boldsymbol{q}) & \boldsymbol{G}(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}(\boldsymbol{q}) \\ \boldsymbol{D}(\boldsymbol{q}) \end{pmatrix}^{-1} = \begin{pmatrix} \dfrac{\cos q_1 - \sin q_1}{2q_2} & -\dfrac{\sin q_1 + \cos q_1}{q_2} \\ \dfrac{\sin q_1 + \cos q_1}{2} & \cos q_1 - \sin q_1 \end{pmatrix}.$$

We obtain then

$$\begin{aligned} \dot{\boldsymbol{q}} &= \boldsymbol{G}(\boldsymbol{q})\, u = \begin{pmatrix} -\dfrac{\sin q_1 + \cos q_1}{q_2} \\ \cos q_1 - \sin q_1 \end{pmatrix} u \\ \ddot{\boldsymbol{q}} &= \boldsymbol{G}(\boldsymbol{q})\, \dot{u} + \dot{\boldsymbol{G}}(\boldsymbol{q}) u \\ &= \begin{pmatrix} -\dfrac{\sin q_1 + \cos q_1}{q_2} \\ \cos q_1 - \sin q_1 \end{pmatrix} \dot{u} + \begin{pmatrix} \dfrac{(\sin q_1 - \cos q_1)\, \dot{q}_1}{q_2} + \dfrac{(\sin q_1 + \cos q_1)\, \dot{q}_2}{q_2^2} \\ -(\sin q_1 + \cos q_1)\, \dot{q}_1 \end{pmatrix} u. \end{aligned} \tag{16}$$

Based on their definitions, the matrix relations $\boldsymbol{AG} = \boldsymbol{O}$ and $\boldsymbol{AE} = \boldsymbol{I}$ hold for all $\boldsymbol{q}$. Note that in this case we have a symbolic expression for $\boldsymbol{G}$, so that we can compute explicitly $\dot{\boldsymbol{G}}$ without the substitution $\dot{\boldsymbol{G}}u = -\left(\boldsymbol{E}\dot{\boldsymbol{A}} + \boldsymbol{G}\dot{\boldsymbol{D}}\right)\boldsymbol{G}u$.

Premultiplying (12) by $\boldsymbol{G}^T(\boldsymbol{q})$ and substituting the acceleration $\ddot{\boldsymbol{q}}$ from (16) yields

$$\boldsymbol{G}^T(\boldsymbol{q})\boldsymbol{M}(\boldsymbol{q})\boldsymbol{G}(\boldsymbol{q})\, \dot{u} = \boldsymbol{G}^T(\boldsymbol{q}) \left(\boldsymbol{\tau} - \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{G}}(\boldsymbol{q})u - \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})\right), \tag{17}$$

which is the reduced dynamics of the constrained RP robot —a scalar first-order differential equation in v. All terms in (17) have been defined, but we can write more explicitly the leading scalar (the reduced inertia)

$$\boldsymbol{G}^T(\boldsymbol{q})\boldsymbol{M}(\boldsymbol{q})\boldsymbol{G}(\boldsymbol{q}) = \left(I_1 + I_2 + m_2(q_2 - d)^2\right) \left(\frac{\sin q_1 + \cos q_1}{q_2}\right)^2 + m_2 \left(\cos q_1 - \sin q_1\right)^2 q_1 > 0 \quad \forall \boldsymbol{q}.$$

Similarly, when premultiplying (12) by $\boldsymbol{E}^T(\boldsymbol{q})$ one isolates the scalar multiplier λ. Substituting the acceleration $\ddot{\boldsymbol{q}}$ from (16) yields

$$\lambda = \boldsymbol{E}^T(\boldsymbol{q}) \left(\boldsymbol{M}(\boldsymbol{q})\boldsymbol{G}(\boldsymbol{q})\dot{u} + \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{G}}(\boldsymbol{q})u + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{\tau}\right). \tag{18}$$

The required inverse dynamics keeps at zero both the tangent acceleration, i.e., $\dot{u} = \dot{u}_d = 0$ in (17) and (18), and the normal force, i.e., $\lambda = \lambda_d = 0$ in (18), as well as sets $u = u_d$ in (17) and (18). This hybrid task is simultaneously achieved by the joint torque

$$\boldsymbol{\tau}_d = \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{G}}(\boldsymbol{q})\, u_d + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}), \tag{19}$$

where the state $(\boldsymbol{q}, \dot{\boldsymbol{q}})$ at the initial time $t = 0$ satisfies

$$h(\boldsymbol{q}(0)) = q_2(0)\,(\sin q_1(0) + \cos q_1(0)) - L = 0 \tag{20}$$

and

$$\boldsymbol{A}(\boldsymbol{q}(0))\, \dot{\boldsymbol{q}}(0) = q_2(0)\,(\cos q_1(0) - \sin q_1(0))\, \dot{q}_1(0) + (\sin q_1(0) + \cos q_1(0))\, \dot{q}_2(0) = 0, \tag{21}$$

as well as

$$\boldsymbol{D}(\boldsymbol{q}(0))\, \dot{\boldsymbol{q}}(0) = -\frac{q_2(0)}{2}\,(\sin q_1(0) + \cos q_1(0))\, \dot{q}_1(0) + \frac{1}{2}\,(\cos q_1(0) - \sin q_1(0))\, \dot{q}_2(0) = u_d. \tag{22}$$

Once the initial configuration $\boldsymbol{q}(0)$ satisfies the constraint (20), the existence of an initial velocity $\dot{\boldsymbol{q}}(0)$ satisfying (21) and (22) is guaranteed by design from the invertibility of the matrix defined in (13). When these initial conditions are met, the torque $\boldsymbol{\tau}_d$ in (19) will always satisfy the task in nominal conditions. Moreover, it will be always $\dot{\boldsymbol{q}} = \boldsymbol{G}(\boldsymbol{q})u_d$, e.g., in (19).

Part b) Note that $\dot{\boldsymbol{q}} = \mathbf{0}$ automatically satisfies $\boldsymbol{A}(\boldsymbol{q})\dot{\boldsymbol{q}} = \mathbf{0}$ at any time. Setting $\dot{\boldsymbol{q}}(0) = \ddot{\boldsymbol{q}}(0) = \mathbf{0}$ implies also $u(0) = \dot{u}(0) = 0$ from (14) and (15). Thus, the robot is at an equilibrium under the action of a suitable joint torque $\boldsymbol{\tau}(0)$. Replacing these conditions and $\lambda(0) = \lambda_0 \neq 0$ in (17) and (18) provides

$$\boldsymbol{G}^T(\boldsymbol{q}(0))\,\boldsymbol{\tau}(0) = 0 \qquad \text{and} \qquad \boldsymbol{E}^T(\boldsymbol{q}(0))\,\boldsymbol{\tau}(0) = -\lambda_0 \neq 0,$$

and thus

$$\begin{aligned}\boldsymbol{\tau}_0 = \boldsymbol{\tau}(0) &= \begin{pmatrix} \boldsymbol{E}^T(\boldsymbol{q}(0)) \\ \boldsymbol{G}^T(\boldsymbol{q}(0)) \end{pmatrix}^{-1} \begin{pmatrix} -\lambda_0 \\ 0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{E}(\boldsymbol{q}(0)) & \boldsymbol{G}(\boldsymbol{q}(0)) \end{pmatrix}^{-T} \begin{pmatrix} -\lambda_0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \boldsymbol{A}(\boldsymbol{q}(0)) \\ \boldsymbol{D}(\boldsymbol{q}(0)) \end{pmatrix}^{T} \begin{pmatrix} -\lambda_0 \\ 0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}^T(\boldsymbol{q}(0)) & \boldsymbol{D}^T(\boldsymbol{q}(0)) \end{pmatrix} \begin{pmatrix} -\lambda_0 \\ 0 \end{pmatrix} = -\boldsymbol{A}^T(\boldsymbol{q}(0))\lambda_0. \end{aligned} \tag{23}$$

Note that this final expression is coherent with the balancing of forces and torques in static conditions. The presence of $\boldsymbol{A}^T(\boldsymbol{q})$ in this mapping is the constrained version of the duality between velocity and force transformations in free space (i.e., where $\boldsymbol{J}^T(\boldsymbol{q})$ appears).

Part c) Being $u_d \neq 0$ and constant ($\dot{u}_d = 0$) and for a constant $\lambda_d \neq 0$, a control law that uses the feedback errors $e_u = u_d - u$ and $e_\lambda = \lambda_d - \lambda$ is

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})\boldsymbol{G}(\boldsymbol{q})\,k_u e_u + \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{G}}(\boldsymbol{q})\,u + \mathbf{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{A}^T(\boldsymbol{q}) \left(\lambda_d + k_\lambda \int_0^t e_\lambda\, ds\right), \tag{24}$$

with $k_u > 0$, $k_\lambda > 0$ and with the state z of the integral initially set to $z(0) = -e_\lambda(0)/k_\lambda$. Using (24) in (17) leads to the exactly linear error dynamics

$$\dot{u} = k_u e_u \quad \Rightarrow \quad \dot{e}_u + k_u e_u = 0 \quad \Rightarrow \quad e_u(t) = e_u(0)\exp(-k_u t) \quad (u \to u_d \text{ for } t \to \infty)\,.$$

Moreover, using (24) in (18) yields

$$\lambda = \lambda_d + k_\lambda \int_0^t e_\lambda\, ds \quad \Rightarrow \quad e_\lambda + k_\lambda \int_0^t e_\lambda\, ds = 0 \quad \Rightarrow \quad e_\lambda(t) = e_\lambda(0)\exp(-k_\lambda t) \;\; (\lambda \to \lambda_d \text{ for } t \to \infty)\,.$$

In principle, there is no need of a force/torque (F/T) sensor for implementing the control law (24) as long as the model is accurate. In particular, the value of λ can be computed using (18). Note however that this implies the presence of an *algebraic loop*[2] for the feedback control law (24). This can be resolved by considering a discrete-time implementation that introduces (at least) a one-step delay by the sampling time T_c, or $\boldsymbol{\tau}(t_k) = \boldsymbol{\tau}(kT_c) = \boldsymbol{\tau}(\boldsymbol{q}_k, \dot{\boldsymbol{q}}_k, \lambda_{k-1})$ with $\lambda_{k-1} = \lambda((k-1)T_c)$. In alternative, one can extract the information about the value of λ from a F/T sensor (measured at t_{k-1}, used in the control law at t_k), which would also help in robustifying the control approach.

* * * * *

[2]Equation (18) is evaluated for $\lambda(t)$ at time t using also the applied torque $\boldsymbol{\tau}(t)$; in turn, the control law (24) computes $\boldsymbol{\tau}(t)$ at time t using the multiplier $\lambda(t)$ at the same instant. This creates an algebraic loop between cause and effect that needs to be resolved.