Robotics 1

June 10, 2026

Exercise 1

The 6-dof lightweight robot Z1 Pro by Unitree is shown in Fig. 1. The robot has six revolute joints, a spherical wrist, and an elbow offset. Assign a set of frames according to the standard Denavit-Hartenberg (DH) convention, so that all linear parameters a_i and d_i are *nonnegative*. Place the first (base) DH frame on the ground and the origin of the last DH frame at the center of the final flange. Draw the frames in the clearest possible way directly on the extra sheets that have been distributed, and then build the corresponding table of parameters. For the robot configuration shown in the *side view* on the extra sheets, provide also the sign (or, in case, the value) of each joint variable.

Figure 1: The Z1 Pro robot by Unitree

Exercise 2

With reference to the generic situation shown in Fig. 2, consider the following coordinated task for two robots, a 3R arm A and a 2R arm B, both having links of unitary length and moving on a horizontal plane (x, y) —thus, we will use a 2D planar notation for Cartesian vectors and velocities, as well as for rotation matrices. The pose of the base frame of robot B with respect to the base frame of robot A is given by

$$^{A}\boldsymbol{T}_B = \begin{pmatrix} {}^{A}\boldsymbol{R}_B & {}^{A}\boldsymbol{p}_{O_B} \\ 0 \quad 0 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 2.5 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0.5 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1}$$

The configuration of robot A is $\boldsymbol{q}_A = (0, \pi/4, \pi/2)$ [rad], with joint velocity $\dot{\boldsymbol{q}}_A = (\pi/4, -\pi/2, 0)$ [rad/s]. Determine the configuration $\boldsymbol{q}_B$ of robot B and its joint velocity $\dot{\boldsymbol{q}}_B$ so that:

- the position of the end-effector point P_B of robot B with respect to point P_A of robot A, when expressed in the end-effector frame EA of robot A, is

$$^{EA}\boldsymbol{p}_{AB} = \begin{pmatrix} -\sqrt{2}/2 \\ -\sqrt{2}/2 \end{pmatrix}; \tag{2}$$

- the end-effector velocity of the two robots is the same, i.e., $\boldsymbol{v}_A = \boldsymbol{v}_B \in \mathbb{R}^2$;
- in this instantaneous situation, there is no collision between the two robots.

List first the logical steps that you have to follow in order to solve the problem, then provide its numerical solution.

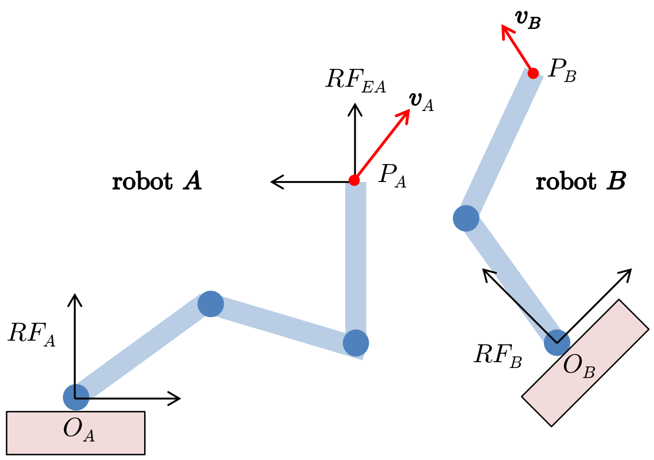

Figure 2: Generic placement and configurations of robots A and B involved in a coordinated task

Exercise 3

A point mass should move on a line in minimum time from rest to rest between $x_i = 0$ and $x_f = 9$ m, under maximum speed and acceleration limits $|\dot{x}| \leq V = 6$ m/s, $|\ddot{x}| \leq A = 4$ m/s^2. The point mass should start immediately at $t_i = 0$ and move away from $x_i = 0$ as fast as possible, since a moving obstacle may possibly enter a small line segment there and produce a collision. Moreover, the mid transfer position $x_m = x_f/2$ is forbidden during the time interval $t \in (t_l, t_u) = (1, 1.6)$ s, since a collision would certainly occur with another moving obstacle traversing orthogonally the line at this point. When the line is permanently free, determine the minimum motion time T and sketch the corresponding optimal speed profile and associated position. Develop then a strategy to complete the assigned task in the least possible total time, without a collision between the point mass and the moving obstacles. Illustrate first qualitatively your solution approach and its motivation. Sketch then the corresponding speed profile and compute the motion time T_{obs}.

[210 minutes (3,5 hours); open books]

Solution

June 10, 2026

Exercise 1

A possible assignment of DH frames that satisfies the additional requirement of nonnegative linear parameters a_i and d_i, for $i = 1, \ldots, 6$, is shown with two different views in Fig. 3. The corresponding parameters are given in Tab. 1, where the last column shows also the signs of the joint variables $q_i = \theta_i$, for $i = 1, \ldots, 6$, in the robot configuration shown in the side view.

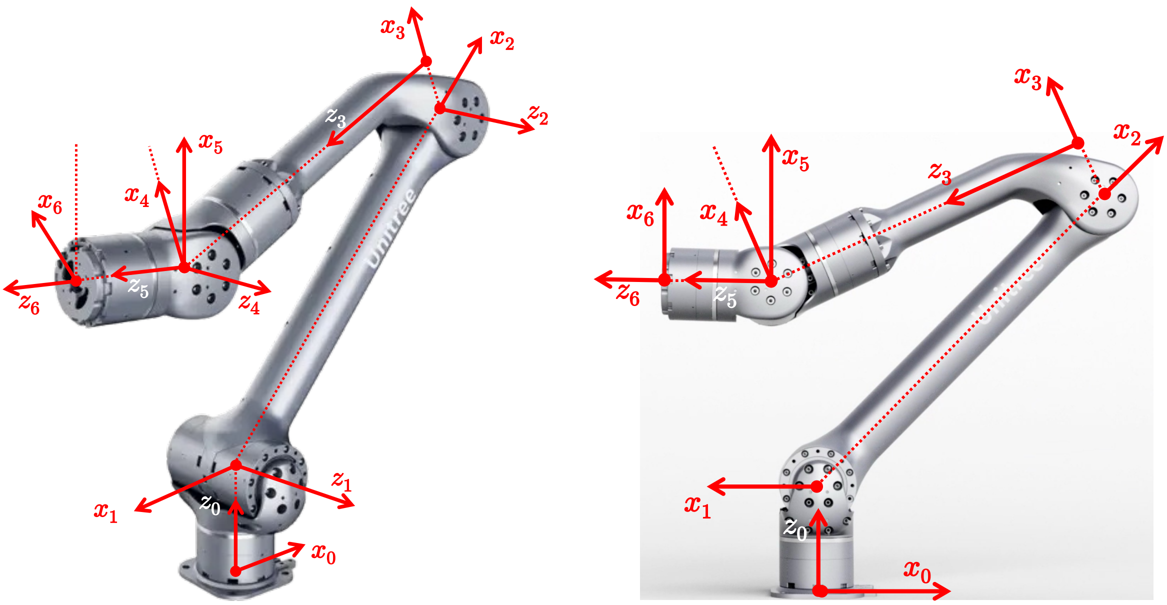

Figure 3: Two views of the assigned DH frames for the Unitree Z1 Pro robot

i	α_i	a_i	d_i	θ_i
1	$-\pi/2$	0	$d_1 > 0$	$q_1\,(= \pi)$
2	0	$a_2 > 0$	0	$q_2\,(< 0)$
3	$-\pi/2$	$a_3 > 0$	0	$q_3\,(> 0)$
4	$\pi/2$	0	$d_4 > 0$	$q_4\,(= 0)$
5	$-\pi/2$	0	0	$q_5\,(< 0)$
6	0	0	$d_6 > 0$	$q_6\,(\geq 0)$

Table 1: DH parameters for the frame assignment in Fig. 3

Exercise 2

Figure 4 shows graphically the solution to the problem. The relevant (applied) position vectors being involved are illustrated in Fig. 5.

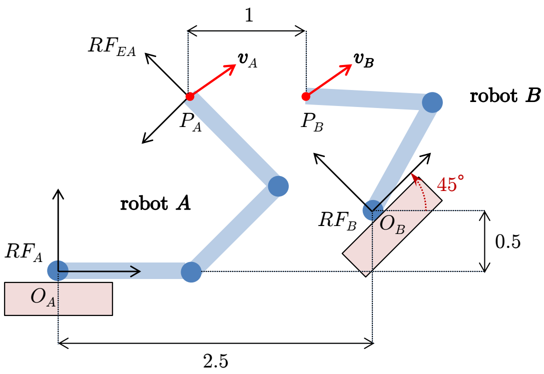

Figure 4: Robots A and B executing the desired coordinated motion with their end-effectors

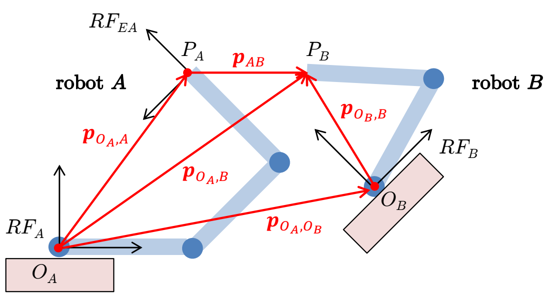

Figure 5: Illustration of the position vectors involved in the solution of the problem

For this, one should[1]:

1. compute through the direct kinematics of robot A the position of P_A as ${}^A\boldsymbol{p}_{O_A,A} = \boldsymbol{k}_A(\boldsymbol{q}_A)$;
2. derive the Jacobian $\boldsymbol{J}_A(\boldsymbol{q}) = \partial \boldsymbol{k}_A / \partial \boldsymbol{q}$ and compute the end-effector velocity of robot A as $\boldsymbol{v}_A = \boldsymbol{J}(\boldsymbol{q}_A)\dot{\boldsymbol{q}}_A$, which is naturally expressed in the base frame RF_A of this robot (i.e., ${}^A\boldsymbol{v}_A$);
3. using the desired ${}^{EA}\boldsymbol{p}_{AB}$ in (2), compute the position ${}^A\boldsymbol{p}_{O_A,B}$ of point P_B expressed in the base frame RF_A (use here homogeneous coordinates and the direct homogeneous transformation matrix $\boldsymbol{T}_A(\boldsymbol{q}_A)$ of robot A); note that ${}^A\boldsymbol{p}_{AB} = {}^A\boldsymbol{p}_{O_A,B} - {}^A\boldsymbol{p}_{O_A,A}$;
4. with ${}^A\boldsymbol{p}_{O_A,B}$ and the constant homogeneous transformation ${}^A\boldsymbol{T}_B$ in (1), compute the desired position ${}^B\boldsymbol{p}_{O_B,B}$ of point P_B expressed in the base frame RF_B of robot B;
5. solve the inverse kinematics problem for robot B is closed form as $\boldsymbol{q}_B = \boldsymbol{k}_B^{-1}({}^B\boldsymbol{p}_{O_B,B})$, choosing the joint solution that avoids collision between the two robots —see again Fig. 4;
6. using $\boldsymbol{v}_A$, write the common velocity of the end-effectors of the two robots as $\boldsymbol{v}_B$ in the base frame RF_B of robot B, i.e., ${}^B\boldsymbol{v}_B$ —use here the rotation matrix ${}^A\boldsymbol{R}_B$ from (1);
7. derive the Jacobian $\boldsymbol{J}_B(\boldsymbol{q}) = \partial \boldsymbol{k}_B / \partial \boldsymbol{q}$ of the direct kinematics ${}^B\boldsymbol{p}_B = \boldsymbol{k}_B(\boldsymbol{q})$ of robot B, evaluate the Jacobian at $\boldsymbol{q}_B$, and solve for the joint velocity as $\dot{\boldsymbol{q}}_B = \boldsymbol{J}_B^{-1}(\boldsymbol{q}_B)\boldsymbol{v}_B$.

[1] For better clarity, we overload the notation of applied vectors, by specifying in their subscript both the starting and the ending point.

Thus, from $\boldsymbol{q}_A = (0, \pi/4, \pi/2)$ [rad] $= (0°, 45°, 90°)$ and $\dot{\boldsymbol{q}}_A = (\pi/4, -\pi/2, 0)$ [rad/s], we compute in sequence:

$$ {}^A\boldsymbol{p}_{O_A,A} = \boldsymbol{k}_A(\boldsymbol{q}_A) = \left.\begin{pmatrix} c_1 + c_{12} + c_{123} \\ s_1 + s_{12} + s_{123} \end{pmatrix}\right|_{\boldsymbol{q}=\boldsymbol{q}_A} = \begin{pmatrix} 1 \\ 1.4142 \end{pmatrix} \text{ [m]} $$

$$ \boldsymbol{J}_A(\boldsymbol{q}_A) = \left.\begin{pmatrix} -(s_1 + s_{12} + s_{123}) & -(s_{12} + s_{123}) & -s_{123} \\ c_1 + c_{12} + c_{123} & c_{12} + c_{123} & c_{123} \end{pmatrix}\right|_{\boldsymbol{q}=\boldsymbol{q}_A} = \begin{pmatrix} -1.4142 & -1.4142 & -0.7071 \\ 1 & 0 & -0.7071 \end{pmatrix} $$

$$ {}^A\boldsymbol{v}_A = \boldsymbol{J}_A(\boldsymbol{q}_A)\dot{\boldsymbol{q}}_A = \boldsymbol{J}_A(\boldsymbol{q}_A)\begin{pmatrix} \pi/4 \\ -\pi/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1.1107 \\ 0.7854 \end{pmatrix} \text{ [m/s]} $$

$$ \boldsymbol{T}_A(\boldsymbol{q}_A) = \begin{pmatrix} \boldsymbol{R}_A(\boldsymbol{q}_A) & \boldsymbol{k}_A(\boldsymbol{q}_A) \\ 0 \quad 0 & 1 \end{pmatrix} = \left.\begin{pmatrix} c_{123} & -s_{123} & p_{A,x} \\ s_{123} & c_{123} & p_{A,y} \\ 0 & 0 & 1 \end{pmatrix}\right|_{\boldsymbol{q}=\boldsymbol{q}_A} = \begin{pmatrix} -0.7071 & -0.7071 & 1 \\ 0.7071 & -0.7071 & 1.4142 \\ 0 & 0 & 1 \end{pmatrix} $$

$$ \begin{aligned} {}^A\boldsymbol{p}_{O_B,B|hom} = \begin{pmatrix} {}^A\boldsymbol{p}_{O_B,B} \\ 1 \end{pmatrix} &= \boldsymbol{T}_A(\boldsymbol{q}_A)\, {}^{EA}\boldsymbol{p}_{AB|hom} = \boldsymbol{T}_A(\boldsymbol{q}_A) \begin{pmatrix} {}^{EA}\boldsymbol{p}_{AB} \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -0.7071 & -0.7071 & 1 \\ 0.7071 & -0.7071 & 1.4142 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -0.7071 \\ -0.7071 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1.4142 \\ 1 \end{pmatrix} \end{aligned} $$

$$ {}^A\boldsymbol{p}_{AB} = {}^A\boldsymbol{p}_{O_A,B} - {}^A\boldsymbol{p}_{O_A,A} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ [m]} \quad \text{(see the relative position of the two end-effectors in Fig. 4)} $$

$$ \begin{aligned} {}^B\boldsymbol{p}_{O_B,B|hom} = \begin{pmatrix} {}^B\boldsymbol{p}_{O_B,B} \\ 1 \end{pmatrix} &= {}^A\boldsymbol{T}_B^{-1}\, {}^A\boldsymbol{p}_{O_A,B|hom} \\ &= \begin{pmatrix} 0.7071 & 0.7071 & -2.1213 \\ -0.7071 & 0.7071 & 1.4142 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1.4142 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.2929 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} p_{B,x} \\ p_{B,y} \\ 1 \end{pmatrix} \end{aligned} $$

Use the closed-form expression of the inverse kinematics of a 2R planar robot:

$$ q_{B,2} = \text{atan2}\{s_2, c_2\} = \text{atan2}\{+\sqrt{1-c_2^2},\, (p_{B,x}^2 + p_{B,y}^2 - 2)/2\} = \text{atan2}\{0.8894, -0.4571\} = 2.0455 \text{ rad} $$

$$ q_{B,1} = \text{atan2}\{p_{B,y}(1+c_2) - p_{B,x}s_2,\, p_{B,x}(1+c_2) - p_{B,y}s_2\} = \text{atan2}\{0.2601, 0.9656\} = 0.2631 \text{ rad} $$

$$ \Rightarrow \qquad \boldsymbol{q}_B = \begin{pmatrix} q_{B,1} \\ q_{B,2} \end{pmatrix} = \begin{pmatrix} 0.2631 \\ 2.0455 \end{pmatrix} \text{ [rad]} = \begin{pmatrix} 15.07° \\ 117.20° \end{pmatrix} $$

Note that the + sign has been chosen in the inverse kinematics solution for $q_{B,2}$. With the − sign, the other inverse solution $\boldsymbol{q}_B = (2.3086, -2.0455)$ [rad] $= (132.28°, -117.20°)$ would let robot B collide with robot A, as it can be inferred from Fig. 4.

$$ {}^B\boldsymbol{v}_B = {}^A\boldsymbol{R}_B^T\, {}^A\boldsymbol{v}_A = \begin{pmatrix} 0.7071 & 0.7071 \\ -0.7071 & 0.7071 \end{pmatrix} \begin{pmatrix} 1.1107 \\ 0.7854 \end{pmatrix} = \begin{pmatrix} 1.3408 \\ -0.2300 \end{pmatrix} \text{ [m/s]} $$

$$ \boldsymbol{J}_B(\boldsymbol{q}_B) = \left.\begin{pmatrix} -(s_1 + s_{12}) & -s_{12} \\ c_1 + c_{12} & c_{12} \end{pmatrix}\right|_{\boldsymbol{q}=\boldsymbol{q}_B} = \begin{pmatrix} -1 & -0.7399 \\ 0.2929 & -0.6727 \end{pmatrix} $$

$$ \Rightarrow \qquad \dot{\boldsymbol{q}}_B = \boldsymbol{J}_B^{-1}(\boldsymbol{q}_B)\, {}^B\boldsymbol{v}_B = \begin{pmatrix} -0.7563 & 0.8319 \\ -0.3293 & -1.1243 \end{pmatrix} \begin{pmatrix} 1.3408 \\ -0.2300 \end{pmatrix} = \begin{pmatrix} -1.2054 \\ -0.1829 \end{pmatrix} \text{ [rad/s]}. $$

Exercise 3

For the given data, the time-optimal motion in the absence of obstacles has a triangular velocity profile (see Fig. 6), since

$$L = x_f - x_i = 9 \qquad \frac{V^2}{A} = \frac{36}{4} = 9 = L \qquad \Rightarrow \qquad T = \sqrt{\frac{4L}{A}} = 2\,\frac{V}{A} = 3 \text{ s}.$$

The velocity profile reaches $V = 4$ m/s at $t_m = T/2 = 1.5$ s, when the path travelled so far is

$$x(T/2) - x_i = \frac{1}{2}\,A\left(\frac{T}{2}\right)^2 = \frac{L}{2} = 4.5 = x_m.$$

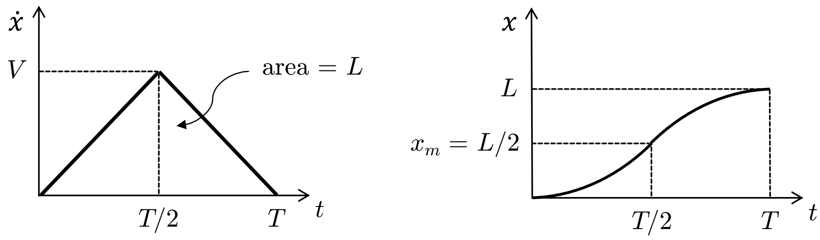

Figure 6: Time-optimal velocity profile [left] and associated position [right] without obstacles

In the presence of moving obstacles, the position x_m cannot be crossed before $t_u = 1.6$ s (the upper limit of the forbidden time interval) to avoid collision with the second obstacle. Note also that the point mass cannot reach x_m sooner than by $t_m = T/2 = 1.5$ s, so that the lower limit ($t_l = 1$ s) of the time interval of collision is irrelevant.

The simplest way to achieve the new task would be to delay the motion start by $t_u - t_m = 1.6 - 1.5 = 0.1$ s and then apply the same triangular velocity profile of the free case. This would give $T_{\text{obs}} = T + (t_u - t_m) = 3.1$ s. However, this solution is discarded by the requirement to move out of $x_i = 0$ as fast as possible and right at $t_i = 0$ (to prevent the potential occurrence of a collision with the first moving obstacle).

To comply with avoidance of both types of collision, one can adopt a uniform time scaling procedure. The idea is to uniformly scale time (and thus motion) so that the point mass will move away from $x_i = 0$ right at $t_i = 0$ and reach the midpoint $x_m = 4.5$ m exactly at $t_u = 1.6$ s, rather than at $T/2 = 1.5$ s. Thus, the scaling factor would be

$$k = \frac{t_u}{T/2} = \frac{1.6}{1.5} = 1.0\bar{6} \qquad \Rightarrow \qquad T_{\text{obs}} = kT = 3.2\,\text{s}.$$

As a result, the speed and acceleration profiles remain of the same type as in Fig. 6, now with

$$\overline{V} = \frac{V}{k} = 5.625 \qquad \overline{A} = \frac{A}{k^2} = 3.515625 \qquad \Rightarrow \qquad \frac{\overline{V}^2}{\overline{A}} = \frac{V^2}{A} = 9 = L.$$

Accordingly, the formula for the completion time become coherently with the time scaling

$$T_{\text{obs}} = 2\,\frac{\overline{V}}{\overline{A}} = 2k\,\frac{V}{A} = kT = 3.2\,\text{s}.$$

While this strategy achieves the goal, it delays unnecessarily the second half of the motion, preventing to obtain the shortest possible completion time.

Another way to complete the task without collisions, and with a reduced increase of motion time, is to accelerate less during the first phase (i.e., at a value $A_u < A$), getting to $x_m = L/2$ at t_u while still reaching the maximum speed V. From there, the solution remains the same as in the free motion case: by decelerating at $-A = -4$ m/s^2, the point mass would get to the final position $x_f = L = 9$ m with zero final velocity in the same additional time $T/2 = 1.5$ s. For this, one should check if the acceleration switching can occur at time t_u. We impose then

$$\dot{x}(t_u) = A_u t_u = V \quad \Rightarrow \quad A_u = \frac{V}{t_u} = \frac{6}{1.6} = 3.75\,\text{m/s}^2.$$

However, the check on the reached position fails since

$$x(t_u) = \frac{1}{2} A_u t_u^2 = 4.8 \neq 4.5 = \frac{L}{2} = x_m,$$

meaning that the point mass has already collided at x_m with the moving obstacle before t_u.

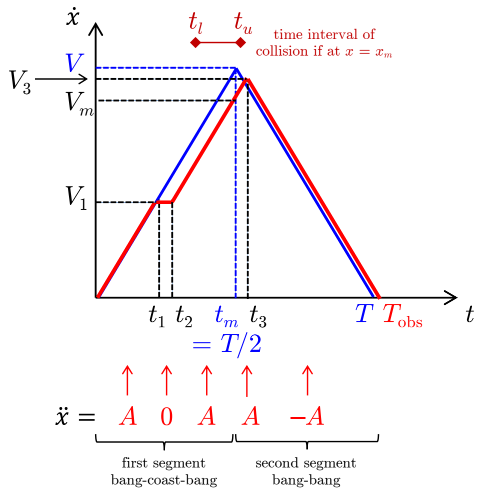

Figure 7: Modified velocity profile [red] in the presence of obstacles, compared to free case [blue]

A better strategy would be to relax the condition of reaching $\dot{x}(t_u) = V$ at t_u, while keeping the condition on the position $x(t_u) = x_m$. In doing so, we should maximize the final velocity V_m when reaching x_m, which will result in a faster way to get to the final goal in the remaining part of the motion. For this, the point mass should keep the maximum feasible acceleration $A = 4$ m/s^2 as long as possible, cruising in between for some interval to avoid collision. As a result, this subproblem is a rest-to-state motion task in which we have to maximize the final speed $V_m \leq V$ under the acceleration limit $|\ddot{x}| \leq A$, with boundary conditions

$$x(t_i) = x(0) = x_i \quad \dot{x}(t_i) = \dot{x}(0) = 0 \quad \Rightarrow \quad x(t_u) = x(1.6) = x_m = 4.5 \quad \dot{x}(t_u) = \dot{x}(1.6) = V_m.$$

The final velocity profile will be of the form shown in Fig. 7.

The solution to the first subproblem takes the form of a bang-coast-bang acceleration profile for $t \in (0, t_u)$, with maximum positive acceleration in both the initial and final phases. Let $t_1 > 0$ and

$t_2 = t_u - t_1 = 1.6 - t_1$ be the (symmetric!) instants of acceleration switching in $(0, t_u)$. One has

$$\ddot{x}(t) = A = 4 \quad \text{for } t \in (0, t_1) \quad \Rightarrow \quad \begin{aligned} \dot{x}_1 &= \dot{x}(t_1) = At_1 = 4t_1 \\ x_1 &= x(t_1) = \tfrac{1}{2}\, At_1^2 = 2t_1^2 \end{aligned}$$

$$\ddot{x}(t) = 0 \quad \text{for } t \in (t_1, t_2) \quad \Rightarrow \quad \begin{aligned} \dot{x}_2 &= \dot{x}(t_2) = \dot{x}_1 = 4t_1 \\ x_2 &= x(t_2) = x_1 + \dot{x}_1(t_2 - t_1) = 2t_1^2 + 4t_1\,(1.6 - 2t_1) \\ &= -6t_1^2 + 6.4t_1 \end{aligned}$$

$$\ddot{x}(t) = A = 4 \quad \text{for } t \in (t_2, t_u) \quad \Rightarrow \quad \begin{aligned} \dot{x}_u &= \dot{x}(t_u) = \dot{x}_2 + A(1.6 - t_2) = 4t_1 + 4t_1 = 8t_1 \\ x_u &= x(t_u) = x_2 + \dot{x}_2(1.6 - t_2) + \tfrac{1}{2}\, A(1.6 - t_2)^2 \\ &= (-6t_1^2 + 6.4t_1) + 4t_1^2 + 2t_1^2 = 6.4t_1. \end{aligned}$$

Imposing the final boundary conditions, we obtain

$$x_u = 6.4t_1 = x_m = 4.5\,\text{m} \quad \Rightarrow \quad t_1 = \frac{4.5}{6.4} = 0.7031 \text{ s} \qquad t_2 = 1.6 - t_1 = 0.8969 \text{ s} \tag{3}$$

$$\dot{x}_u = 8t_1 = 5.625 \quad \Rightarrow \quad V_m = 5.625 \text{ m/s } (< V = 6 \text{ m/s}).$$

In addition, we have

$$V_1 = \dot{x}_1 = \dot{x}_2 = 4t_1 = 2.8125 \text{ m/s} \qquad x_1 = 2t_1^2 = 0.9888 \text{ m} \qquad x_2 = -6t_1^2 + 6.4t_1 = 1.5337 \text{ m}.$$

For the remaining part of the motion, we can still accelerate so to reach an even larger speed, and then decelerate with $\ddot{x} = -A$ to reach the final goal at rest. This is a state-to-rest motion task without further constraints, which is solved by a bang-bang acceleration profile with boundary conditions

$$x(t_u) = x(1.6) = x_m = 4.5 \qquad \dot{x}(t_u) = V_m = 5.625 \quad \Rightarrow \quad x(T_{\text{obs}}) = x_f = 9 \qquad \dot{x}(T_{\text{obs}}) = 0,$$

by switching at some instant $t_3 \in (t_u, T_{\text{obs}})$. The unknowns t_3 and T_{obs} are both found by imposing the above conditions. One has

$$\ddot{x}(t) = A = 4 \text{ for } t \in (t_u, t_3) \ \Rightarrow \ \begin{aligned} &\dot{x}_3 = \dot{x}(t_3) = V_m + A(t_3 - t_u) = 5.625 + 4(t_3 - 1.6) = 4t_3 - 0.775 \\ &x_3 = x(t_3) = x_m + V_m(t_3 - t_u) + \tfrac{1}{2}\, A(t_3 - t_u)^2 \\ &= 4.5 + 5.625(t_3 - 1.6) + 2(t_3 - 1.6)^2 = 2\,t_3^2 - 0.775\,t_3 + 0.62 \end{aligned}$$

$$\ddot{x}(t) = -A = -4 \text{ for } t \in (t_3, T_{\text{obs}}) \ \Rightarrow \ \begin{aligned} &\dot{x}(T_{\text{obs}}) = \dot{x}_3 - A(T_{\text{obs}} - t_3) = (4t_3 - 0.775) - 4(T_{\text{obs}} - t_3) \\ &x(T_{\text{obs}}) = x_3 + \dot{x}_3(T_{\text{obs}} - t_3) - \tfrac{1}{2}\, A(T_{\text{obs}} - t_3)^2 \\ &= (2\,t_3^2 - 0.775\,t_3 + 0.62) + (4t_3 - 0.775)(T_{\text{obs}} - t_3) - 2(T_{\text{obs}} - t_3)^2. \end{aligned}$$

Imposing the final boundary conditions, we obtain

$$\dot{x}(T_{\text{obs}}) = 0 \ \Rightarrow \ (4t_3 - 0.775) - 4(T_{\text{obs}} - t_3) = 0 \quad \Rightarrow \quad T_{\text{obs}} - t_3 = t_3 - 0.19375$$

$$x(T_{\text{obs}}) = x_f = 9 \ \Rightarrow \ \begin{aligned} &(2\,t_3^2 - 0.775\,t_3 + 0.62) + (4t_3 - 0.775)(T_{\text{obs}} - t_3) - 2(T_{\text{obs}} - t_3)^2 = 9 \ \Rightarrow \\ &(2\,t_3^2 - 0.775\,t_3 + 0.62) + (4t_3 - 0.775)(t_3 - 0.19375) - 2(t_3 - 0.19375)^2 - 9 = 0 \ \Rightarrow \\ &4\,t_3^2 - 1.55t_3 - 8.3049 = 0 \ \Rightarrow \ t_3 = 1.6476 \text{ s} \ \text{(discarding the negative root } -1.2601\text{)}, \end{aligned}$$

and thus finally

$$T_{\text{obs}} = 2t_3 - 0.19375 = 3.1015 \text{ s},$$

which is about the same final time that would have been obtained by simply delaying the start of the free motion case by 0.1 s (which is not admissible, because of the possible collision at start). In addition, we have

$$V_3 = \dot{x}_3 = \dot{x}(t_3) = 4t_3 - 0.775 = 5.8155 \text{ m/s} \qquad x_3 = x(t_3) = 2\,t_3^2 - 0.775\,t_3 + 0.62 = 4.7725 \text{ m}.$$

We note that, even in this second part of the motion, the point mass will not reach the maximum feasible speed $V = 6$ m/s.

Summarizing, the presence of the second moving obstacle has slowed down the total motion time. However, the increase in percentage is very small:

$$\Delta T\,\% = \frac{T_{\rm obs} - T}{T} = \frac{0.1015}{3} = 0.0338 \simeq 3.4\%.$$

This is because the forbidden interval extends by only $t_u - t_m = 0.1$ seconds beyond the middle instant of the free motion.

Note finally that the solution can also be obtained, perhaps more directly, by reasoning on the complete profiles of the speed in Fig. 7 and of the corresponding acceleration. The area below the speed profile should equal the total displacement

$$\frac{V_1}{2}\,t_1 + V_1\,(t_2 - t_1) + \frac{V_1 + V_3}{2}\,(t_3 - t_2) + \frac{V_3}{2}\,(T_{obs} - t_3) = L, \tag{4}$$

with $t_2 = t_u - t_1$, and the (signed) area of the acceleration profile should sum to zero (rest-to-rest motion)

$$A\,t_1 + 0\,(t_2 - t_1) + A\,(t_3 - t_2) - A\,(T_{obs} - t_3) = 0. \tag{5}$$

In addition, the position at $t = t_u$ should equal x_m, or

$$\frac{A}{2}\,t_1^2 + V_1\,(t_2 - t_1) + V_1\,(t_u - t_2) + \frac{A}{2}\,(t_u - t_2)^2 = x_m. \tag{6}$$

With the three equations (4)–(6), one can solve for t_1, t_2, and t_3, and then evaluate all remaining quantities —including $T_{\rm obs}$. In particular, using $V_1 = At_1$ and $t_2 = t_u - t_1$, equation (6) becomes

$$\frac{A}{2}\,t_1^2 + At_1\,(t_u - 2t_1) + At_1^2 + \frac{A}{2}\,t_1^2 = x_m \quad \Rightarrow \quad A\,t_1 t_u = x_m \quad \Rightarrow \quad t_1 = \frac{x_m}{A\,t_u} = 0.7031,$$

which is the condition already found in (3).

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