Robotics 1

February 16, 2026

[students with midterm]

Exercise 1

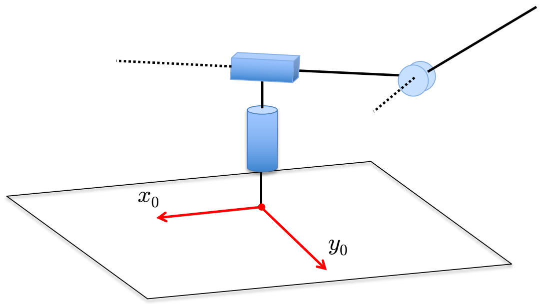

Figure 1: A 3-dof spatial robot

Figure 1 shows the kinematic skeleton of a 3-dof spatial robot. Determine the primary and secondary workspaces, respectively WS_1 and WS_2, of this robot when the revolute joints q_1 and q_3 are unlimited while the prismatic joint has $q_2 \in [0, L]$. Sketch a vertical section of WS_1 and locate explicitly therein the regions having a different number of inverse kinematics solutions.

Exercise 2

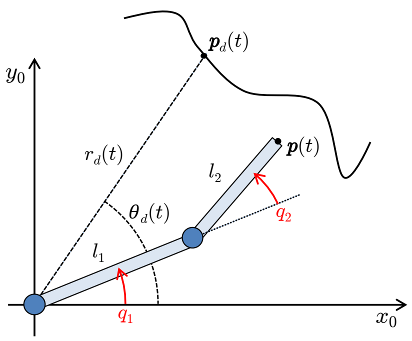

Figure 2: A 2R planar robot tracking a Cartesian trajectory

With reference to Fig. 2, the end-effector of a 2R planar robot having different link lengths ($l_1 \neq l_2$) should track a smooth desired Cartesian trajectory $\boldsymbol{p}_d(t) = (p_{xd}(t), p_{yd}(t))$. Design a control law for the joint velocity commands $\dot{\boldsymbol{q}} \in \mathbb{R}^2$ such that, for any initial configuration $\boldsymbol{q}(0)$, the trajectory tracking error converges exponentially to zero in a decoupled way when expressed in polar coordinates $\boldsymbol{z} = (r, \theta)$, i.e., independently along the radial direction r and with respect to the pointing angle θ, with assigned time constants $\tau_r > 0$ and $\tau_\theta > 0$, respectively for $e_r = r_d - r$ and $e_\theta = \theta_d - \theta$. Determine if and when the control law may run into a singularity.

Finally, compute the value of $\dot{\boldsymbol{q}}(0)$ at the initial time $t = 0$ with the following data and conditions:

$$l_1 = 2 \quad l_2 = 1.5 \text{ [m]} \quad \boldsymbol{q}(0) = \begin{pmatrix} 0 \\ \pi/2 \end{pmatrix} \text{ [rad]} \quad \boldsymbol{p}_d(t) = \begin{pmatrix} 0.5\,(2 + \sin t) \\ 1.5 + 0.5 \cos t \end{pmatrix} \text{ [m]} \quad \tau_r = 4 \quad \tau_\theta = 2 \text{ [s]}.$$

Exercise 3

Prove the relation $(\boldsymbol{J}^T)^\# = (\boldsymbol{J}^\#)^T$ for a general Jacobian matrix $\boldsymbol{J}$ of any size and rank.

Exercise 4

The end-effector of a planar robot should trace an arc of a circle with center $C = (C_x, C_y)$ and radius R, from point $\boldsymbol{p}_A = (C_x + R, C_y)$ to point $\boldsymbol{p}_B = (C_x + 0.5\sqrt{2}R, C_y + 0.5\sqrt{2}R)$. Parametrize this path with its arc length as $\boldsymbol{p} = \boldsymbol{p}(\sigma)$. Determine then a timing law $\sigma(t)$ within the class of bang-coast-bang acceleration profiles (including possibly the bang-bang case) that achieves rest-to-rest motion in minimum time under the Cartesian bounds on the norms of velocity and acceleration

$$\|\dot{\boldsymbol{p}}\| \leq v_{\max} \qquad \|\ddot{\boldsymbol{p}}\| \leq a_{\max}.$$

Consider the possible cases that may arise and provide for each case the minimum time T in symbolic form, expressed in terms of the data C, R, $v_{\max}$, and $a_{\max}$. Is there a case in which the optimal solution cannot be determined in closed form?

Apply then your results to the following numerical data:

i) $C = (1, 1)$ [m], $R = 1$ m, $v_{\max} = 1.2$ m/s, $a_{\max} = 3$ m/s^2;

ii) $C = (2, 1)$ [m], $R = 0.5$ m, $v_{\max} = 1.6$ m/s, $a_{\max} = 6$ m/s^2.

Compute the numerical value of the minimum time T in the two cases and sketch the time profiles of the speed $\dot{\sigma}(t)$ and of the (scalar) acceleration $\ddot{\sigma}(t)$ of the optimal timing law.

[180 minutes (3 hours); open books]

Solution

February 16, 2026

Exercise 1

The 3-dof spatial robot in Fig. 1 is a RPR manipulator. The motion of the prismatic joint is bounded as $q_2 \in [0, L]$, while the revolute joints are free to rotate. Assume for simplicity that $L > a_3$, which is the length of the last link.

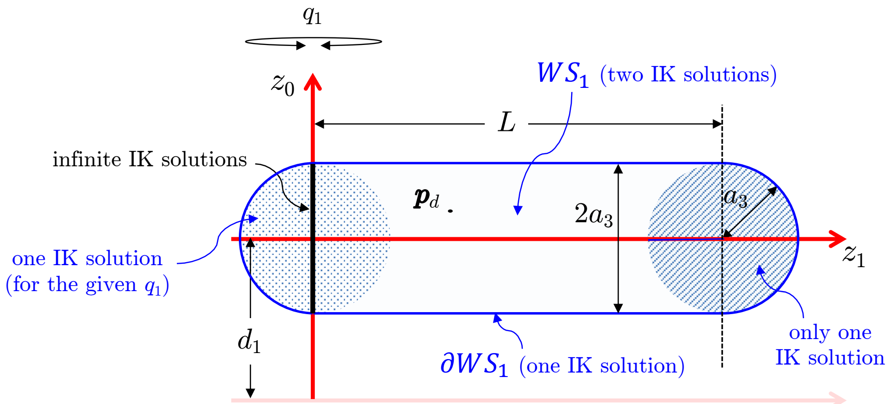

Figure 3: A section of the primary workspace for the RPR robot when q_2 is bounded in $[0, L]$

The primary workspace WS_1 is then a "tortilla" of thickness $2a_3$, with rounded external border and a maximum radius of $L + a_3$ from the z_0 axis. Figure 3 shows a vertical section of WS_1 cut at a given q_1. In the white area strictly inside WS_1, there are two inverse kinematics (IK) solutions (the other two are excluded, since they would require $q_2 < 0$). On the boundary ∂WS_1 of the primary workspace there is only one IK solution. The same happens when $\boldsymbol{p}_d$ is strictly inside the outer dashed circular area. As for the inner dotted circular area around the z_0 axis, when removing the segment in black, there is one IK solution for the given q_1. However, a second IK solution is available by rotating q_1 by π. Finally, on the black segment on the z_0 axis, characterized by $p_{xd} = p_{yd} = 0$, $p_{zd} \in [d_1 - a_3, d_1 + a_3]$, there are infinite IK solutions (obtained by rotating q_1).

On the other hand, the secondary workspace is empty, $WS_2 = \emptyset$. In fact, the robot end-effector can point in any direction of the 3D space (a two-dimensional task), but there is no single position in WS_1 such that all pointing directions can be realized.

Exercise 2

The problem is solved by a kinematic control law defined using the end-effector error expressed in polar coordinates for the stabilizing feedback term and with the feedforward velocity term converted in the same coordinates.

The direct kinematics in Cartesian coordinates is

$$\boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} l_1 \cos q_1 + l_2 \cos(q_1 + q_2) \\ l_1 \sin q_1 + l_2 \sin(q_1 + q_2) \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}), \tag{1}$$

with associated Jacobian

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}} = \begin{pmatrix} -l_1 \sin q_1 - l_2 \sin(q_1 + q_2) & -l_2 \sin(q_1 + q_2) \\ l_1 \cos q_1 + l_2 \cos(q_1 + q_2) & l_2 \cos(q_1 + q_2) \end{pmatrix} = \begin{pmatrix} -p_y & l_1 \sin q_1 - p_y \\ p_x & p_x - l_1 \cos q_1 \end{pmatrix}, \tag{2}$$

where we have used also a second alternative form for this matrix that mixes $\boldsymbol{p}$ and q_1. The end-effector position can be expressed also using the polar coordinates $\boldsymbol{z} = (r, \theta)$ as

$$\boldsymbol{p} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} = \boldsymbol{g}(\boldsymbol{z}). \tag{3}$$

Without loss of generality, by restricting ourselves to $r \geq 0$ we have

$$\boldsymbol{z} = \begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} \sqrt{p_x^2 + p_y^2} \\ \mathrm{ATAN2}\{p_y, p_x\} \end{pmatrix} = \boldsymbol{g}^{-1}(\boldsymbol{p}), \tag{4}$$

and from the identity $\boldsymbol{f}(\boldsymbol{q}) = \boldsymbol{g}(\boldsymbol{z})$,

$$\boldsymbol{z} = \begin{pmatrix} \sqrt{l_1^2 + l_2^2 + 2l_1l_2\cos q_2} \\ \mathrm{ATAN2}\{l_1\sin q_1 + l_2\sin(q_1+q_2), l_1\cos q_1 + l_2\cos(q_1+q_2)\} \end{pmatrix} = \boldsymbol{g}^{-1}(\boldsymbol{f}(\boldsymbol{q})) = \boldsymbol{f}_z(\boldsymbol{q}). \tag{5}$$

A key element of the control law will be the Jacobian $\boldsymbol{J}_z(\boldsymbol{q}) = \partial \boldsymbol{f}_z / \partial \boldsymbol{q}$ of this mapping.

Compute first the time derivative of the two components in (4). One has

$$\dot{r} = \frac{p_x\,\dot{p}_x + p_y\,\dot{p}_y}{\sqrt{p_x^2 + p_y^2}}$$

and, replacing the ATAN2 function with the arctan function,

$$\dot{\theta} = \frac{d}{dt}\arctan\left(\frac{p_y}{p_x}\right) = \frac{1}{1 + \left(\frac{p_y}{p_x}\right)^2}\,\frac{\dot{p}_y\,p_x - \dot{p}_x\,p_y}{p_x^2} = \frac{\dot{p}_y\,p_x - \dot{p}_x\,p_y}{p_x^2 + p_y^2}.$$

In matrix form, we obtain

$$\dot{\boldsymbol{z}} = \begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \dfrac{p_x}{\sqrt{p_x^2 + p_y^2}} & \dfrac{p_y}{\sqrt{p_x^2 + p_y^2}} \\ \dfrac{-p_y}{p_x^2 + p_y^2} & \dfrac{p_x}{p_x^2 + p_y^2} \end{pmatrix} \begin{pmatrix} \dot{p}_x \\ \dot{p}_y \end{pmatrix} = \boldsymbol{J}_g(\boldsymbol{p})\,\dot{\boldsymbol{p}} \tag{6}$$

Thanks to (3), the following identity holds

$$\boldsymbol{J}_g(\boldsymbol{p}) = \boldsymbol{J}_g(\boldsymbol{g}(\boldsymbol{z})) = \begin{pmatrix} \cos\theta & \sin\theta \\ \dfrac{-\sin\theta}{r} & \dfrac{\cos\theta}{r} \end{pmatrix}.$$

Note that the Jacobian $\boldsymbol{J}_g$ is not defined when $p_x = p_y = 0$ (or $r = 0$), in agreement with the undefined value taken in this case by θ in the inverse polar mapping (4); thus, this should be considered as a representation singularity. In any event, such a condition is never encountered since the origin of the plane is outside the robot workspace (being $l_1 \neq l_2$).

Using now (1), the following expressions are obtained after simple manipulations:

$$\begin{aligned} p_x^2 + p_y^2 &= l_1^2 + l_2^2 + 2l_1l_2\cos q_2 \\ p_x\,\dot{p}_x + p_y\,\dot{p}_y &= -l_1l_2\sin q_2\,\dot{q}_2 \\ \dot{p}_y\,p_x - \dot{p}_x\,p_y &= \left(l_1^2 + l_2^2 + 2l_1l_2\cos q_2\right)\dot{q}_1 + \left(l_2^2 + l_1l_2\cos q_2\right)\dot{q}_2. \end{aligned}$$

Then, the time derivatives of r and θ can be rewritten as

$$\dot{r} = \frac{-l_1 l_2 \sin q_2}{\sqrt{l_1^2 + l_2^2 + 2 l_1 l_2 \cos q_2}}\, \dot{q}_2$$

and

$$\dot{\theta} = \dot{q}_1 + \frac{l_2^2 + l_1 l_2 \cos q_2}{l_1^2 + l_2^2 + 2 l_1 l_2 \cos q_2}\, \dot{q}_2.$$

As a result, the time derivative of (5) in matrix form is[1]

$$\dot{\boldsymbol{z}} = \begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} 0 & -\dfrac{l_1 l_2 \sin q_2}{\sqrt{l_1^2 + l_2^2 + 2 l_1 l_2 \cos q_2}} \\ 1 & \dfrac{l_2^2 + l_1 l_2 \cos q_2}{l_1^2 + l_2^2 + 2 l_1 l_2 \cos q_2} \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} = \boldsymbol{J}_z(\boldsymbol{q})\, \dot{\boldsymbol{q}}. \tag{7}$$

The Jacobian $\boldsymbol{J}_z$ is always well defined, except when $l_1 = l_2$ and $q_2 = \pi$ —the second column goes to infinity, since the inverse of the polar transformation (3) is undefined for $r = 0$ ($p_x = p_y = 0$). In the present case, we have $l_1 \neq l_2$ and also this situation never occurs. The determinant of $\boldsymbol{J}_z$ is

$$\det \boldsymbol{J}_z(\boldsymbol{q}) = \frac{l_1 l_2 \sin q_2}{\sqrt{l_1^2 + l_2^2 + 2 l_1 l_2 \cos q_2}}$$

and has a singularity when $q_2 = 0$ or $q_2 = \pi$.

With the above in mind, consider now the kinematic control law

$$\dot{\boldsymbol{q}} = \boldsymbol{J}_z^{-1}(\boldsymbol{q})\, (\dot{\boldsymbol{z}}_d + \boldsymbol{K}_P\, \boldsymbol{e}_z)\,, \tag{8}$$

with

$$\boldsymbol{e}_z = \begin{pmatrix} e_r \\ e_\theta \end{pmatrix} = \begin{pmatrix} r_d - r \\ \theta_d - \theta \end{pmatrix} = \boldsymbol{z}_d - \boldsymbol{z} \qquad \boldsymbol{K}_P = \begin{pmatrix} k_r & 0 \\ 0 & k_\theta \end{pmatrix} > 0.$$

The term $\dot{\boldsymbol{z}}_d$ in (8) is the nominal feedforward velocity expressed in polar coordinates. It is obtained from $\boldsymbol{p}_d(t)$ as

$$\dot{\boldsymbol{z}}_d = \boldsymbol{J}_g(\boldsymbol{p}_d)\, \dot{\boldsymbol{p}}_d,$$

The feedback term $\boldsymbol{K}_P\, \boldsymbol{e}_z$ in (8) uses the trajectory error as defined in polar coordinates (see Fig. 4), weighted by a diagonal matrix with positive gains k_r and k_θ.

When using the control law (8), the closed-loop error dynamics in polar coordinates is

$$\dot{\boldsymbol{e}}_z = \dot{\boldsymbol{z}}_d - \dot{\boldsymbol{z}} = \dot{\boldsymbol{z}}_d - \boldsymbol{J}_z(\boldsymbol{q})\dot{\boldsymbol{q}} = \dot{\boldsymbol{z}}_d - \boldsymbol{J}_z(\boldsymbol{q})\big(\boldsymbol{J}_z^{-1}(\boldsymbol{q})\, (\dot{\boldsymbol{z}}_d + \boldsymbol{K}_P\, \boldsymbol{e}_z)\big). = -\boldsymbol{K}_P\, \boldsymbol{e}_z,$$

Componentwise, a decoupled dynamic behavior is obtained with each error exponentially converging to zero at the prescribed rate:

$$\begin{aligned} \dot{e}_r &= -k_r\, e_r \qquad \Rightarrow \qquad e_r(t) = e_r(0) \exp\left(-k_r t\right) = e_r(0) \exp\left(-t/\tau_r\right) \\ \dot{e}_\theta &= -k_\theta\, e_\theta \qquad \Rightarrow \qquad e_\theta(t) = e_\theta(0) \exp\left(-k_\theta t\right) = e_\theta(0) \exp\left(-t/\tau_\theta\right), \end{aligned}$$

with the time constants $\tau_r = 1/k_r$ and $\tau_\theta = 1/k_\theta$.

[1] The Jacobian in (7) can also be obtained as $\boldsymbol{J}_z(\boldsymbol{q}) = \boldsymbol{J}_g(\boldsymbol{p})\boldsymbol{J}(\boldsymbol{q})$, using the second form in (2), the matrix in (6), and expressing then $\boldsymbol{p}$ as a function of $\boldsymbol{q}$ via (1).

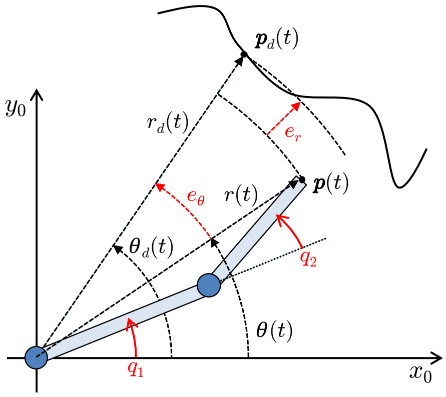

Figure 4: The polar errors $e_r = r_d - r$ and $e_\theta = \theta_d - \theta$ along the Cartesian trajectory

The control law (8) can be applied only out the singularities of $\boldsymbol{J}_z(\boldsymbol{q})$, which occur when $q_2 = 0$ or $q_2 = \pi$, namely at the boundaries of the primary workspace of the robot. For the given values of l_1 and l_2, such situation can only happen during the initial transient; in fact, when the tracking error has vanished, the reference Cartesian trajectory $\boldsymbol{p}_d(t)$ remains fully inside the robot workspace.

Replacing the numerical values, we compute for the robot in the initial configuration

$$\boldsymbol{p}(0) = \begin{pmatrix} 2 \\ 1.5 \end{pmatrix} \text{[m]} \quad \Rightarrow \quad \boldsymbol{z}(0) = \begin{pmatrix} 2.5 \\ 0.6435 \end{pmatrix} \text{[m,rad]} \qquad \boldsymbol{J}_z(\boldsymbol{q}(0)) = \begin{pmatrix} 0 & -1.2 \\ 1 & 0.36 \end{pmatrix}$$

and from the desired trajectory

$$\boldsymbol{p}_d(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{[m]} \;\Rightarrow\; \boldsymbol{z}_d(0) = \begin{pmatrix} \sqrt{5} \\ 1.1071 \end{pmatrix} \text{[m,rad]} \;\Rightarrow\; \boldsymbol{e}_z(0) = \boldsymbol{z}_d(0) - \boldsymbol{z}_(0) = \begin{pmatrix} -0.2639 \\ 0.4636 \end{pmatrix} \text{[m,rad]}$$

$$\dot{\boldsymbol{p}}_d(t) = \begin{pmatrix} 0.5\cos t \\ -0.5\sin t \end{pmatrix} \quad \Rightarrow \quad \dot{\boldsymbol{p}}_d(0) = \begin{pmatrix} 0.5 \\ 0 \end{pmatrix} \text{[m/s]}$$

$$\Rightarrow \quad \dot{\boldsymbol{z}}_d(0) = \boldsymbol{J}_g(\boldsymbol{p}_d(0))\,\dot{\boldsymbol{p}}_d(0) = \begin{pmatrix} 1/\sqrt{5} & * \\ -0.4 & * \end{pmatrix} \begin{pmatrix} 0.5 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.2236 \\ -0.2 \end{pmatrix} \text{[m/s,rad/s]}.$$

Finally, the gain matrix is

$$\boldsymbol{K}_P = \begin{pmatrix} 1/\tau_r & 0 \\ 0 & 1/\tau_\theta \end{pmatrix} = \begin{pmatrix} 0.25 & 0 \\ 0 & 0.5 \end{pmatrix} \text{[s}^{-1}\text{]}.$$

Therefore, from (8) we have at $t = 0$

$$\dot{\boldsymbol{q}}(0) = \boldsymbol{J}_z^{-1}(\boldsymbol{q}(0))\,(\dot{\boldsymbol{z}}_d(0) + \boldsymbol{K}_P\,\boldsymbol{e}_z(0)) = \begin{pmatrix} 0.3 & 1 \\ -0.8333 & 0 \end{pmatrix} \begin{pmatrix} 0.1576 \\ 0.0318 \end{pmatrix} = \begin{pmatrix} 0.0791 \\ -0.1314 \end{pmatrix} \text{[rad/s]}.$$

Exercise 3

The proof of the identity $(\boldsymbol{J}^T)^\# = (\boldsymbol{J}^\#)^T$, i.e., the fact that the order of pseudoinversion and transposition of a matrix can be exchanged, uses the four defining properties of the pseudoinverse of a matrix and the fact that $\boldsymbol{J}^\#$ is the pseudoinverse of $\boldsymbol{J}$.

1. $\boldsymbol{J}^T(\boldsymbol{J}^T)^\#\,\boldsymbol{J}^T = \boldsymbol{J}^T$

$$\boldsymbol{J}^T(\boldsymbol{J}^T)^\#\,\boldsymbol{J}^T = \boldsymbol{J}^T(\boldsymbol{J}^\#)^T\,\boldsymbol{J}^T = \left(\boldsymbol{J}\,\boldsymbol{J}^\#\,\boldsymbol{J}\right)^T = \boldsymbol{J}^T$$

2. $(\boldsymbol{J}^T)^{\#}\,\boldsymbol{J}^T(\boldsymbol{J}^T)^{\#} = (\boldsymbol{J}^T)^{\#}$

$$(\boldsymbol{J}^T)^{\#}\,\boldsymbol{J}^T(\boldsymbol{J}^T)^{\#} = (\boldsymbol{J}^{\#})^T\,\boldsymbol{J}^T(\boldsymbol{J}^{\#})^T = \left(\boldsymbol{J}^{\#}\boldsymbol{J}\,\boldsymbol{J}^{\#}\right)^T = (\boldsymbol{J}^{\#})^T = (\boldsymbol{J}^T)^{\#}$$

3. $\left(\boldsymbol{J}^T(\boldsymbol{J}^T)^{\#}\right)^T = \boldsymbol{J}^T(\boldsymbol{J}^T)^{\#}$

$$\left(\boldsymbol{J}^T(\boldsymbol{J}^T)^{\#}\right)^T = \left(\boldsymbol{J}^T(\boldsymbol{J}^{\#})^T\right)^T = \boldsymbol{J}^{\#}\boldsymbol{J} = \left(\boldsymbol{J}^{\#}\boldsymbol{J}\right)^T = \boldsymbol{J}^T(\boldsymbol{J}^{\#})^T = \boldsymbol{J}^T(\boldsymbol{J}^T)^{\#}$$

4. $\left((\boldsymbol{J}^T)^{\#}\,\boldsymbol{J}^T\right)^T = (\boldsymbol{J}^T)^{\#}\,\boldsymbol{J}^T$

$$\left((\boldsymbol{J}^T)^{\#}\,\boldsymbol{J}^T\right)^T = \left((\boldsymbol{J}^{\#})^T\boldsymbol{J}^T\right)^T = \boldsymbol{J}\,\boldsymbol{J}^{\#} = \left(\boldsymbol{J}\,\boldsymbol{J}^{\#}\right)^T = (\boldsymbol{J}^{\#})^T\boldsymbol{J}^T = (\boldsymbol{J}^T)^{\#}\,\boldsymbol{J}^T.$$

Another possibility is to use the Singular Value Decomposition (SVD) of matrix $\boldsymbol{J}$. However, we should formally distinguish then the two cases when $m < n$ (or $m \le n$) and when $m \ge n$ (or $m > n$). Consider for instance the first case, with

$$\boldsymbol{J} = \boldsymbol{U\Sigma V}^T \qquad \boldsymbol{\Sigma} = \left(\begin{array}{ccccccc|c} \sigma_1 & 0 & 0 & & \dots & & 0 & \\ 0 & \sigma_2 & 0 & & \dots & & 0 & \\ 0 & \dots & \ddots & & \dots & & 0 & \boldsymbol{O}_{m\times(n-m)} \\ 0 & \dots & 0 & \sigma_\rho & 0 & \dots & 0 & \\ 0 & & \dots & & 0 & \dots & 0 & \\ 0 & & & \dots & & & 0 & \end{array}\right),$$

where $\rho = \text{rank}\,\boldsymbol{J} \le m$. We know that

$$\boldsymbol{J}^{\#} = \boldsymbol{V\Sigma}^{\#}\boldsymbol{U}^T \qquad \boldsymbol{\Sigma}^{\#} = \begin{pmatrix} \frac{1}{\sigma_1} & 0 & & \dots & & 0 \\ 0 & \frac{1}{\sigma_2} & 0 & \dots & & 0 \\ 0 & & \ddots & & \dots & 0 \\ 0 & & \dots & \frac{1}{\sigma_\rho} & \dots & 0 \\ 0 & & & \dots & & 0 \\ & & & \boldsymbol{O}_{(n-m)\times m} & & \end{pmatrix},$$

Therefore, being

$$\boldsymbol{J}^T = \boldsymbol{V\Sigma}^T\boldsymbol{U}^T \quad \Rightarrow \quad \left(\boldsymbol{J}^T\right)^{\#} = \boldsymbol{U}\left(\boldsymbol{\Sigma}^T\right)^{\#}\boldsymbol{V}^T \qquad \text{and} \qquad \left(\boldsymbol{J}^{\#}\right)^T = \boldsymbol{U}\left(\boldsymbol{\Sigma}^{\#}\right)^T\boldsymbol{V}^T,$$

we only need to show that

$$\left(\boldsymbol{\Sigma}^T\right)^{\#} = \left(\boldsymbol{\Sigma}^{\#}\right)^T,$$

which follows by direct inspection from

$$\left(\boldsymbol{\Sigma}^T\right)^{\#} = \left(\begin{array}{ccccccc|c} \frac{1}{\sigma_1} & 0 & 0 & & \dots & & 0 & \\ 0 & \frac{1}{\sigma_2} & 0 & & \dots & & 0 & \\ 0 & \dots & \ddots & & \dots & & 0 & \boldsymbol{O}_{m\times(n-m)} \\ 0 & \dots & 0 & \frac{1}{\sigma_\rho} & 0 & \dots & 0 & \\ 0 & & \dots & & 0 & \dots & 0 & \\ 0 & & & \dots & & & 0 & \end{array}\right).$$

The same procedure can be followed in case $m \geq n$.

Exercise 4

The parametrization of the given path (an arc of a circle of radius R) with its arc length σ is

$$\boldsymbol{p}(\sigma) = \boldsymbol{p}_C + R \begin{pmatrix} \cos \dfrac{\sigma}{R} \\ \sin \dfrac{\sigma}{R} \end{pmatrix} \qquad \sigma \in \left[0, \frac{\pi R}{4}\right]. \tag{9}$$

In fact

$$\boldsymbol{p}(0) = \boldsymbol{p}_C + R \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} C_x + R \\ C_y \end{pmatrix} = \boldsymbol{p}_A$$

$$\boldsymbol{p}\Big(\frac{\pi R}{4}\Big) = \boldsymbol{p}_C + R \begin{pmatrix} \cos \dfrac{\pi}{4} \\ \sin \dfrac{\pi}{4} \end{pmatrix} = \begin{pmatrix} C_x + \sqrt{2}\, R/2 \\ C_y + \sqrt{2}\, R/2 \end{pmatrix} = \boldsymbol{p}_B,$$

and the length of the path is $L = (\pi/4)\, R$. The first and second time derivatives of (9) are

$$\dot{\boldsymbol{p}} = \begin{pmatrix} -\sin \dfrac{\sigma}{R} \\ \cos \dfrac{\sigma}{R} \end{pmatrix} \dot{\sigma}$$

$$\ddot{\boldsymbol{p}} = \begin{pmatrix} -\sin \dfrac{\sigma}{R} \\ \cos \dfrac{\sigma}{R} \end{pmatrix} \ddot{\sigma} - \frac{1}{R} \begin{pmatrix} \cos \dfrac{\sigma}{R} \\ \sin \dfrac{\sigma}{R} \end{pmatrix} \dot{\sigma}^2 = Rot(\sigma/R) \begin{pmatrix} -\dfrac{\dot{\sigma}^2}{R} \\ \ddot{\sigma} \end{pmatrix},$$

where

$$Rot(\sigma/R) = \begin{pmatrix} \cos \dfrac{\sigma}{R} & -\sin \dfrac{\sigma}{R} \\ \sin \dfrac{\sigma}{R} & \cos \dfrac{\sigma}{R} \end{pmatrix}$$

is an element of $SO(2)$ —thus, $Rot^{-1}(\alpha) = Rot^T(\alpha)$ and $\det Rot(\alpha) = +1$ for any α. Therefore,

$$\|\dot{\boldsymbol{p}}\| = |\dot{\sigma}| \qquad \|\ddot{\boldsymbol{p}}\| = \sqrt{\ddot{\boldsymbol{p}}^T \ddot{\boldsymbol{p}}} = \sqrt{\Big(\frac{\dot{\sigma}^2}{R}\Big)^2 + \ddot{\sigma}^2}$$

and the two bounds on the norms of the Cartesian velocity and acceleration become

$$|\dot{\sigma}| \leq v_{\max} \qquad \sqrt{\Big(\frac{\dot{\sigma}^2}{R}\Big)^2 + \ddot{\sigma}^2} \leq a_{\max}. \tag{10}$$

A rest-to-rest timing law $\sigma(t)$ with bang-coast-bang (b-c-b) acceleration and a corresponding trapezoidal speed profile is defined by the laws

$$\ddot{\sigma}(t) = \begin{cases} A & t \in [0, T_r] \\ 0 & t \in [T_r, T - T_r] \\ -A & t \in [T - T_r, T] \end{cases} \qquad \dot{\sigma}(t) = \begin{cases} A\, t & t \in [0, T_r] \\ V & t \in [T_r, T - T_r] \\ V - A\,(t - T + T_r) & t \in [T - T_r, T], \end{cases}$$

with rising time $T_r = V/A$ and total motion time T. The two values A (constant absolute acceleration/deceleration) and V (constant cruising speed) have to be chosen so as to minimize the motion time T while complying with the bounds (10). The length of the path is $L = \pi R/4$. The above profiles hold as long as $L > V^2/A$ (condition of existence of the cruising phase at constant speed V). The travel time is then

$$T = \frac{AL + V^2}{AV} = \frac{L}{V} + \frac{V}{A}. \tag{11}$$

However, if the path is too short ($L \leq V^2/A$) to allow reaching the chosen cruising speed V, the timing law will collapse into a bang-bang (b-b) acceleration with triangular profile for the speed,

$$\ddot{\sigma}(t) = \begin{cases} A & t \in [0, \bar{T}/2] \\ -A & t \in [\bar{T}/2, \bar{T}] \end{cases} \qquad \dot{\sigma}(t) = \begin{cases} At & t \in [0, \bar{T}/2] \\ \bar{V} - At\left(t - \bar{T}/2\right) & t \in [\bar{T}/2, \bar{T}], \end{cases}$$

where

$$\bar{T} = \sqrt{\frac{4L}{A}} \qquad \bar{V} = \frac{A\bar{T}}{2} \; (\leq V) \tag{12}$$

are, respectively, the travel time and the maximum speed reached along the path.

In order to determine the optimal values of V and A, consider first the b-c-b case. From (10), taking only into account the phase at constant cruising speed V where $\ddot{\sigma} = 0$, we get

$$|\dot{\sigma}| \leq V = \min\left\{ v_{\max}, \sqrt{R\, a_{\max}} \right\}, \tag{13}$$

where the second limitation comes from the centripetal Cartesian acceleration.[2] Further, the worst case for the acceleration bound is when both $\dot{\sigma}$ and $\ddot{\sigma}$ take their maximum (absolute) value, or

$$\sqrt{\left(\frac{V^2}{R}\right)^2 + A^2} = a_{\max} \qquad \Rightarrow \qquad A = \sqrt{a_{\max}^2 - \left(\frac{V^2}{R}\right)^2}.$$

Therefore, if the values of the bounds on the norms return $v_{\max}$ as the minimum in (13), then

$$V = v_{\max} \qquad A = \sqrt{a_{\max}^2 - \left(\frac{v_{\max}^2}{R}\right)^2} \tag{14}$$

and the b-c-b solution will occur if and only if

$$L = \frac{\pi R}{4} > \frac{v_{\max}^2}{\sqrt{a_{\max}^2 - \left(\frac{v_{\max}^2}{R}\right)^2}} = \frac{V^2}{A}, \tag{15}$$

yielding from (11)

$$T = \frac{\pi R}{4\, v_{\max}} + \frac{v_{\max}}{\sqrt{a_{\max}^2 - \left(\frac{v_{\max}^2}{R}\right)^2}}.$$

On the other hand, when inequality (15) is violated, the timing law will assume a b-b acceleration profile. From (12), we have then

$$\bar{T} = \frac{\sqrt{\pi R}}{\sqrt[4]{a_{\max}^2 - \left(\frac{v_{\max}^2}{R}\right)^2}} \qquad \bar{V} = \frac{\sqrt{\pi R}\, \sqrt[4]{a_{\max}^2 - \left(\frac{v_{\max}^2}{R}\right)^2}}{2}.$$

[2] For a linear path, $R \to \infty$ and the minimum is always given by the direct bound $v_{\max}$ on the Cartesian velocity. By continuity, the same holds true for a circular path with sufficiently small curvature $\kappa = 1/R$.

If instead the minimum in (13) is given by $\sqrt{R\,a_{\max}}$, then the previous b-c-b solution in closed form cannot be applied since the second Cartesian bound in (10) would return an upper bound $A = 0$ for $\ddot{\sigma}$ —which is clearly impossible (a similar issue occurs for the b-b solution). In fact, the problem should be reformulated in the positive quadrant of the $(\dot{\sigma}, \ddot{\sigma})$ plane (or, even better, of the $(\dot{\sigma}^2, \ddot{\sigma})$ plane), finding the feasible pair $(\dot{\sigma}, \ddot{\sigma}) = (V, A)$ that satisfies the bounds (10) and provides the minimum motion time T. Unfortunately, there is no closed formula for the optimal solution in that case. Thus, we skip any further analysis of this situation and simply turn our attention to the two numerical examples.

For case *i)*, we have $V = 1.2 = v_{max}$, $A = 2.6318$, and from (15)

$$L = 0.7854 > 0.5472 = \frac{V^2}{A}.$$

Thus, the minimum-time solution has a bang-coast-bang acceleration profile with

$$T = 1.1105\,\text{s} \qquad\qquad T_r = 0.4560\,\text{s}.$$

The speed and acceleration profiles are shown in Fig. 5.

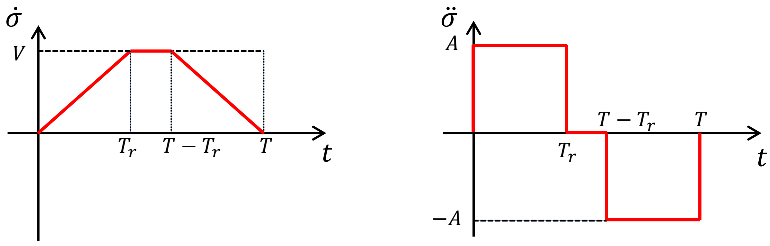

Figure 5: Time profiles of $\dot{\sigma}(t)$ and $\ddot{\sigma}(t)$ for case *i)*

For case *ii)*, we have $V = 1.6 = v_{max}$, $A = 2.1282$, and from (15)

$$L = 0.3927 < 0.8184 = \frac{V^2}{A}.$$

Thus, the minimum-time solution has a bang-bang acceleration profile with

$$\bar{T} = 0.7086\,\text{s} \qquad\qquad \bar{T}_r = \frac{\bar{T}}{2} = 0.3543\,\text{s} \qquad\qquad \bar{V} = 1.1083\,\text{m/s}.$$

The speed and acceleration profiles are shown in Fig. 6.

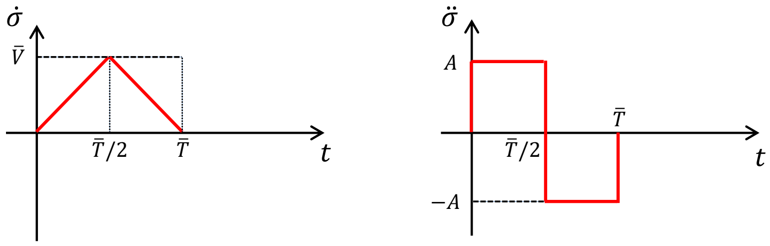

Figure 6: Time profiles of $\dot{\sigma}(t)$ and $\ddot{\sigma}(t)$ for case *ii)*

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