Robotics 1
February 16, 2026

Exercise 1

For localizing an underwater target, the unmanned underwater vehicle (UUV) sketched in Fig. 1 is assisted by a surface vessel, that carries a bathymeter that measures the depth D of the target. The UUV robot should place the target at the center of the field of view of its tilting camera. Assume the problem to be restricted to the vertical plane (x_s, z_s) of a fixed reference frame RF_s located at the sea level. Knowing the pose of the UUV robot, namely of a frame RF_u placed at its center of mass, and the position of the surface vessel with respect to RF_s, determine the analytic expression of the angle β assumed by the camera when the task is satisfied. Define the problem using explicitly homogeneous transformation matrices (conveniently, in their planar version as 3×3 matrices). Choose then some realistic input data and give the corresponding numerical value of β.

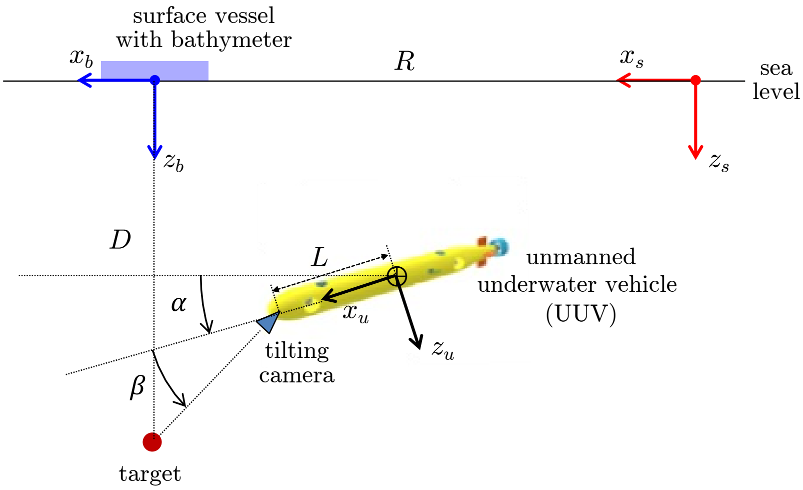

Figure 1: An UUV pointing its camera to an underwater target (planar view)

Exercise 2

Figure 2 shows the kinematic skeleton of a 3-dof spatial robot.

- Assign a set of frames according to the standard Denavit-Hartenberg (DH) convention and determine the corresponding table of parameters in such a way that *all twist angles are non-positive*. Place the origin of the last DH frame at the end of the last link. Draw the frames directly on the figure.
- Derive the direct kinematics $\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q})$ of the position of the origin O_3 of the last DH frame.
- For a given position $\boldsymbol{p}_d \in \mathbb{R}^3$, solve the inverse kinematics problem in closed form, $\boldsymbol{q} = \boldsymbol{f}^{-1}(\boldsymbol{p}_d)$, specifying the number of solutions in the different possible cases.
- Let $\boldsymbol{v} \in \mathbb{R}^3$ be the velocity of the origin O_3. Determine all configurations $\boldsymbol{q}_s$ at which the square Jacobian in $\boldsymbol{v} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$ is singular. In such a $\boldsymbol{q}_s$, provide a joint velocity $\dot{\boldsymbol{q}}_s \neq \boldsymbol{0}$ that produces $\boldsymbol{v} = \boldsymbol{0}$ and find the Cartesian direction(s) along which no instantaneous velocity $\boldsymbol{v} \neq \boldsymbol{0}$ can be realized.
- Determine the primary and secondary workspaces, respectively WS_1 and WS_2, of this robot when q_1 and q_3 are unlimited while $q_2 \in [0, L]$. Sketch a vertical section of WS_1 and locate explicitly therein the regions with a different number of inverse kinematics solutions.

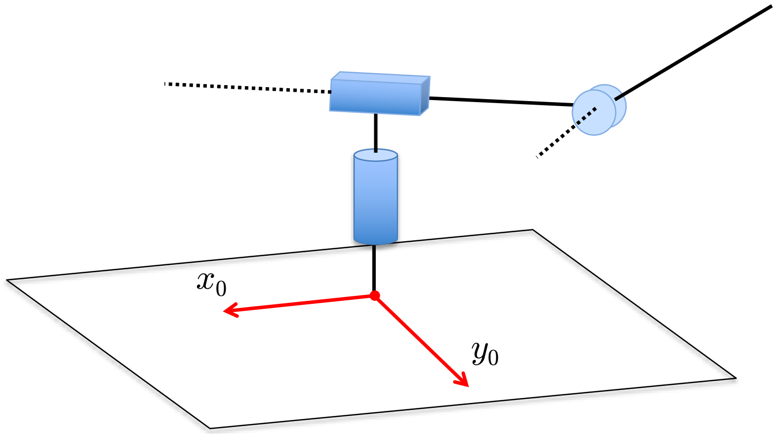

Figure 2: A 3-dof spatial robot

Exercise 3

Prove the relation $(\boldsymbol{J}^T)^\# = (\boldsymbol{J}^\#)^T$ for a general Jacobian matrix $\boldsymbol{J}$ of any size and rank.

Exercise 4

The end-effector of a planar robot should trace an arc of a circle with center $C = (C_x, C_y)$ and radius R, from point $\boldsymbol{p}_A = (C_x + R, C_y)$ to point $\boldsymbol{p}_B = (C_x + 0.5\sqrt{2}R, C_y + 0.5\sqrt{2}R)$. Parametrize this path with its arc length as $\boldsymbol{p} = \boldsymbol{p}(\sigma)$. Determine then a timing law $\sigma(t)$ within the class of bang-coast-bang acceleration profiles (including possibly the bang-bang case) that achieves rest-to-rest motion in minimum time under the Cartesian bounds on the norms of velocity and acceleration

$$\|\dot{\boldsymbol{p}}\| \leq v_{\max} \qquad \|\ddot{\boldsymbol{p}}\| \leq a_{\max}.$$

Consider the possible cases that may arise and provide for each case the minimum time T in symbolic form, expressed in terms of the data C, R, $v_{\max}$, and $a_{\max}$. Is there a case in which the optimal solution cannot be determined in closed form?

Apply then your results to the following numerical data:

i) $C = (1, 1)$ [m], $R = 1$ m, $v_{\max} = 1.2$ m/s, $a_{\max} = 3$ m/s^2;

ii) $C = (2, 1)$ [m], $R = 0.5$ m, $v_{\max} = 1.6$ m/s, $a_{\max} = 6$ m/s^2.

Compute the numerical value of the minimum time T in the two cases and sketch the time profiles of the speed $\dot{\sigma}(t)$ and of the (scalar) acceleration $\ddot{\sigma}(t)$ of the optimal timing law.

[240 minutes (4 hours); open books]

Solution

February 16, 2026

Exercise 1

Using the 3×3 homogeneous transformation matrices of the planar case, the mappings between RF_s and RF_u and between RF_u and the camera frame RF_c (not shown in Fig. 1) are given by

$$
{}^s\boldsymbol{T}_u = \begin{pmatrix} \cos\alpha & -\sin\alpha & p_{u,x} \\ \sin\alpha & \cos\alpha & p_{u,z} \\ 0 & 0 & 1 \end{pmatrix} \qquad {}^u\boldsymbol{T}_c = \begin{pmatrix} \cos\beta & -\sin\beta & L \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix},
$$

where $\boldsymbol{p}_u = (p_{u,x}, p_{u,z})$ is the position of the center of mass of the UUV robot. Thus

$$
{}^s\boldsymbol{T}_c = \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) & p_{u,x} + L\cos\alpha \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) & p_{u,z} + L\sin\alpha \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^s\boldsymbol{R}_c(\alpha+\beta) & {}^s\boldsymbol{p}_c(\boldsymbol{p}_u, \alpha) \\ \boldsymbol{0}^T & 1 \end{pmatrix}.
$$

The position of the target, as expressed in RF_s, is known from the surface vessel position and the depth measurement

$$
{}^s\boldsymbol{p}_t = \begin{pmatrix} p_{t,x} \\ p_{t,z} \end{pmatrix} = \begin{pmatrix} R \\ D \end{pmatrix}.
$$

The position vector from the camera (i.e., the origin of RF_c) on the UUV robot to the target is

$$
{}^s\boldsymbol{p}_{ct} = \begin{pmatrix} p_{ct,x} \\ p_{ct,z} \end{pmatrix} = {}^s\boldsymbol{p}_t - {}^s\boldsymbol{p}_c = \begin{pmatrix} R - p_{u,x} - L\cos\alpha \\ D - p_{u,z} - L\sin\alpha \end{pmatrix}.
$$

Therefore, being

$$
p_{ct,x} = \Delta\,\cos(\alpha+\beta) \qquad p_{ct,z} = \Delta\,\sin(\alpha+\beta),
$$

where $\Delta > 0$ is the distance (unknown so far, but irrelevant) between the camera and the target, we obtain

$$
\alpha + \beta = \text{ATAN2}\left\{\frac{p_{ct,z}}{\Delta}, \frac{p_{ct,x}}{\Delta}\right\} = \text{ATAN2}\left\{p_{ct,z}, p_{ct,x}\right\},
$$

and thus finally

$$
\beta = \text{ATAN2}\left\{D - p_{u,z} - L\sin\alpha, R - p_{u,x} - L\cos\alpha\right\} - \alpha = f(R, D, \boldsymbol{p}_u, \alpha)
$$

as a function of measurements and data. Indeed, this formula could have been found also using pure geometric reasoning. For instance, with the data

$$
L = 1.5\,\text{m} \qquad R = 50\,\text{m} \qquad D = 100\,\text{m} \qquad \boldsymbol{p}_u = (25, 40)\,[\text{m}] \qquad \alpha = 30° = \frac{\pi}{6}\,\text{rad},
$$

one obtains $\beta = 0.6667$ rad $= 38.19°$.

Exercise 2

The 3-dof spatial robot in Fig. 2 is a RPR manipulator. A correct assignment of DH frames satisfying the additional requirement of non-positivity of the twist angles α_i, $i = 1, 2, 3$, is shown in Fig. 3. The corresponding parameters are given in Tab. 1, where the signs of the constant parameters d_1 and a_3 are also specified.

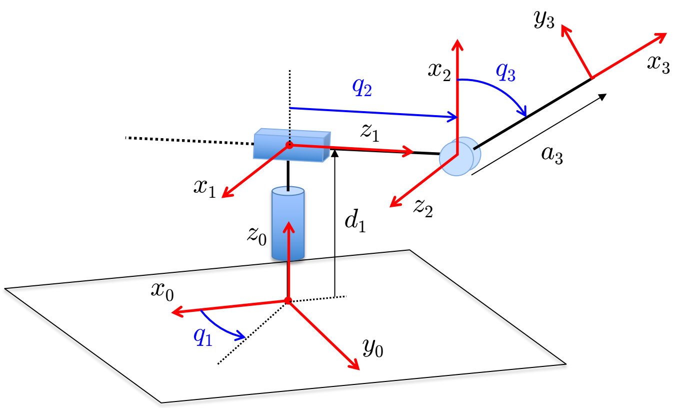

Figure 3: DH frames for the RPR robot in Fig. 2

i	α_i	a_i	d_i	θ_i
1	$-\pi/2$	0	$d_1 > 0$	q_1
2	$-\pi/2$	0	q_2	$-\pi/2$
3	0	$a_3 > 0$	0	q_3

Table 1: DH parameters for the frame assignment in Fig. 3

The direct kinematics for the position of the origin O_3 of the last DH frame is

$$\boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} -\sin q_1\,(q_2 - a_3 \sin q_3) \\ \cos q_1\,(q_2 - a_3 \sin q_3) \\ d_1 + a_3 \cos q_3 \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}). \tag{1}$$

For a given value $\boldsymbol{p}_d = (p_{xd}, p_{yd}, p_{zd})$ of $\boldsymbol{p}$, the inverse kinematics problem is solved as follows. From the third equation in (1), one has

$$\cos q_3 = \frac{p_{zd} - d_1}{a_3} \qquad \sin q_3 = \pm\sqrt{1 - \cos^2 q_3} \qquad \Rightarrow \qquad q_3^{(+,-)} = \text{ATAN2}\{\sin q_3, \cos q_3\}, \tag{2}$$

i.e., with one solution for each of the two signs in $\sin q_3$. Indeed, if $\cos q_3 \notin [-1, 1]$ there is no solution ($\boldsymbol{p}_d$ is out of the robot workspace). Because the change of sign occurs in the first argument of ATAN2, the two solutions are symmetric w.r.t. 0 when $\cos q_3 \in (0, 1)$, and symmetric w.r.t. π when $\cos q_3 \in (-1, 0)$. When $\cos q_3 = 0$ ($p_{zd} = d_1$), the two solutions are $\pm\pi/2$. Moreover, if $|p_{zd} - d_1| = a_3$, there is only one solution value: $q_3 = 0$, if $p_{zd} - d_1 > 0$, or $q_3 = \pi$, if $p_{zd} - d_1 < 0$.

Next, squaring and summing the first two equations in (1) leads to

$$p_{xd}^2 + p_{yd}^2 = (q_2 - a_3 \sin q_3)^2 .$$

Extracting the square root, which introduces the signs $\pm$, and using $\sin q_3$ from (2), which introduces another pair of ± signs, leads to[1]

$$q_2 = \pm\sqrt{a_3^2 - (p_{zd} - d_1)^2} \ \pm \ \sqrt{p_{xd}^2 + p_{yd}^2}, \tag{3}$$

and thus to four solutions, one for each choice of sign pairs,

$$q_2^{(+,+)} \quad q_2^{(+,-)} \quad q_2^{(-,+)} = -q_2^{(+,-)} \quad q_2^{(-,-)} = -q_2^{(+,+)}. \tag{4}$$

Because of their structure, two of these solutions are always eliminated when q_2 has to be strictly positive only. Moreover, the number of solutions for q_2 collapses to two when either $p_{xd} = p_{yd} = 0$ or $p_{zd} = d_1 \pm a_3$. Finally, $q_2 = 0$ is left as the unique solution if and only if both conditions hold at the same time.

As for the first joint variable, from the first two equations in (1) one has

$$q_1 = \text{ATAN2}\{-p_{xd}\,\text{sign}(q_2 - a_3 \sin q_3), p_{yd}\,\text{sign}(q_2 - a_3 \sin q_3)\} \qquad \text{if } q_2 - a_3 \sin q_3 \neq 0. \tag{5}$$

Plugging in the two values of q_3 and for each the corresponding pairs of values of q_2, we obtain from (5) four values of q_1, which are equal two by two. When $q_2 - a_3 \sin q_3 = 0$, q_1 is instead undefined (there are infinite solutions to the inverse kinematics problem).

Summarizing, the four regular solutions are:

$$\begin{aligned}
\boldsymbol{q}^{(+,+)} &= \left(q_1^{(+,+)}, q_2^{(+,+)}, q_3^{(+)}\right)\\
\boldsymbol{q}^{(+,-)} &= \left(q_1^{(+,-)}, q_2^{(+,-)}, q_3^{(+)}\right)\\
\boldsymbol{q}^{(-,+)} &= \left(q_1^{(-,+)}, q_2^{(-,+)}, q_3^{(-)}\right)\\
\boldsymbol{q}^{(-,-)} &= \left(q_1^{(-,-)}, q_2^{(-,-)}, q_3^{(-)}\right).
\end{aligned}$$

For completeness, some numerical cases (not requested by the exercise) are reported here to check the correctness of the above formulas in various situations. Table 2 shows the obtained results, using $d_1 = a_3 = 1$ m as robot data. Joint values are in [rad,m,rad] units. The third case has $\boldsymbol{p}_d$ at the upper boundary of the workspace (the four solutions collapse in pairs).

$\boldsymbol{p}_d$ [m]	$\boldsymbol{q}^{(+,+)}$	$\boldsymbol{q}^{(+,-)}$	$\boldsymbol{q}^{(-,+)}$	$\boldsymbol{q}^{(-,-)}$
$\begin{pmatrix}1.5\\1.5\\1.5\end{pmatrix}$	$\begin{pmatrix}-0.7854\\2.9873\\1.0472\end{pmatrix}$	$\begin{pmatrix}2.3562\\-1.2553\\1.0472\end{pmatrix}$	$\begin{pmatrix}-0.7854\\1.2553\\-1.0472\end{pmatrix}$	$\begin{pmatrix}2.3562\\-2.9873\\-1.0472\end{pmatrix}$
$\begin{pmatrix}-1.5\\0\\1\end{pmatrix}$	$\begin{pmatrix}1.5708\\2.5000\\1.5708\end{pmatrix}$	$\begin{pmatrix}-1.5708\\-0.5000\\1.5708\end{pmatrix}$	$\begin{pmatrix}1.5708\\0.5000\\-1.5708\end{pmatrix}$	$\begin{pmatrix}-1.5708\\-2.5000\\-1.5708\end{pmatrix}$
$\begin{pmatrix}1.5\\0\\2\end{pmatrix}$	$\begin{pmatrix}-1.5708\\1.5000\\0\end{pmatrix}$	$\begin{pmatrix}1.5708\\-1.5000\\0\end{pmatrix}$	$\begin{pmatrix}-1.5708\\1.5000\\0\end{pmatrix}$	$\begin{pmatrix}1.5708\\-1.5000\\0\end{pmatrix}$

Table 2: Results of three feasible inverse kinematics problems for the RPR spatial robot

[1]The same expression is obtained when squaring and summing all three equations in (1); using again $\sin q_3$ from (2), leads then to a second-order polynomial equation in q_2, whose solutions are given by (3).

The velocity $\boldsymbol{v} \in \mathbb{R}^3$ of the origin O_3 is obtained differentiating (1) with respect to time,

$$\boldsymbol{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} -\cos q_1 (q_2 - a_3 \sin q_3) & -\sin q_1 & a_3 \sin q_1 \cos q_3 \\ -\sin q_1 (q_2 - a_3 \sin q_3) & \cos q_1 & -a_3 \cos q_1 \cos q_3 \\ 0 & 0 & -a_3 \sin q_3 \end{pmatrix} \dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}. \tag{6}$$

The 3×3 Jacobian matrix $\boldsymbol{J}(\boldsymbol{q})$ has determinant

$$\det \boldsymbol{J}(\boldsymbol{q}) = a_3 \sin q_3 \left(q_2 - a_3 \sin q_3\right),$$

which is zero either when $\sin q_3 = 0$ ($q_3 = 0$ or π) or when $q_2 = a_3 \sin q_3$, or when both hold, i.e., $q_2 = 0$ and $q_3 = 0$ or π. In a singular configuration $\boldsymbol{q}_s = (q_1, q_2, 0)$. with $q_2 \neq 0$, one has

$$\boldsymbol{J}(\boldsymbol{q}_s) = \begin{pmatrix} -q_2 \cos q_1 & -\sin q_1 & a_3 \sin q_1 \\ -q_2 \sin q_1 & \cos q_1 & -a_3 \cos q_1 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \dot{\boldsymbol{q}}_s = \beta \begin{pmatrix} 0 \\ a_3 \\ 1 \end{pmatrix} \in \mathcal{N}\{\boldsymbol{J}(\boldsymbol{q}_s)\}.$$

All end-effector velocities $\boldsymbol{v}_s \in \mathbb{R}^3$ that cannot be instantaneously realized have the form, for $\gamma \neq 0$,

$$\boldsymbol{v}_s = \begin{pmatrix} * \\ * \\ \gamma \end{pmatrix} \notin \mathcal{R}\{\boldsymbol{J}(\boldsymbol{q}_s)\}.$$

When in addition $q_2 = 0$, or $\boldsymbol{q}_s = (q_1, 0, 0)$ for an arbitrary q_1, one has similarly

$$\boldsymbol{J}(\boldsymbol{q}_s) = \begin{pmatrix} 0 & -\sin q_1 & a_3 \sin q_1 \\ 0 & \cos q_1 & -a_3 \cos q_1 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \dot{\boldsymbol{q}}_s = \beta_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta_2 \begin{pmatrix} 0 \\ a_3 \\ 1 \end{pmatrix} \in \mathcal{N}\{\boldsymbol{J}(\boldsymbol{q}_s)\},$$

being $\operatorname{rank} \boldsymbol{J}(\boldsymbol{q}_s) = 1$. The end-effector velocities that cannot be instantaneously realized have one of the following forms[2], both $\notin \mathcal{R}\{\boldsymbol{J}(\boldsymbol{q}_s)\}$ for $\gamma_1 \neq 0, \gamma_2 \neq 0$:

$$\boldsymbol{v}_{s1} = \gamma_1 \begin{pmatrix} * \\ * \\ 1 \end{pmatrix} \qquad \boldsymbol{v}_{s2} = \gamma_2 \begin{pmatrix} \cos q_1 \\ \sin q_1 \\ * \end{pmatrix}.$$

Consider now that the motion of the prismatic joint is bounded as $q_2 \in [0, L]$, while the revolute joints are free to rotate. For simplicity, assume that $L > a_3$. The primary workspace WS_1 is then a "tortilla" of thickness $2a_3$, with rounded external border and a maximum radius of $L + a_3$ from the z_0 axis. Figure 4 shows a vertical section of WS_1 cut at a given q_1. In the white area strictly inside WS_1, there are two inverse kinematics (IK) solutions (the other two are excluded, since they would require $q_2 < 0$). On the boundary ∂WS_1 of the primary workspace there is only one IK solution. The same happens when $\boldsymbol{p}_d$ is strictly inside the outer dashed circular area. As for the inner dotted circular area around the z_0 axis, when removing the segment in black, there is one IK solution for the given q_1. However, a second IK solution is available by rotating q_1 by π. Finally, on the black segment on the z_0 axis, characterized by $p_{xd} = p_{yd} = 0$, $p_{zd} \in [d_1 - a_3, d_1 + a_3]$, there are infinite IK solutions (obtained by rotating q_1). It is also interesting to relate the drop/increase of the number of IK solutions to the singularities of the Jacobian matrix in (6) [this is left to the reader].

[2] While $\mathcal{R}\{\boldsymbol{J}(\boldsymbol{q}_s)\}$ is a subspace, the set of vectors not belonging to a given subspace is *not* a subspace itself.

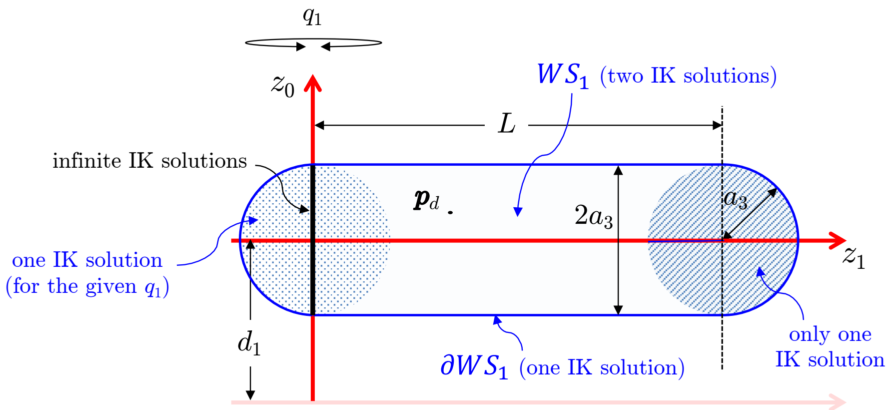

Figure 4: A section of the primary workspace for the RPR robot when q_2 is bounded in $[0, L]$

On the other hand, the secondary workspace is empty, $WS_2 = \emptyset$. In fact, the robot end-effector can point in any direction of the 3D space (a two-dimensional task), but there is no single position in WS_1 such that all pointing directions can be realized.

Exercise 3

The proof of the identity $(\boldsymbol{J}^T)^\# = (\boldsymbol{J}^\#)^T$, i.e., the fact that the order of pseudoinversion and transposition of a matrix can be exchanged, uses the four defining properties of the pseudoinverse of a matrix and the fact that $\boldsymbol{J}^\#$ is the pseudoinverse of $\boldsymbol{J}$.

1. $\boldsymbol{J}^T(\boldsymbol{J}^T)^\#\,\boldsymbol{J}^T = \boldsymbol{J}^T$

$$\boldsymbol{J}^T(\boldsymbol{J}^T)^\#\,\boldsymbol{J}^T = \boldsymbol{J}^T(\boldsymbol{J}^\#)^T\,\boldsymbol{J}^T = \left(\boldsymbol{J}\,\boldsymbol{J}^\#\,\boldsymbol{J}\right)^T = \boldsymbol{J}^T$$

2. $(\boldsymbol{J}^T)^\#\,\boldsymbol{J}^T(\boldsymbol{J}^T)^\# = (\boldsymbol{J}^T)^\#$

$$(\boldsymbol{J}^T)^\#\,\boldsymbol{J}^T(\boldsymbol{J}^T)^\# = (\boldsymbol{J}^\#)^T\,\boldsymbol{J}^T(\boldsymbol{J}^\#)^T = \left(\boldsymbol{J}^\#\boldsymbol{J}\,\boldsymbol{J}^\#\right)^T = (\boldsymbol{J}^\#)^T = (\boldsymbol{J}^T)^\#$$

3. $\left(\boldsymbol{J}^T(\boldsymbol{J}^T)^\#\right)^T = \boldsymbol{J}^T(\boldsymbol{J}^T)^\#$

$$\left(\boldsymbol{J}^T(\boldsymbol{J}^T)^\#\right)^T = \left(\boldsymbol{J}^T(\boldsymbol{J}^\#)^T\right)^T = \boldsymbol{J}^\#\boldsymbol{J} = \left(\boldsymbol{J}^\#\boldsymbol{J}\right)^T = \boldsymbol{J}^T(\boldsymbol{J}^\#)^T = \boldsymbol{J}^T(\boldsymbol{J}^T)^\#$$

4. $\left((\boldsymbol{J}^T)^\#\,\boldsymbol{J}^T\right)^T = (\boldsymbol{J}^T)^\#\,\boldsymbol{J}^T$

$$\left((\boldsymbol{J}^T)^\#\,\boldsymbol{J}^T\right)^T = \left((\boldsymbol{J}^\#)^T\boldsymbol{J}^T\right)^T = \boldsymbol{J}\,\boldsymbol{J}^\# = \left(\boldsymbol{J}\,\boldsymbol{J}^\#\right)^T = (\boldsymbol{J}^\#)^T\boldsymbol{J}^T = (\boldsymbol{J}^T)^\#\,\boldsymbol{J}^T.$$

Another possibility is to use the Singular Value Decomposition (SVD) of matrix $\boldsymbol{J}$. However, we should formally distinguish then the two cases when $m < n$ (or $m \le n$) and when $m \ge n$ (or $m > n$). Consider for instance the first case, with

$$\boldsymbol{J} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T \qquad\qquad \boldsymbol{\Sigma} = \left(\begin{array}{cccccc|c} \sigma_1 & 0 & 0 & & \dots & & 0 & \\ 0 & \sigma_2 & 0 & & \dots & & 0 & \\ 0 & \dots & \ddots & & \dots & & 0 & \boldsymbol{O}_{m\times(n-m)} \\ 0 & \dots & 0 & \sigma_\rho & 0 & \dots & 0 & \\ 0 & & \dots & & 0 & \dots & 0 & \\ 0 & & & \dots & & & 0 & \end{array}\right),$$

where $\rho = \text{rank}\, \boldsymbol{J} \leq m$. We know that

$$\boldsymbol{J}^{\#} = \boldsymbol{V}\boldsymbol{\Sigma}^{\#}\boldsymbol{U}^T \qquad \boldsymbol{\Sigma}^{\#} = \begin{pmatrix} \frac{1}{\sigma_1} & 0 & & \dots & 0 \\ 0 & \frac{1}{\sigma_2} & 0 & \dots & 0 \\ 0 & & \ddots & & \dots\ 0 \\ 0 & & \dots & \frac{1}{\sigma_\rho} & \dots\ 0 \\ 0 & & & \dots & 0 \\ & & & \boldsymbol{O}_{(n-m)\times m} & \end{pmatrix},$$

Therefore, being

$$\boldsymbol{J}^T = \boldsymbol{V}\boldsymbol{\Sigma}^T\boldsymbol{U}^T \quad \Rightarrow \quad \left(\boldsymbol{J}^T\right)^{\#} = \boldsymbol{U}\left(\boldsymbol{\Sigma}^T\right)^{\#}\boldsymbol{V}^T \qquad \text{and} \qquad \left(\boldsymbol{J}^{\#}\right)^T = \boldsymbol{U}\left(\boldsymbol{\Sigma}^{\#}\right)^T\boldsymbol{V}^T,$$

we only need to show that

$$\left(\boldsymbol{\Sigma}^T\right)^{\#} = \left(\boldsymbol{\Sigma}^{\#}\right)^T,$$

which follows by direct inspection from

$$\left(\boldsymbol{\Sigma}^T\right)^{\#} = \left(\begin{array}{cccccccc|c} \frac{1}{\sigma_1} & 0 & 0 & & \dots & & & 0 & \\ 0 & \frac{1}{\sigma_2} & 0 & & \dots & & & 0 & \\ 0 & \dots & \ddots & & \dots & & & 0 & \boldsymbol{O}_{m\times(n-m)} \\ 0 & \dots & 0 & \frac{1}{\sigma_\rho} & 0 & \dots & & 0 & \\ 0 & & \dots & & 0 & \dots & & 0 & \\ 0 & & & \dots & & & & 0 & \end{array}\right).$$

The same procedure can be followed in case $m \geq n$.

Exercise 4

The parametrization of the given path (an arc of a circle of radius R) with its arc length σ is

$$\boldsymbol{p}(\sigma) = \boldsymbol{p}_C + R\begin{pmatrix} \cos\frac{\sigma}{R} \\ \sin\frac{\sigma}{R} \end{pmatrix} \qquad \sigma \in \left[0, \frac{\pi R}{4}\right]. \tag{7}$$

In fact

$$\boldsymbol{p}(0) = \boldsymbol{p}_C + R\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} C_x + R \\ C_y \end{pmatrix} = \boldsymbol{p}_A$$

$$\boldsymbol{p}\left(\frac{\pi R}{4}\right) = \boldsymbol{p}_C + R\begin{pmatrix} \cos\frac{\pi}{4} \\ \sin\frac{\pi}{4} \end{pmatrix} = \begin{pmatrix} C_x + \sqrt{2}\,R/2 \\ C_y + \sqrt{2}\,R/2 \end{pmatrix} = \boldsymbol{p}_B,$$

and the length of the path is $L = (\pi/4)\, R$. The first and second time derivatives of (7) are

$$\dot{\boldsymbol{p}} = \begin{pmatrix} -\sin\dfrac{\sigma}{R} \\ \cos\dfrac{\sigma}{R} \end{pmatrix} \dot{\sigma}$$

$$\ddot{\boldsymbol{p}} = \begin{pmatrix} -\sin\dfrac{\sigma}{R} \\ \cos\dfrac{\sigma}{R} \end{pmatrix} \ddot{\sigma} - \frac{1}{R} \begin{pmatrix} \cos\dfrac{\sigma}{R} \\ \sin\dfrac{\sigma}{R} \end{pmatrix} \dot{\sigma}^2 = Rot(\sigma/R) \begin{pmatrix} -\dfrac{\dot{\sigma}^2}{R} \\ \ddot{\sigma} \end{pmatrix},$$

where

$$Rot(\sigma/R) = \begin{pmatrix} \cos\dfrac{\sigma}{R} & -\sin\dfrac{\sigma}{R} \\ \sin\dfrac{\sigma}{R} & \cos\dfrac{\sigma}{R} \end{pmatrix}$$

is an element of $SO(2)$ (thus, $Rot^{-1}(\alpha) = Rot^T(\alpha)$ and $\det Rot(\alpha) = +1$ for any α). Therefore,

$$\|\dot{\boldsymbol{p}}\| = |\dot{\sigma}| \qquad \|\ddot{\boldsymbol{p}}\| = \sqrt{\ddot{\boldsymbol{p}}^T \ddot{\boldsymbol{p}}} = \sqrt{\left(\frac{\dot{\sigma}^2}{R}\right)^2 + \ddot{\sigma}^2}$$

and the two bounds on the norms of the Cartesian velocity and acceleration become

$$|\dot{\sigma}| \leq v_{\max} \qquad \sqrt{\left(\frac{\dot{\sigma}^2}{R}\right)^2 + \ddot{\sigma}^2} \leq a_{\max}. \tag{8}$$

A rest-to-rest timing law $\sigma(t)$ with bang-coast-bang (b-c-b) acceleration and a corresponding trapezoidal speed profile is defined by the laws

$$\ddot{\sigma}(t) = \begin{cases} A & t \in [0, T_r] \\ 0 & t \in [T_r, T - T_r] \\ -A & t \in [T - T_r, T] \end{cases} \qquad \dot{\sigma}(t) = \begin{cases} A\, t & t \in [0, T_r] \\ V & t \in [T_r, T - T_r] \\ V - A\,(t - T + T_r) & t \in [T - T_r, T], \end{cases}$$

with rising time $T_r = V/A$ and total motion time T. The two values A (constant absolute acceleration/deceleration) and V (constant cruising speed) have to be chosen so as to minimize the motion time T while complying with the bounds (8). The length of the path is $L = \pi R/4$. The above profiles hold as long as $L > V^2/A$ (condition of existence of the cruising phase at constant speed V). The travel time is then

$$T = \frac{AL + V^2}{AV} = \frac{L}{V} + \frac{V}{A}. \tag{9}$$

However, if the path is too short ($L \leq V^2/A$) to allow reaching the chosen cruising speed V, the timing law will collapse into a bang-bang (b-b) acceleration with triangular profile for the speed,

$$\ddot{\sigma}(t) = \begin{cases} A & t \in [0, \bar{T}/2] \\ -A & t \in [\bar{T}/2, \bar{T}] \end{cases} \qquad \dot{\sigma}(t) = \begin{cases} At & t \in [0, \bar{T}/2] \\ \bar{V} - A\left(t - \bar{T}/2\right) & t \in [\bar{T}/2, \bar{T}], \end{cases}$$

where

$$\bar{T} = \sqrt{\frac{4L}{A}} \qquad \bar{V} = \frac{A\bar{T}}{2}\ (\leq V) \tag{10}$$

are, respectively, the travel time and the maximum speed reached along the path.

In order to determine the optimal values of V and A, consider first the b-c-b case. From (8), taking only into account the phase at constant cruising speed V where $\ddot{\sigma} = 0$, we get

$$|\dot{\sigma}| \leq V = \min\left\{v_{\max}, \sqrt{R\, a_{\max}}\right\}, \tag{11}$$

where the second limitation comes from the centripetal Cartesian acceleration.[3] Further, the worst case for the acceleration bound is when both $\dot{\sigma}$ and $\ddot{\sigma}$ take their maximum (absolute) value, or

$$\sqrt{\left(\frac{V^2}{R}\right)^2 + A^2} = a_{\max} \qquad \Rightarrow \qquad A = \sqrt{a_{\max}^2 - \left(\frac{V^2}{R}\right)^2}.$$

Therefore, if the values of the bounds on the norms return $v_{\max}$ as the minimum in (11), then

$$V = v_{\max} \qquad\qquad A = \sqrt{a_{\max}^2 - \left(\frac{v_{\max}^2}{R}\right)^2} \tag{12}$$

and the b-c-b solution will occur if and only if

$$L = \frac{\pi R}{4} > \frac{v_{\max}^2}{\sqrt{a_{\max}^2 - \left(\frac{v_{\max}^2}{R}\right)^2}} = \frac{V^2}{A}, \tag{13}$$

yielding from (9)

$$T = \frac{\pi R}{4\, v_{\max}} + \frac{v_{\max}}{\sqrt{a_{\max}^2 - \left(\frac{v_{\max}^2}{R}\right)^2}}.$$

On the other hand, when inequality (13) is violated, the timing law will assume a b-b acceleration profile. From (10), we have then

$$\bar{T} = \frac{\sqrt{\pi R}}{\sqrt[4]{a_{\max}^2 - \left(\frac{v_{\max}^2}{R}\right)^2}} \qquad\qquad \bar{V} = \frac{\sqrt{\pi R}\,\sqrt[4]{a_{\max}^2 - \left(\frac{v_{\max}^2}{R}\right)^2}}{2}.$$

If instead the minimum in (11) is given by $\sqrt{R\, a_{\max}}$, then the previous b-c-b solution in closed form cannot be applied since the second Cartesian bound in (8) would return an upper bound $A = 0$ for $\ddot{\sigma}$ —which is clearly impossible (a similar issue occurs for the b-b solution). In fact, the problem should be reformulated in the positive quadrant of the $(\dot{\sigma}, \ddot{\sigma})$ plane (or, even better, of the $(\dot{\sigma}^2, \ddot{\sigma})$ plane), finding the feasible pair $(\dot{\sigma}, \ddot{\sigma}) = (V, A)$ that satisfies the bounds (8) and provides the minimum motion time T. Unfortunately, there is no closed formula for the optimal solution in that case. Thus, we skip any further analysis of this situation and simply turn our attention to the two numerical examples.

For case *i)*, we have $V = 1.2 = v_{max}$, $A = 2.6318$, and from (13)

$$L = 0.7854 > 0.5472 = \frac{V^2}{A}.$$

Thus, the minimum-time solution has a bang-coast-bang acceleration profile with

$$T = 1.1105\,\text{s} \qquad\qquad T_r = 0.4560\,\text{s}.$$

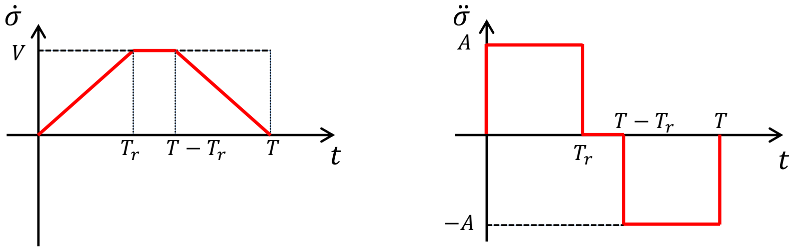

Figure 5: Time profiles of $\dot{\sigma}(t)$ and $\ddot{\sigma}(t)$ for case *i)*

The speed and acceleration profiles are shown in Fig. 5.

For case *ii)*, we have $V = 1.6 = v_{max}$, $A = 2.1282$, and from (13)

$$L = 0.3927 < 0.8184 = \frac{V^2}{A}.$$

Thus, the minimum-time solution has a bang-bang acceleration profile with

$$\bar{T} = 0.7086\,\text{s} \qquad \bar{T}_r = \frac{\bar{T}}{2} = 0.3543\,\text{s} \qquad \bar{V} = 1.1083\,\text{m/s}.$$

The speed and acceleration profiles are shown in Fig. 6.

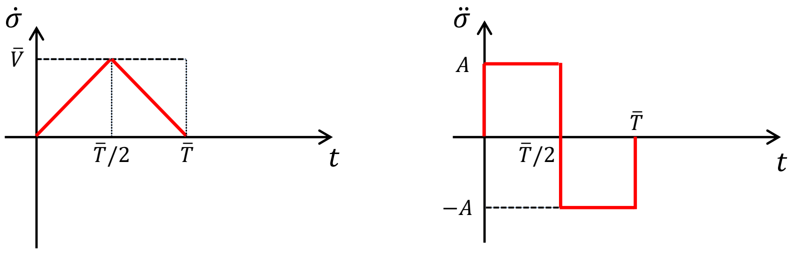

Figure 6: Time profiles of $\dot{\sigma}(t)$ and $\ddot{\sigma}(t)$ for case *ii)*

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[3]For a linear path, $R \to \infty$ and the minimum is always given by the direct bound $v_{\max}$ on the Cartesian velocity. By continuity, the same holds true for a circular path with sufficiently small curvature $\kappa = 1/R$.