Dipartimento di Informatica e Sistemistica "Antonio Ruberti"
SAPIENZA Università di Roma

Logics of Programs

Dipartimento di Informatica e Sistemistica "Antonio Ruberti"
SAPIENZA Università di Roma

Logics of Programs

- Are modal logics that allow to describe properties of transition systems
- Examples:
  - HennesyMilner Logic
  - Propositional Dynamic Logics
  - Modal (Propositional) Mu-calculus
- Perfectly suited for describing transition systems: they can tell apart transition systems modulo bisimulation

HennessyMilner Logic

HM Logic aka (multi) modal logic Ki

- Syntax:

Φ := Final | P (atomic propositions)
[a]Φ | <a>Φ (modal operators)
$\neg\Phi$ | $\Phi_1 \wedge \Phi_2$ | $\Phi_1 \vee \Phi_2$ | true | false (closed under booleans)

- Propositions are used to denote final states and other TS atomic properties
- <a>Φ means there exists an a-transition that leads to a state where Φ holds; i.e., expresses the capability of executing action a bringing about Φ
- [a]Φ means that all a-transitions lead to states where Φ holds; i.e., express that executing action a brings about Φ

HennessyMilner Logic

- Semantics: assigns meaning to the formulas.
- Given a TS T = < A, S, S^0, δ, F>, a state s $\in$ S, and a formula Φ, we define (by structural induction) the "truth relation"

$$T,s \models \Phi$$

- T,s ⊨ Final if s ∈ F (similarly T,s ⊨ P if s ∈ P);
- T,s ⊨ [a] Φ if **for all** s' such that s $\rightarrow_a$ s' we have T,s' ⊨ Φ;
- T,s ⊨ ⟨a⟩Φ if **exists** s' such that s $\rightarrow_a$ s' and T,s' ⊨ Φ;
- T,s ⊨ $\neg\Phi$ if it is not the case that T,s ⊨ Φ;
- T,s ⊨ $\Phi_1 \vee \Phi_2$ if T,s ⊨ Φ_1 or T,s ⊨ Φ_2 ;
- T,s ⊨ $\Phi_1 \wedge \Phi_2$ if T,s ⊨ Φ_1 and T,s ⊨ Φ_2 ;
- T,s ⊨ true always;
- T,s ⊨ false never.

HennessyMilner Logic

- Another way to give the same semantics to formulas: formulas extension in a transition system assigns meaning to the formulas.

- Given a TS T = < A, S, S^0, δ, F> "the extension of a formula Φ in T", denote by $(\Phi)^T$, is defined as follows:
  - $(Final)^T$ = F (similarly $P^T = \{s \mid s \in P\}$);
  - $([a]\,\Phi)^T$ = $\{s \mid \forall s'.\ s \to_a s'$ implies $s' \in (\Phi)^T\}$;
  - $(\langle a\rangle\Phi)^T$ = $\{s \mid \exists s'.\ s \to_a s'$ and $s' \in (\Phi)^T\}$;
  - $(\neg\Phi)^T$ = $S - (\Phi)^T$;
  - $(\Phi_1 \vee \Phi_2)^T$ = $(\Phi_1)^T \cup (\Phi_2)^T$;
  - $(\Phi_1 \wedge \Phi_2)^T$ = $(\Phi_1)^T \cap (\Phi_2)^T$;
  - $(true)^T$ = S;
  - $(false)^T$ = $\emptyset$.

- Note: $T,s \models \Phi$ now written as $s \in (\Phi)^T$

Dipartimento di Informatica e Sistemistica "Antonio Ruberti"
SAPIENZA Università di Roma

Model Checking

- Given a TS T, one of its states s, and a formula Φ verify whether the formula holds in s. Formally:

$$T,s \models \Phi \quad \text{or} \quad s \in (\Phi)^T$$

- Examples (TS is our vending machine):
  - $S_0 \models$ Final
  - $S_0 \models$ <10c>true — *capability of performing action 10c*
  - $S_2 \models$ [big]false — *inability of performing action big*
  - $S_0 \models$ [10c][big]false — *after 10c cannot execute big*

- Model checking variant (aka "query answering"):
  - Given a TS T … — *the database*
  - … compute the extension of Φ — *the query*

Formally: compute the set $(\Phi)^T$ which is equal to $\{s \mid T,s \models \Phi\}$

Satisfiability

- Satisfiability: given a formula Φ verify whether there exists a (finite/ infinite) TS T and a state of T such that the formula holds in s.

  SAT: check the existence of T,s such that $T,s \models \Phi$

- Validity: given a formula Φ verify whether in every (finite/infinite) TS T and in every state of T the formula holds in s.

  VAL: check the non existence of T,s such that $T,s \models \neg\Phi$

  Note: VAL = non SAT

Examples: check the satifiability / validity of the following formulas:

- $\langle 10p\rangle\langle small\rangle\langle collect_s\rangle Final$
- $Final \rightarrow ((\langle 10p\rangle\langle small\rangle\langle collect_s\rangle Final) \land (\langle 20p\rangle\langle big\rangle\langle collect_b\rangle Final))$
- $\langle 10p\rangle\langle small\rangle\langle collect_s\rangle Final \land [10p]false$

Dipartimento di Informatica e Sistemistica "Antonio Ruberti"
SAPIENZA Università di Roma

HennessyMilner Logic and Bisimulation

- Consider two TS, $T = (A,S,s_0,\delta, F)$ and $T' = (A,S',t_0,\delta', F')$.
- Let L be the language formed by all HennessyMilner Logic formulas.
- We define:
  - $\sim_L = \{(s,t) \mid$ for all Φ of L we have $T,s \models \Phi$ iff $T',t \models \Phi\}$
  - $\sim = \{(s,t) \mid$ exists a bisimulation R s.t., $R(s,t)\}$
- **Theorem**: $s \sim_L t$ iff $s \sim t$
- Proof: we show that
  - $s \sim t$ implies $s \sim_L t$ by structural induction on formulas of L.
  - $s \sim_L t$ implies $s \sim t$ by coinduction showing that $s \sim_L t$ is a bisimulation.

This theorem says that HennessyMilner Logic has exactly the same distinguishing power of bisimulation.
So L is the right logic to predicate on transition systems.

An same results holds also for the PDL and Modal Mu-Calculus introduced below.

Examples

- Usefull abbreviation (let actions $A = \{a_{1,\ldots,} a_n\}$):
  - $<any>\ \Phi$ stands for $<a_1>\Phi \vee \cdots \vee <a_n>\Phi$
  - $[any]\ \Phi$ stands for $[a_1]\Phi \wedge \cdots \wedge [a_n]\Phi$
  - $<any - a_1>\ \Phi$ stands for $<a_2>\Phi \vee \cdots \vee <a_v>\Phi$
  - $[any - a_1]\ \Phi$ stands for $[a_2]\Phi \wedge \cdots \wedge [a_v]\Phi$

- Examples:
  - $<a>true$ *capability of performing action a*
  - $[a]false$ *inability of performing action a*
  - $\neg Final \wedge <any>true \wedge [any\text{-}a]false$ *necessity/inevitability of performing action a (i.e., action a is the only action possible)*
  - $\neg Final \wedge [any]false$ *deadlock!*

Propositional Dynamic Logic

- $\Phi := P \mid$ *(atomic propositions)*
  $\neg \Phi \mid \Phi_1 \wedge \Phi_2 \mid \Phi_1 \vee \Phi_2 \mid$ *(closed under boolean operators)*
  $[r]\Phi \mid <r>\Phi$ *(modal operators)*

  $r := a \mid r_1 + r_2 \mid r_1;r_2 \mid r^* \mid P?$ *(complex actions as regular expressions)*

- Essentially add the capability of expressing partial correctness assertions via formulas of the form
  - $\Phi_1 \rightarrow [r]\Phi_2$ *under the conditions Φ_1 all possible executions of r that terminate reach a state of the TS where Φ_2 holds*

- Also add the ability of asserting that a property holds in all nodes of the transition system
  - $[(a_1 + \cdots + a_v)^*]\Phi$ *in every reachable state of the TS Φ holds*

- Useful abbereviations:
  - any stands for $(a_1 + \cdots + a_v)$ *Note that + can be expressed also in HM Logic*
  - u stands for any* *This is the so called master/universal modality*

Modal Mu-Calculus

- $\Phi := P \mid$ *(atomic propositions)*
  $\neg \Phi \mid \Phi_1 \wedge \Phi_2 \mid \Phi_1 \vee \Phi_2 \mid$ *(closed under boolean operators)*
  $[r]\Phi \mid \langle r\rangle\Phi$ *(modal operators)*

  $\mu X.\Phi(X) \mid \nu X.\Phi(X)$ *(fixpoint operators)*

- It is the most expressive logic of the family of logics of programs.
- It subsumes
  - PDL (modalities involving complex actions are translated into formulas involving fixpoints)
  - LTL (linear time temporal logic),
  - CTS, CTS* (branching time temporal logics)

- Examples:
- $[any^*]\Phi$ can be expressed as $\nu X.\ \Phi \wedge [any]X$

- $\mu X.\ \Phi \vee [any]X$ — *along all runs eventually Φ*
- $\mu X.\ \Phi \vee \langle any\rangle X$ — *along some run eventually Φ*
- $\nu X.\ [a](\mu Y.\ \langle any\rangle true \wedge [any-b]Y) \wedge X$
  every run that contains a contains later b

Modal Mu-Calculus

- To understand fixpoint operators one has to consider them as fixpoint of equations:

- Namely given $\mu X.\Phi(X)$ and $\nu X.\Phi(X)$ consider the equation

$$X \equiv \Phi(X)$$

Then:
- $\mu X.\Phi(X)$ stands for the smallest predicate X such that $X \equiv \Phi(X)$ – or $\Phi(X) \rightarrow X$
- $\nu X.\Phi(X)$ stands for the largest predicate X such that $X \equiv \Phi(X)$ – or $X \rightarrow \Phi(X)$

Notice:
- $\mu X.\Phi(X)$ is defined by induction and computed by least fixpoint algorithm over the TS
- $\nu X.\Phi(X)$ is defined by coinduction and computed by greatest fixpoint algorithm over the TS

- Examples:
  - $\nu X.\ \Phi \wedge [any]X$ - gfp of $X \equiv \Phi \wedge [any]X$
  - $\mu X.\ \Phi \vee [any]X$ - lfp of $X \equiv \Phi \vee [any]X$
  - $\mu X.\ \Phi \vee \langle any\rangle X$ - lfp of $X \equiv \Phi \vee \langle any\rangle X$
  - $\nu X.\ [a](\mu Y.\ \langle any\rangle true \wedge [any-b]Y) \wedge X$
    - lfp of $y \equiv \langle any\rangle true \wedge [any-b]Y$
    - gfp of $X \equiv [a]($*lfp above*$) \wedge X$

Examples of Modal Mu-Calculus

Dipartimento di Informatica e Sistemistica "Antonio Ruberti"
SAPIENZA Università di Roma

- Examples (TS is our vending machine):
  - $S_0 \models$ Final
  - $S_0 \models$ <10c>true *capability of performing action 10c*
  - $S_2 \models$ [big]false *inability of performing action big*
  - $S_0 \models$ [10c][big]false *after 10c cannot execute big*
  - $S_i \models \mu$ X. Final $\vee$ [any] X *eventually a final state is reached*
  - $S_0 \models \nu$ Z. (μ X. Final $\vee$ [any] X) $\wedge$ [any] Z *or equivalently*
    $S_0 \models$ [any*](μ X. Final $\vee$ [any] X) *from everywhere eventually final*

Model Checking/Satisfiability

Dipartimento di Informatica e Sistemistica "Antonio Ruberti"
SAPIENZA Università di Roma

- Model checking is polynomial in the size of the TS for
  - HennessyMilner Logic
  - PDL
  - Modal Mu-Calculus
- Also model checking is wrt the formula
  - Polynomial for HennessyMiner Logic
  - Polynomial for PDL
  - Polynomial for Modal Mu-Calculus with bounded alternation of nested fixpoints, and NP$\cap$coNP in general
- Satisfiability is decidable for the three logics, and the complexity (in the size of the formula) is as follows:
  - HennessyMilner Logic: PSPACE-complete
  - PDL: EXPTIME-complete
  - Modal Mu-Calculus: EXPTIME-complete

AI Planning as Model Checking

Dipartimento di Informatica e Sistemistica "Antonio Ruberti"
SAPIENZA Università di Roma

- **Build the TS of the domain:**
  - Consider the set of states formed all possible truth value of the propositions (this works only for propositional setting).
  - Use Pre's and Post of actions for determining the transitions

  Note: the TS is exponential in the size od the description.

- **Write the goal in a logic of program**
  - typically a single least fixpoint formula of Mu-Calculus (compute ***reachable*** states intersection states where goal true)
- **Planning:**
  - model check the formula on the TS starting from the given initial state.
  - use the path (paths) used in the above model checking for returning the plan.

- *This basic technique works only when we have complete information (or at least total observability on state):*
  - *Sequential plans if initial state known and actions are deterministic*
  - *Conditional plans if many possible initial states and/or actions are nondeterministic*

Example

Dipartimento di Informatica e Sistemistica "Antonio Ruberti"
SAPIENZA Università di Roma

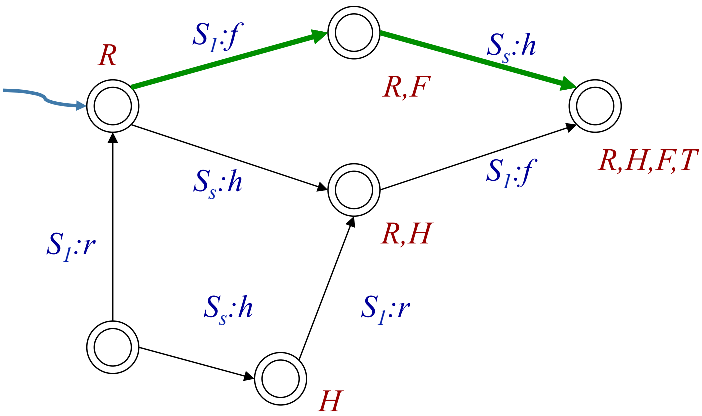
- Operators (Services + Mappings)
  - Registered ∧ ¬FlightBooked → [S_1:bookFlight] FlightBooked
  - ¬Registered → [S_1:register] Registered
  - ¬HotelBooked → [S_2:bookHotel] HotelBooked

- Additional constraints (Community Ontology):
  - TravelSettledUp ≡
    FlightBooked ∧ HotelBooked ∧ EventBooked

- Goals (Client Service Requests):
  - Starting from ***the*** state
    Registered ∧ ¬FlightBooked ∧ ¬ HotelBooked ∧ ¬EventBooked
    check <any*>TravelSettedUp

  - Starting from ***all*** states such that
    ¬FlightBooked ∧ ¬ HotelBooked ∧ ¬EventBooked
    check <any*>TravelSettledUp

Example



Plan:
S_1:bookFlight;
S_2:bookHotel

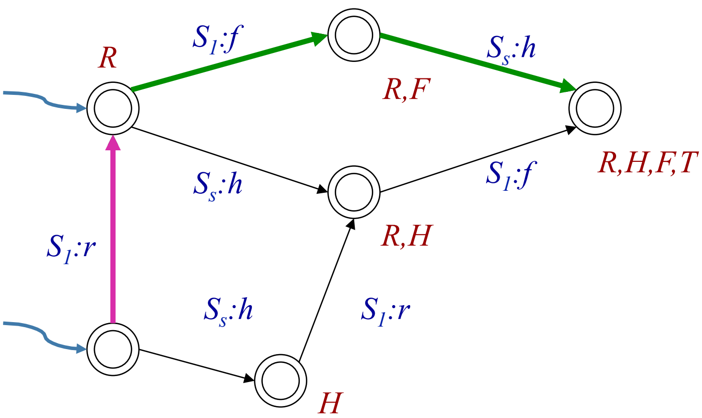
Starting from the state

Registered ∧ ¬ FlightBooked ∧ ¬ HotelBooked ∧ ¬ EventBooked

check

<any*>TravelSettledUp

Example



Plan:
if(¬Registered) {
S_1:register;
}
S_1:bookFlight;
S_2:bookHotel

Starting from all states where

¬ FlightBooked ∧ ¬ HotelBooked ∧ ¬ EventBooked

check

<any*>TravelSettledUp

Satisfiability

- Observe that a formula Φ may be used to select among all TS T those such that for a given state s we have that T,s $\models \Phi$
- SATISFIABILITY: Given a formula Φ verify whether there exists a TS T and a state s such that. Formally:

  check whether exists T, s such that T,s $\models \Phi$

- Satisfiability is:
  - PSPACE for HennesyMilner Logic
  - EXPTIME for PDL
  - EXPTIME for Mu-Calculus

References

[Stirling Banff96] C. Stirling: Modal and temporal logics for processes. Banff Higher Order Workshop LNCS 1043, 149-237, Springer 1996

[Bradfield&Stirling HPA01] J. Bradfield, C. Stirling: Modal logics and mu-calculi. Handbook of Process Algebra, 293-332, Elsevier, 2001.

[Stirling 2001] C. Stirling: Modal and Temporal Properties of Processes. Texts in Computer Science, Springer 2001

[Kozen&Tiuryn HTCS90] D. Kozen, J. Tiuryn: Logics of programs. Handbook of Theoretical Computer Science, Vol. B, 789–840. North Holland, 1990.

[HKT2000] D. Harel, D. Kozen, J. Tiuryn: Dynamic Logic. MIT Press, 2000.

[Clarke& Schlingloff HAR01] E. M. Clarke, B. Schlingloff: Model Checking. Handbook of Automated Reasoning 2001: 1635-1790

[CGP 2000] E.M. Clarke, O. Grumberg, D. Peled: Model Checking. MIT Press, 2000.

[Emerson HTCS90] E. A. Emerson. Temporal and Modal Logic. Handbook of Theoretical Computer Science, Vol B: 995-1072. North Holland, 1990.

[Emerson Banff96] E. A. Emerson. Automated Temporal Reasoning about Reactive Systems. Banff Higher Order Workshop, LNCS 1043, 111-120, Springer 1996

[Vardi CST] M. Vardi: Alternating automata and program verification. Computer Science Today -Recent Trends and Developments, LNCS Vol. 1000, Springer, 1995.

[Vardi etal CAV94] M. Vardi, O. Kupferman and P. Wolper: An Automata-Theoretic Approach to Branching-Time Model Checking (full version of CAV'94 paper).

[Schneider 2004] K. Schenider: Verification of Reactive Systems, Springer 2004.