A LATTICE-THEORETICAL FIXPOINT THEOREM AND ITS APPLICATIONS

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1. A lattice-theoretical fixpoint theorem. In this section we formulate and prove an elementary fixpoint theorem which holds in arbitrary complete lattices. In the following sections we give various applications (and extensions) of this result in the theories of simply ordered sets, real functions, Boolean algebras, as well as in general set theory and topology.[1]

By a *lattice* we understand as usual a system $\mathfrak{A} = \langle A, \leq \rangle$ formed by a non-empty set A and a binary relation $\leq$; it is assumed that $\leq$ establishes a partial order in A and that for any two elements $a,b \in A$ there is a least upper bound (join) $a \cup b$ and a greatest lower bound (meet) $a \cap b$. The relations $\geq$, $<$, and $>$ are defined in the usual way in terms of $\leq$.

The lattice $\mathfrak{A} = \langle A, \leq \rangle$ is called *complete* if every subset B of A has a least upper bound $\bigcup B$ and a greatest lower bound $\bigcap B$. Such a lattice has in particular two elements 0 and 1 defined by the formulas

$$0 = \bigcap A \quad \text{and} \quad 1 = \bigcup A .$$

Given any two elements $a,b \in A$ with $a \leq b$, we denote by $[a,b]$ the *interval* with the endpoints a and b, that is, the set of all elements $x \in A$ for which $a \leq x \leq b$; in symbols,

$$[a,b] = \mathsf{E}_x[x \in A \quad \text{and} \quad a \leq x \leq b].$$

The system $\langle [a,b], \leq \rangle$ is clearly a lattice; it is a complete if $\mathfrak{A}$ is complete.

We shall consider functions on A to A and, more generally, on a subset B of A to another subset C of A. Such a function f is called *increasing* if, for any

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[1] For notions and facts concerning lattices, simply ordered systems, and Boolean algebras consult [1].

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elements $x,y \in B$, $x \leq y$ implies $f(x) \leq f(y)$. By a *fixpoint* of a function f we understand, of course, an element x of the domain of f such that $f(x) = x$.

Throughout the discussion the variables $a,b,\cdots,x,y,\cdots$ are assumed to represent arbitrary elements of a lattice (or another algebraic system involved).

THEOREM 1 (LATTICE-THEORETICAL FIXPOINT THEOREM). *Let*

(i) $\mathfrak{A} = \langle A, \leq \rangle$ *be a complete lattice,*

(ii) f *be an increasing function on* A *to* A,

(iii) P *be the set of all fixpoints of* f.

Then the set P *is not empty and the system* $\langle P, \leq \rangle$ *is a complete lattice; in particular we have*

$$\bigcup P = \bigcup E_x[f(x) \geq x] \in P$$

and

$$\bigcap P = \bigcap E_x[f(x) \leq x] \in P.$$ [2]

Proof. Let

$$u = \bigcup E_x[f(x) \geq x]. \tag{1}$$

We clearly have $x \leq u$ for every element x with $f(x) \geq x$; hence, the function f being increasing,

$$f(x) \leq f(u) \text{ and } x \leq f(u).$$

By (1) we conclude that

$$u \leq f(u). \tag{2}$$

[2] In 1927 Knaster and the author proved a set-theoretical fixpoint theorem by which every function, on and to the family of all subsets of a set, which is increasing under set-theoretical inclusion has at least one fixpoint; see [**3**], where some applications of this result in set theory (a generalization of the Cantor-Bernstein theorem) and topology are also mentioned. A generalization of this result is the lattice-theoretical fixpoint theorem stated above as Theorem 1. The theorem in its present form and its various applications and extensions were found by the author in 1939 and discussed by him in a few public lectures in 1939-1942. (See, for example, a reference in the American Mathematical Monthly **49** (1942), 402.) An essential part of Theorem 1 was included in [**1**, p. 54]; however, the author was informed by Professor Garrett Birkhoff that a proper historical reference to this result was omitted by mistake.

Therefore

$$f(u) \leq f(f(u)),$$

so that $f(u)$ belongs to the set $\mathsf{E}_x[f(x) \geq x]$; consequently, by (1),

(3) $$f(u) \leq u.$$

Formulas (2) and (3) imply that u is a fixpoint of f; hence we conclude by (1) that u is the join of all fixpoints of f, so that

(4) $$\bigcup P = \bigcup \mathsf{E}_x[f(x) \geq x] \in P.$$

Consider the dual lattice $\mathfrak{A}' = \langle A, \geq \rangle$. $\mathfrak{A}'$, like $\mathfrak{A}$, is complete, and f is again an increasing function in $\mathfrak{A}'$. The join of any elements in $\mathfrak{A}'$ obviously coincides with the meet of these elements in $\mathfrak{A}$. Hence, by applying to $\mathfrak{A}'$ the result established for $\mathfrak{A}$ in (4), we conclude that

(5) $$\bigcap P = \bigcap \mathsf{E}_x[f(x) \leq x] \in P.$$

Now let Y be any subset of P. The system

$$\mathfrak{B} = \langle [\bigcup Y, 1], \leq \rangle$$

is a complete lattice. For any $x \in Y$ we have $x \leq \bigcup Y$ and hence

$$x = f(x) \leq f(\bigcup Y);$$

therefore $\bigcup Y \leq f(\bigcup Y)$. Consequently, $\bigcup Y \leq z$ implies

$$\bigcup Y \leq f(\bigcup Y) \leq f(z).$$

Thus, by restricting the domain of f to the interval $[\bigcup Y, 1]$, we obtain an increasing function f' on $[\bigcup Y, 1]$ to $[\bigcup Y, 1]$. By applying formula (5) established above to the lattice $\mathfrak{B}$ and to the function f', we conclude that the greatest lower bound v of all fixpoints of f' is itself a fixpoint of f'. Obviously, v is a fixpoint of f, and in fact the least fixpoint of f which is an upper bound of all elements of Y; in other words, v is the least upper bound of Y in the system $\langle P, \leq \rangle$. Hence, by passing to the dual lattices $\mathfrak{A}'$ and $\mathfrak{B}'$, we see that there exists also a greatest lower bound of Y in $[P, \leq]$. Since Y is an arbitrary subset of P, we finally conclude that

(6) the system $\langle P, \leq \rangle$ is a complete lattice.

In view of (4)-(6), the proof has been completed.

By the theorem just proved, the existence of a fixpoint for every increasing function is a necessary condition for the completeness of a lattice. The question naturally arises whether this condition is also sufficient. It has been shown that the answer to this question is affirmative.[3]

A set F of functions is called *commutative* if

(i) all the functions of F have a common domain, say B, and the ranges of all functions of F are subsets of B;

(ii) for any $f, g \in F$ we have $fg = gf$, that is,

$$f(g(x)) = g(f(x)) \text{ for every } x \in B.$$

Using this notion we can improve Theorem 1 in the following way:

THEOREM 2 (GENERALIZED LATTICE-THEORETICAL FIXPOINT THEOREM). *Let*

(i) $\mathfrak{A} = \langle A, \leq \rangle$ *be a complete lattice,*

(ii) F *be any commutative set of increasing functions on* A *to* A,

(iii) P *be the set of all common fixpoints of all the functions* $f \in F$.

Then the set P *is not empty and the system* $\langle P, \leq \rangle$ *is a complete lattice; in particular, we have*

$$\cup P = \cup \mathsf{E}_x [f(x) \geq x \textit{ for every } f \in F] \in P$$

and

$$\cap P = \cap \mathsf{E}_x [f(x) \leq x \textit{ for every } f \in F] \in P.$$

Proof. Let

(1) $$u = \cup \mathsf{E}_x [f(x) \geq x \textit{ for every } f \in F].$$

As in the proof of Theorem 1 we show that

(2) $$u \leq f(u) \text{ for every } f \in F.$$

Given any function $g \in F$, we have, by (2),

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[3] This is a result of Anne C. Davis; see her note [2] immediately following this this paper.

$$g(u) \leq g(f(u))$$

and hence, the set F being commutative,

$$g(u) \leq f(g(u))$$

for every $f \in F$. Thus

$$g(u) \in \mathsf{E}_x[f(x) \geq x \textit{ for every } f \in F].$$

Therefore, by (1),

$$g(u) \leq u;$$

since g is an arbitrary function of F, we have

$$(3) \qquad f(u) \leq u \text{ for every } f \in F.$$

From (1)-(3) we conclude that u is a common fixpoint of all functions $f \in F$, and, in fact, the least upper bound of all such common fixpoints. In other words,

$$\mathsf{U}P = \mathsf{U}\mathsf{E}_x[f(x) \geq x \textit{ for every } f \in F] \in P.$$

In its remaining part the proof is entirely analogous to that of Theorem 1.

Since every set consisting of a single function is obviously commutative, Theorem 2 comprehends Theorem 1 as a particular case. Theorem 2 will not be involved in our further discussion.

2. Applications and extensions in the theories of simply ordered sets and real functions. A *simply ordered system* $\mathfrak{A} = \langle A, \leq \rangle$, that is, a system formed by a nonempty set A and a binary relation $\leq$ which establishes a simple order in A, is obviously a lattice. If it is a complete lattice, it is called a *continuously* (or *completely*) *ordered system*. The system $\mathfrak{A}$ is said to be a *densely ordered system* if, for all $x,y \in A$ with $x < y$, there is a $z \in A$ with $x < z < y$.

Theorems 1 and 2 obviously apply to every continuously ordered system $\mathfrak{A}$. Under the additional assumption that $\mathfrak{A}$ is densely ordered we can improve Theorem 1 by introducing the notions of quasi-increasing and quasi-decreasing functions.

Given a function f and a subset X of its domain, we denote by $f^*(X)$ the set of all elements $f(x)$ correlated with elements $x \in X$. A function f on B to C, where B and C are any two subsets of A, is called *quasi-increasing* if it satisfies the formulas

$$f(\bigcup X) \geq \bigcap f^*(X) \quad \text{and} \quad f(\bigcap X) \leq \bigcup f^*(X)$$

for every nonempty subset X of B. It is called *quasi-decreasing* if it satisfies the formulas

$$f(\bigcup X) \leq \bigcup f^*(X) \quad \text{and} \quad f(\bigcap X) \geq \bigcap f^*(X)$$

for every nonempty subset X of A. A function which is both quasi-increasing and quasi-decreasing is called *continuous*.

THEOREM 3. *Let*

(i) $\mathfrak{A} = \langle A, \leq \rangle$ *be a continuously and densely ordered set,*

(ii) f *be a quasi-increasing function and* g *a quasi-decreasing function on* A *to* A *such that*

$$f(0) \geq g(0) \text{ and } f(1) \leq g(1),$$

(iii) $P = \mathsf{E}_x[f(x) = g(x)]$.

Then P *is not empty and* $\langle P, \leq \rangle$ *is a continuously ordered system; in particular we have*

$$\bigcup P = \bigcup \mathsf{E}_x[f(x) \geq g(x)] \in P$$

and

$$\bigcap P = \bigcap \mathsf{E}_x[f(x) \leq g(x)] \in P.$$

Proof. Let B be any subset of A such that

(1) $$f(x) \geq g(x) \text{ for } x \in B.$$

Assume that

(2) $$f(\bigcup B) < g(\bigcup B).$$

Since, by hypothesis, $f(0) \geq g(0)$, we conclude that

(3) $$\bigcup B \neq 0.$$

The system $\mathfrak{A}$ being densely ordered, we also conclude from (2) that there is an element $a \in A$ for which

(4) $$f(\mathsf{U}B) < a < g(\mathsf{U}B).$$

Let

(5) $$D = \mathsf{E}_x[\, x \leq \mathsf{U}B \text{ and } g(x) \leq a],$$

whence

(6) $$\mathsf{U}D \leq \mathsf{U}B$$

and

(7) $$\mathsf{U}g^*(D) \leq a.$$

If $\mathsf{U}D = \mathsf{U}B$, we see from (3) that $\mathsf{U}D \neq 0$ and that consequently the set D is not empty; hence, the function g being by hypothesis quasi-decreasing, we obtain

$$g(\mathsf{U}B) = g(\mathsf{U}D) \leq \mathsf{U}g^*(D),$$

and therefore, by (7),

$$g(\mathsf{U}B) \leq a.$$

Since this formula clearly contradicts (4) we conclude that $\mathsf{U}D \neq \mathsf{U}B$ and thus, by (6),

(8) $$\mathsf{U}D < \mathsf{U}B.$$

Let

(9) $$E = \mathsf{E}_x[\mathsf{U}D < x \text{ and } x \in B].$$

If the set E were empty, we would have $x \leq \mathsf{U}D$ for every $x \in B$ and consequently $\mathsf{U}B \leq \mathsf{U}D$, in contradiction to (8). Hence E is not empty. We easily conclude by (9) that $\mathsf{U}E = \mathsf{U}B$. Since, by hypothesis, the function f is quasi-increasing, we have

$$f(\mathsf{U}B) = f(\mathsf{U}E) \geq \bigcap f^*(E)$$

and therefore, by (4),

$$a > \bigcap f^*(E).$$

Hence we must have $a > f(z)$ for some $z \in E$, for otherwise

$$a \leq \bigcap f^*(E).$$

Thus, by (1) and (9),

$$\bigcup D < z, \quad z \in B, \quad \text{and} \quad g(z) \leq a;$$

therefore, by (5), $z \in D$. The formulas

$$\bigcup D < z \quad \text{and} \quad z \in D$$

clearly contradict each other.

We have thus shown that formula (2) cannot hold for any non-empty set B satisfying (1). In other words, we have

(10) $f(\bigcup B) \geq g(\bigcup B)$ for every non-empty subset B of

$$\mathsf{E}_x[f(x) \geq g(x)].$$

By applying the result just obtained to the dual system $\mathfrak{A}' = \langle A, \geq \rangle$, we conclude that

(11) $f(\bigcap C) \leq g(\bigcap C)$ for every subset C of

$$\mathsf{E}_x[f(x) \leq g(x)].$$

Now let Y be any subset (whether empty or not) of the set

$$P = \mathsf{E}_x[f(x) = g(x)],$$

and let

(12) $$u = \bigcup \mathsf{E}_x[f(x) \geq g(x) \textit{ and } x \leq \bigcap Y].$$

By (10) and (11) we have

(13) $$f(u) \geq g(u) \text{ and } f(\bigcap Y) \leq g(\bigcap Y).$$

Hence, in case $u = \bigcap Y$, we obtain at once

(14) $f(u) = g(u)$, that is, $u \in P$.

In case $u \neq \bigcap Y$ we see from (12) that $u < \bigcap Y$. The system $\mathfrak{A}$ being densely ordered, we conclude that

$$(15) \qquad u = \bigcap \mathsf{E}_x[u < x \leq \bigcap Y].$$

We also see from (12) that $f(x) < g(x)$ for every element x of the set

$$\mathsf{E}_x[u < x \leq \bigcap Y].$$

Hence, by (11) and (15), we obtain

$$f(u) \leq g(u),$$

and this formula, together with (13), implies (14) again. Thus we have shown that

(16) for every subset Y of P, if $u = \bigcup \mathsf{E}_x[f(x) \geq g(x)$ *and* $x \leq \bigcup Y]$, then $u \in P$.

Dually we have

(17) for every subset Y of P, if $v = \bigcap \mathsf{E}_x[f(x \leq g(x)$ *and* $x \geq \bigcap Y]$, then $v \in P$.

We see immediately that the element u in (16) is the largest element of P which is a lower bound of all elements of Y; in other words, u is the greatest lower bound of Y in the system $\langle P, \leq \rangle$. Similarly, the element v in (17) is the least upper bound of Y in $\langle P, \leq \rangle$. Consequently,

(18) $\langle P, \leq \rangle$ is a continuously ordered system.

Finally, let us take in (16) and (17) the empty set for Y, so that $\bigcap Y = 1$ and $\bigcup Y = 0$. We then easily arrive at formulas

$$(19) \qquad \bigcup P = \bigcup \mathsf{E}_x[f(x) \geq g(x)] \in P$$

and

$$(20) \qquad \bigcap P = \bigcap \mathsf{E}_x[f(x) \leq g(x)] \in P.$$

By (18)-(20) the proof is complete.

Every increasing function is clearly quasi-increasing. The identity function, $g(x) = x$ for every $x \in A$, is continuous, that is, both quasi-increasing and quasi-decreasing, and the same applies to every constant function, $g(x) = c \in A$ for every $x \in A$. Hence we can take in Theorem 3 an arbitrary increasing function for f and the identity function for g; we thus obtain Theorem 1 in its application to continuously and densely ordered systems. On the other hand, by taking for g a constant function, we arrive at:

THEOREM 4 (GENERALIZED WEIERSTRASS THEOREM). *Let*

(i) $\mathfrak{A} = \langle A, \leq \rangle$ *be a continuously and densely ordered system,*

(ii) f *be a quasi-increasing function on* A *to* A *and* c *be an element of* A *such that*

$$f(0) \geq c \geq f(1),$$

(iii) $P = \mathsf{E}_x[f(x) = c]$.

Then P *is not empty and* $\langle P, \leq \rangle$ *is a continuously ordered system; in particular, we have*

$$\bigcup P = \bigcup \mathsf{E}_x[f(x) \geq c] \in P$$

and

$$\bigcap P = \bigcap \mathsf{E}_x[f(x) \leq c] \in P.$$

An analogous theorem for pseudo-decreasing functions can be derived from Theorem 3 by taking an arbitrary constant function for f.

It can be shown by means of simple examples that Theorems 3 and 4 do not extend either to arbitrary continuously ordered systems or to arbitrary complete lattices which satisfy the density condition (that is, in which, for any elements x and y, $x < y$ implies the existence of an element z with $x < z < y$).

We can generalize Theorem 3 by considering two simply ordered systems,

$$\mathfrak{A} = \langle A, \leq \rangle \text{ and } \mathfrak{B} = \langle B, \leq \rangle,$$

as well as two functions on A to B, a quasi-increasing function f and a quasi-decreasing function g. The system $\mathfrak{A}$ is assumed to be continuously and densely ordered. No such assumptions regarding B are needed. Instead, the definitions of quasi-increasing and quasi-decreasing functions must be slightly modified. For example, a function f on A to B will be called quasi-increasing if, for every non-empty subset X of A and for every $b \in B$ we have

$$f(\bigcup X) \geq b \text{ whenever } f(x) \geq b \text{ for every } x \in X$$

and

$$f(\bigcap X) \leq b \text{ whenever } f(x) \leq b \text{ for every } x \in X.$$

By repeating with small changes the proof of Theorem 3, we see that under these assumptions the conclusions of the theorem remain valid. (The only change which is not obvious is connected with the fact that the system $\mathfrak{B}$ is not assumed to be densely ordered; therefore we cannot claim the existence of an element $a \in B$ which satisfies (4), and we have to distinguish two cases, dependent on whether an element a with this property exists or not.) Theorem 4 can of course be generalized in the same way.

Theorems 3 and 4 thus generalized can be applied in particular to real functions defined on a closed interval $[a,b]$ of real numbers. In application to real functions Theorem 3 can easily be derived from Theorem 4. In fact, if f is a quasi-increasing real function and g a quasi-decreasing real function on the interval $[a,b]$, then the function f' defined by the formula

$$f'(x) = f(x) - g(x)$$

is clearly quasi-increasing; by applying Theorem 4 to this function, we obtain the conclusions of Theorem 3 for f and g. Hence the fixpoint theorem (Theorem 1) for increasing real functions is also a simple consequence of Theorem 4. Finally, since every continuous function is quasi-increasing, and since, in the real domain, continuous functions in our terminology coincide with continuous functions in the usual sense, Theorem 4 is a generalization of the well-known Weierstrass theorem on continuous real functions.[4])

Returning to Theorem 3 for simply ordered systems, if we assume that both functions f and g are continuous, we can strengthen the conclusion of the theorem; in fact we can show, not only that the system $\langle P, \leq \rangle$ is continuously

[4]Theorem 3 (for both simply ordered systems and real functions) was originally stated under the assumption that the function f is increasing and the function g is continuous; see [3]. In 1949 A. P. Morse noticed that this result in the real domain could be improved; in fact, he obtained Theorem 4 for real functions—under a different, though equivalent, definition of a quasi-increasing function. By his definition, a real function f on an interval $[a,b]$ is quasi-increasing if it is upper semicontinuous on the left and lower semicontinuous on the right, that is, if

(i) $$\overline{\lim_{x \to d-}} f(x) \leq f(d) \leq \underline{\lim_{x \to d+}} f(x) \text{ for every } d \in [a,b].$$

By generalizing this observation, the author arrived at the present abstract formulations of Theorems 3 and 4. According to a recent remark of Morse, the first part of the conclusion of Theorem 4, that is, the statement that the set P is not empty, holds in the real domain for a still more comprehensive class of functions; in fact, for all real functions which satisfy the condition obtained from (i) by replacing $\underline{\lim}$ by $\overline{\lim}$ on the right side of the double inequality (or else by replacing $\overline{\lim}$ by $\underline{\lim}$ on the left side).

ordered, but also that, for every nonempty subset X of P, the least upper bound of X in $\langle P, \leq \rangle$ coincides with the least upper bound of X in $\langle A, \leq \rangle$, and similarly for the greatest lower bound. In application to real functions this means that the set P of real numbers is, not only continuously ordered, but also closed in the topological sense. Analogous remarks apply to Theorem 4.

3. Applications to Boolean algebras and the theory of set-theoretical equivalence. As is known, a *Boolean algebra* can be defined as a lattice $\mathfrak{A} = \langle A, \leq \rangle$, with 0 and 1, in which for every element $b \in A$ there is a uniquely determined element $\bar{b} \in A$ (called the *complement* of b), such that

$$b \cup \bar{b} = 1 \text{ and } b \cap \bar{b} = 0.$$

Given any two elements $a, b \in A$, we shall denote by $a - b$ their difference, that is, the element $a \cap \bar{b}$. If $\mathfrak{A} = \langle A, \leq \rangle$ is a Boolean algebra and $a \in A$, then $\mathfrak{A}' = \langle [0,a], \leq \rangle$ is also a Boolean algebra, though the complement of an element b in $\mathfrak{A}'$ does not coincide with the complement of b in $\mathfrak{A}$.

By applying the lattice-theoretical fixpoint theorem we obtain:

THEOREM 5. *Let*

(i) $\mathfrak{A} = \langle A, \leq \rangle$ *be a complete Boolean algebra,*

(ii) a, b *be any elements of* A, f *be an increasing function on* $[0,a]$ *to* A, *and* g *an increasing function on* $[0,b]$ *to* A.

Then there are elements $a', b' \in A$ *such that*

$$f(a - a') = b' \text{ and } g(b - b') = a'.$$

Proof. Consider the function h defined by the formula

$$(1) \qquad h(x) = f(a - g(b - x)) \text{ for every } x \in A.$$

Let x and y be any elements in A such that

$$x \leq y.$$

We have then

$$b - x \geq b - y;$$

and since $b - x$ and $b - y$ are in $[0,b]$, and g is an increasing function on $[0,b]$ to A, we conclude that

$$g(b-x) \geq g(b-y)$$

and

$$a-g(b-x) \leq a-g(b-y).$$

Hence, the elements $a-g(b-x)$ and $a-g(b-y)$ being in $[0,a]$, and f being an increasing function on $[0,a]$ to A, we obtain

$$f(a-g(b-x)) \leq f(a-g(b-y)),$$

that is, by (1),

$$h(x) \leq h(y).$$

Thus h is an increasing function on A to A, and consequently, by Theorem 1, it has a fixpoint b'. Hence, by (1),

$$f(a-g(b-b')) = b'. \tag{2}$$

We put

$$g(b-b') = a'. \tag{3}$$

From (2) and (3) we see at once that the elements a' and b' satisfy the conclusion of our theorem.

If in the hypothesis of Theorem 5 we assume in addition that $f(a) \leq b$ and $g(b) \leq a$, we can obviously improve the conclusion by stating that there are elements $a', a'', b', b'' \in A$ for which

$$a = a' \cup a'', \quad b = b' \cup b'', \quad a' \cap a'' = b' \cap b'' = 0,$$

$$f(a'') = b' \text{ and } g(b'') = a'.$$ [5]

Theorem 5 has interesting applications in the discussion of homogeneous elements. Given a Boolean algebra $\mathfrak{A} = \langle A, \leq \rangle$, two elements $a, b \in A$ are called *homogeneous*, in symbols $a \approx b$, if the Boolean algebras $\langle [0,a], \leq \rangle$ and $\langle [0,b], \leq \rangle$ are isomorphic. In other words, $a \approx b$ if and only if there is a function f satisfying the following conditions: the domain of f is $[0,a]$; the range of f is $[0,b]$; the formulas $x \leq y$ and $f(x) \leq f(y)$ are equivalent for any

[5] In this more special form Theorem 5 is a generalization of a set-theoretical theorem obtained by Knaster and the author; see [3].

$x, y \in [0, a]$. Various fundamental properties of the homogeneity relation easily follow from this definition; for example, we have:

THEOREM 6. *$\mathfrak{A} = \langle A, \leq \rangle$ being an arbitrary Boolean algebra,*

(i) *$a \approx a$ for every $a \in A$;*

(ii) *if $a, b \in A$ and $a \approx b$, then $b \approx a$;*

(iii) *if $a, b, c \in A$, $a \approx b$, and $b \approx c$, then $a \approx c$;*

(iv) *if $a_1, a_2, b_1, b_2 \in A$, $a_1 \cap a_2 = 0 = b_1 \cap b_2$,*
$a_1 \approx b_1$, and $a_2 \approx b_2$, then $a_1 \cup a_2 \approx b_1 \cup b_2$;

(v) *if $a, b_1, b_2 \in A$, $b_1 \cap b_2 = 0$, and $a \approx b_1 \cup b_2$,*
then there are elements $a_1, a_2 \in A$ such that $a_1 \cup a_2 = a$, $a_1 \cap a_2 = 0$, $a_1 \approx b_1$, and $a_2 \approx b_2$.

In what follows we shall use parts (i)-(iii) of Theorem 6 without referring to them explicitly. If now we restrict our attention to complete Boolean algebras, we can establish various deeper properties of the homogeneity relation by applying Theorem 5. We start with the following:

THEOREM 7. *$\mathfrak{A} = \langle A, \leq \rangle$ being a complete Boolean algebra, if*

$$a, b_1, b_2, c, d \in A, \quad b_1 \cap b_2 = 0, \quad c \approx d, \quad \text{and } a \cup c \approx b_1 \cup b_2 \cup d,$$

then there are elements $a_1, a_2 \in A$ such that

$$a_1 \cup a_2 = a, \quad a_1 \cap a_2 = 0, \quad a_1 \cup c \approx b_1 \cup d, \quad \text{and} \quad a_2 \cup c \approx b_2 \cup d.$$

Proof. By the definition of homogeneity, the formula $c \approx d$ implies the existence of a function f which maps isomorphically the Boolean algebra $\langle [0, c], \leq \rangle$ onto the Boolean algebra $\langle [0, d], \leq \rangle$; we have in particular

$$(1) \qquad f(c) = d.$$

Similarly, the formula $a \cup c \approx b_1 \cup b_2 \cup d$ implies the existence of a function g which maps isomorphically $\langle [0, b_1 \cup b_2 \cup d], \leq \rangle$ onto $\langle [0, a \cup c], \leq \rangle$, and we have

$$(2) \qquad g(b_1 \cup b_2 \cup d) = a \cup c.$$

We can assume for a while that the domain of g has been restricted to the interval $[0, b_1 \cup d]$. Thus, f is an increasing function on $[0,c]$ to A, g is an increasing function on $[0, b_1 \cup d]$ to A, and by applying Theorem 5 we obtain two elements c', d' such that

$$(3) \qquad f(c - c') = d' \text{ and } g((b_1 \cup d) - d') = c'.$$

The functions f and g being increasing, formulas (1)-(3) imply

$$(4) \qquad d' \leq d \text{ and } c' \leq a \cup c.$$

We now let

$$(5) \qquad a_1 = c' - c \text{ and } a_2 = a - a_1.$$

By (4) we have $c' - c \leq a$, and hence, by (5),

$$(6) \qquad a_1 \cup a_2 = a \text{ and } a_1 \cap a_2 = 0.$$

From (4) and (5) we also obtain

$$(7) \qquad (c - c') \cup c' = a_1 \cup c \text{ and } (c - c') \cap c' = 0,$$

$$(8) \qquad d' \cup [(b_1 \cup d) - d'] = b_1 \cup d \text{ and } d' \cap [(b_1 \cup d) - d'] = 0.$$

Since f maps isomorphically $\langle [0,c], \leq \rangle$ onto $\langle [0,d], \leq \rangle$, we conclude from (3) that it also maps isomorphically $\langle [0,c - c'], \leq \rangle$ onto $\langle [0,d'], \leq \rangle$ and that consequently

$$(9) \qquad c - c' \approx d'.$$

Analogously, by (3),

$$(10) \qquad c' \approx (b_1 \cup d) - d'.$$

By Theorem 6 (iv), formulas (7)-(10) imply

$$(11) \qquad a_1 \cup c \approx b_1 \cup d.$$

Furthermore, from (4) and (5) we derive

$$(12) \qquad (c \cap c') \cup [(a \cup c) - c'] = a_2 \cup c \text{ and } (c \cap c') \cap [(a \cup c) - c'] = 0,$$

(13) $(d - d') \cup [(b_2 - d) \cup d'] = b_2 \cup d$ and $(d - d') \cap [(b_2 - d) \cup d'] = 0.$

The function f being an isomorphic transformation, we obtain, with the help of (1) and (3),

$$f(c \cap c') = f(c - (c - c')) = f(c) - f(c - c') = d - d',$$

and hence, by arguing as above in the proof of (9),

(14) $$c \cap c' \approx d - d'.$$

Since, by (4) and the hypothesis,

$$(b_2 - d) \cup d' = (b_1 \cup b_2 \cup d) - [(b_1 \cup d) - d'],$$

we conclude analogously, with the help of (2) and (3), that

$$g(b_2 - d) \cup d') = g(b_1 \cup b_2 \cup d) - g((b_1 \cup d) - d') = (a \cup c) - c'$$

and therefore

(15) $$(a \cup c) - c' \approx (b_2 - d) \cup d'.$$

From (12)-(15), by applying Theorem 6 (iv) again, we get

(16) $$a_2 \cup c \approx b_2 \cup d.$$

By (6), (11), and (16), the proof is complete.

In deriving the remaining theorems of this section we shall apply exclusively those properties of the homogeneity relation which have been established in Theorems 6 and 7; thus the results obtained will apply to every binary relation (between elements of a complete Boolean algebra) for which these two theorems hold. It may be noticed in this connection that Theorem 6 (v) restricted to complete Boolean algebras is a simple consequence of Theorems 6 (i) and 7.

THEOREM 8 (MEAN-VALUE THEOREM). *$\mathfrak{A} = \langle A, \leq \rangle$ being a complete Boolean algebra, if $a,b,c,a',c' \in A$, $a \leq b \leq c$, $a' \leq c'$, $a \approx a'$, and $c \approx c'$, then there is an element $b' \in A$ such that $a' \leq b' \leq c'$ and $b \approx b'$.*

Proof. We apply Theorem 7, with a,b_1,b_2,c,d respectively replaced by $c' - a'$, $b - a$, $c - b$, a', a, and we conclude that there are elements $a_1,a_2 \in A$

such that

$$c' - a' = a_1 \cup a_2 \quad \text{and} \quad (b - a) \cup a \approx a_1 \cup a'.$$

The element $b' = a_1 \cup a'$ clearly satisfies the conclusion of our theorem.

THEOREM 9. *$\mathfrak{A} = \langle A, \leq \rangle$ being a complete Boolean algebra, for any elements $a, b \in A$ the following two conditions are equivalent:*

(i) *there is an element $a_1 \in A$ such that $a \approx a_1 \leq b$;*

(ii) *there is an element $b_1 \in A$ such that $a \leq b_1 \approx b$.*

Proof. To derive (ii) from (i), we consider an arbitrary element a_1 satisfying (i), and we apply Theorem 8 with a, c, a', c' respectively replaced by a_1, 1, a, 1. The implication in the opposite direction follows immediately from Theorem 6 (v) (and hence holds in an arbitrary Boolean algebra).

THEOREM 10 (EQUIVALENCE THEOREM). *$\mathfrak{A} = \langle A, \leq \rangle$ being a complete Boolean algebra, if $a, b, c \in A$, $a \leq b \leq c$, and $a \approx c$, then $a \approx b \approx c$.*

Proof. This follows immediately from Theorem 8 with $a' = c' = c$.

THEOREM 11. *$\mathfrak{A} = \langle A, \leq \rangle$ being a complete Boolean algebra, for any elements $a_1, a_2, b \in A$ the formulas*

(i) $a_1 \cup b \approx a_2 \cup b \approx b$ *and*

and

(ii) $a_1 \cup a_2 \cup b \approx b$

are equivalent.

Proof. Obviously,

$$b \leq a_1 \cup b \leq a_1 \cup a_2 \cup b \quad \text{and} \quad b \leq a_2 \cup b \leq a_1 \cup a_2 \cup b.$$

Hence (ii) implies (i) by Theorem 10.

Assume now, conversely, that (i) holds. We clearly have

$$[a_2 - (a_1 \cup b)] \cap (a_1 \cup b) = [a_2 - (a_1 \cup b)] \cap b = 0$$

and

$$a_2 - (a_1 \cup b) \approx a_2 - (a_1 \cup b).$$

By Theorem 6 (iv), these two formulas together with (i) imply

(1) $$a_1 \cup a_2 \cup b = [a_2 - (a_1 \cup b)] \cup (a_1 \cup b) \approx [a_2 - (a_1 \cup b)] \cup b.$$

Since

$$[a_2 - (a_1 \cup b)] \cup b \leq a_2 \cup b \leq a_1 \cup a_2 \cup b,$$

we derive from (1), by applying Theorem 10,

(2) $$a_2 \cup b \approx a_1 \cup a_2 \cup b.$$

Formulas (i) and (2) obviously imply (ii), and the proof is complete.

Various properties of the relation of homogeneity can conveniently be expressed in terms of another, related relation which is denoted by $\preceq$. Thus $\mathfrak{A} = \langle A, \leq \rangle$ being a Boolean algebra, and a,b being any elements of A, we write $a \preceq b$ if there is an element $a_1 \in A$ such that $a \approx a_1 \leq b$; in case the algebra $\mathfrak{A}$ is complete, an equivalent formulation of this condition is given in Theorem 9(ii). Theorems 8 and 10 can now be put in a somewhat simpler, though essentially equivalent, form:

MEAN-VALUE THEOREM. *$\mathfrak{A} = \langle A, \leq \rangle$ being a complete Boolean algebra, if $a,b,c \in A$, $a \leq c$, and $a \preceq b \preceq c$, then there is an element $b' \in A$ such that $a \leq b' \leq c$ and $b \approx b'$.*

EQUIVALENCE THEOREM. *$\mathfrak{A} = \langle A, \leq \rangle$ being a complete Boolean algebra, if $a,b \in A$, $a \preceq b$, and $b \preceq a$, then $a \approx b$.*

We shall give two further results formulated in terms of $\preceq$.

THEOREM 12. *$\mathfrak{A} = \langle A, \leq \rangle$ being a complete Boolean algebra, if*

$$a_1, a_2, c_1, c_2 \in A,\ a_1 \leq c_1,\ a_1 \preceq c_2,\ a_2 \preceq c_1, \text{ and } a_2 \leq c_2,$$

then there are elements $b_1, b_2 \in A$ such that $a_1 \leq b_1 \leq c_1$, $a_2 \leq b_2 \leq c_2$, and $b_1 \approx b_2$.

Proof. The hypothesis implies the existence of two elements a_1', a_2' such that

(1) $$a_1 \approx a_1' \leq c_2 \text{ and } a_2 \approx a_2' \leq c_1.$$

Since, by (1),

$$a_1 \cap a_2' \leq a_1 \approx a_1',$$

we conclude from Theorem 9 that there is an element d for which

(2) $$a_1 \cap a_2' \approx d \leq a_1'.$$

We have, by (1),

$$a_2 \approx (a_1 \cap a_2') \cup (a_2' - a_1) \text{ and } (a_1 \cap a_2') \cap (a_2' - a_1) = 0;$$

hence, by Theorem 6(v), there are elements e_1, e_2 such that

(3) $$a_2 = e_1 \cup e_2 \text{ and } e_1 \cap e_2 = 0,$$

(4) $$e_1 \approx a_1 \cap a_2' \text{ and } e_2 \approx a_2' - a_1.$$

By (1)-(4) and the hypothesis,

$$d \leq a_1' \leq c_2, \; e_1 < c_2, \; d \approx e_1, \text{ and } c_2 \approx c_2;$$

hence, by Theorem 8, there is an element f for which

(5) $$e_1 \leq f \leq c_2 \text{ and } a_1' \approx f.$$

Since, by (4),

$$e_2 - f \leq e_2 \approx a_2' - a_1,$$

Theorem 9 implies the existence of an element g with

(6) $$e_2 - f \approx g \leq a_2' - a_1.$$

We now put

(7) $$b_1 = a_1 \cup g \text{ and } b_2 = f \cup (e_2 - f) = f \cup e_2.$$

By (1), (3), (5), (6), (7), and the hypothesis, we obtain

(8) $$a_1 \leq b_1 \leq c_1 \text{ and } a_2 \leq b_2 \leq c_2.$$

By (5) and (6) we have

$$a_1 \cap g = f \cap (e_2 - f) = 0, \; a_1 \approx f, \; g \approx e_2 - f;$$

hence, by (7) and Theorem 6(iv), we get

(9) $$b_1 \approx b_2.$$

From (8) and (9) we see that the elements b_1 and b_2 satisfy the conclusion of our theorem.

From the theorem just proved, by letting $a_1 = c_1$, we derive as an immediate consequence the mean-value theorem; if we put $a_1 = c_1$ and $a_2 = c_2$, we obtain the equivalence theorem. A further consequence of Theorem 12 is:

THEOREM 13 (INTERPOLATION THEOREM). *$\mathfrak{A} = \langle A, \leq \rangle$ being a complete Boolean algebra, if $a_1, a_2, c_1, c_2 \in A$ and $a_i \preceq c_j$ for $i,j = 1,2$, then there is an element $b \in A$ such that $a_i \preceq b \preceq c_j$ for $i,j = 1,2$.*

Proof. The hypothesis implies the existence of two elements a_1' and a_2' for which

$$(1) \qquad a_1 \approx a_1' \leq c_1 \text{ and } a_2 \approx a_2' \leq c_2 .$$

Hence, as is easily seen,

$$a_1' \leq c_1,\ a_1' \preceq c_2,\ a_2' \preceq c_1,\ a_2' \leq c_2 .$$

Consequently, by Theorem 12, there are elements b_1, b_2 such that

$$(2) \qquad a_1' \leq b_1 \leq c_1,\ a_2' \leq b_2 \leq c_2, \text{ and } b_1 \approx b_2 .$$

From (1) and (2), with the help of Theorem 9, we obtain

$$a_i \preceq b_1 \preceq c_j \qquad \text{for} \qquad i,j = 1,2 .$$

Thus the element $b = b_1$ satisfies the conclusion of our theorem.

From Theorems 7 and 11-13 we obtain by induction more general results in which the couples $\langle a_1, a_2 \rangle$, $\langle b_1, b_2 \rangle$ $\langle c_1, c_2 \rangle$ are replaced by finite sequences

$$\langle a_1, \cdots, a_n \rangle,\ \langle b_1, \cdots, b_n \rangle,\ \langle c_1, \cdots, c_n \rangle$$

with an arbitrary number n of terms; in Theorem 13 the couples $\langle a_1, a_2 \rangle$ and $\langle c_1, c_2 \rangle$ can be replaced by two finite sequences with different numbers of terms. The results discussed can be further extended to infinite sequences; however, these extensions seem to require a different method of proof, and we see no way of deriving them by means of elementary arguments from the fixpoint theorem of § 1.[6]

[6] Theorems 6-13 concerning the relation of homogeneity and their applications to cardinal products of Boolean algebras and to the theory of set-theoretical equivalence are not essentially new. (Theorem 12 is new, but it can be regarded simply as a new formulation of the interpolation theorem 13.) All these results are stated explicitly or implicitly in [7, §§ 11, 12, 15-17], where historical references to earlier publications can also be found. However, the method applied in [7] is different from that in the present paper and is not directly related to any fixpoint theorem. Also, the axiom of choice is not involved at all in the present discussion, while the situation in [7] is in this respect more complicated (compare, for instance, the remarks starting on page 239).

All the results of this section, except Theorem 5, remain valid if the Boolean algebra $\mathfrak{A} = \langle A, \leq \rangle$ is assumed to be not necessarily complete, but only *countably-complete* (*σ-complete*). This can be seen in the following way. To prove Theorem 5 we have constructed, in terms of two given increasing functions f and g, a new function h, and we have shown that this function h is increasing and hence has a fixpoint. In the subsequent discussion, Theorem 5 has been applied only once, namely in the proof of Theorem 7. The functions f and g involved in this application not only are increasing, but have much stronger properties, in fact, the *distributive properties* under countable joins and meets; that is, for every infinite sequence $\langle a_1, \cdots, a_n, \cdots \rangle$ we have

$$f(a_1 \cup \cdots \cup a_n \cup \cdots) = f(a_1) \cup \cdots \cup f(a_n) \cup \cdots,$$

$$f(a_1 \cap \cdots \cap a_n \cap \cdots) = f(a_1) \cap \cdots \cap f(a_n) \cap \cdots,$$

and similarly for g. It can be shown that the function h constructed from f and g in the way indicated in the proof of Theorem 5 also has these distributive properties. It is also easily seen that, in any countably-complete Boolean algebra (and, more generally, in any countably-complete lattice with 0), every function h which is distributive under countable joins has at least one fixpoint a; in fact,

$$a = 0 \cup h(0) \cup h(h(0)) \cup \cdots.$$

The results obtained in this section have interesting consequences concerning isomorphism of cardinal (direct) products of Boolean algebras. To obtain these consequences it suffices to notice that every system of Boolean algebras $\langle \mathfrak{A}_i \rangle$ can be represented by means of a system of disjoint elements $\langle a_i \rangle$ of a single Boolean algebra $\mathfrak{A}$ (in fact, of the cardinal product of all algebras $\mathfrak{A}_i$) in such a way that (i) each algebra $\mathfrak{A}_i$ is isomorphic to the subalgebra $\langle [0,a_i], \leq \rangle$ of $\mathfrak{A}$; hence (ii) two algebras $\mathfrak{A}_i$ and $\mathfrak{A}_j$ are isomorphic ($\mathfrak{A}_i \cong \mathfrak{A}_j$) if and only if the elements a_i and a_j are homogeneous ($a_i \approx a_j$); (iii) for $i \neq j$, we have $\mathfrak{A}_i \times \mathfrak{A}_j = \mathfrak{A}_k$ if and only if $a_i \cup a_j \approx a_k$; (iv) $\mathfrak{A}_i$ is isomorphic to a factor of $\mathfrak{A}_k$ if and only if $a_i \preceq a_k$. Keeping this in mind, we derive, for example, the following corollary from Theorem 11:

$\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{B}$ being three complete Boolean algebras, we have

$$\mathfrak{A}_1 \times \mathfrak{B} \cong \mathfrak{A}_2 \times \mathfrak{B} \cong \mathfrak{B}$$

if and only if

$$\mathfrak{A}_1 \times \mathfrak{A}_2 \times \mathfrak{B} \cong \mathfrak{B}.$$

Results of this type can again be extended to countably-complete Boolean algebras.

Any given sets A, B, $C, \cdots$ can be regarded as elements of a complete Boolean algebra; in fact, of the algebra formed by all subsets of the union $A \cup B \cup C \cup \cdots$, with set-theoretical inclusion as the fundamental relation. As is easily seen, two sets A and B treated this way are homogeneous in the Boolean-algebraic sense if and only if they are set-theoretically equivalent, that is, have the same power. Hence, as particular cases of theorems on homogeneous elements, we obtain various results concerning set-theoretical equivalence; for instance, Theorem 10 yields the well-known Cantor-Bernstein theorem. [7]

4. Applications to topology. [8] By a *derivative algebra* we understand a system $\mathfrak{A} = \langle A, \leq, \mathrm{D} \rangle$ in which $\langle A, \leq \rangle$ is a Boolean algebra and D is a unary operation (function) on A to A assumed to satisfy certain simple postulates; the main consequence of these postulates which is involved in our further discussion is the fact that D is increasing. The element $\mathrm{D}x$ (for any given $x \in A$) is referred to as the *derivative* of x. The derivative algebra $\mathfrak{A}$ is called *complete* if the Boolean algebra $\langle A, \leq \rangle$ is complete.

In topology the notion of the derivative of a set is either treated as a fundamental notion in terms of which the notion of a topological space is characterized, or else it is defined in terms of other fundamental notions (for example, the derivative of a point set X is defined as the set of all limit points of X). At any rate, all point sets of a topological space form a complete derivative algebra under the set-theoretical relation of inclusion and the topological operation of derivative. Hence the theorems on complete derivative algebras can be applied to arbitrary topological spaces.

$\mathfrak{A} = \langle A, \leq, \mathrm{D} \rangle$ being a derivative algebra, an element $a \in A$ is called *closed* if $\mathrm{D}a \leq a$; it is called *dense-in-itself* if $\mathrm{D}a \geq a$, and *perfect* if $\mathrm{D}a = a$; it is called *scattered* if there is no element $x \leq a$ different from 0 which is dense-in-itself.

As a consequence of the fixpoint theorem we obtain:

THEOREM 14 (GENERALIZED CANTOR-BENDIXON THEOREM).

[7] These extensions can be found in [7]. The proof of Theorems 12 and 13 extended to infinite sequences requires an application of the axiom of choice (to denumerable families of sets). Compare the preceding footnote.

[8] In connection with this section see [4, pp. 182 f.]; compare also [5], in particular pp. 38 f. and 44.

$$\mathfrak{A} = \langle A, \leq, \mathrm{D} \rangle$$

being a complete derivative algebra, every closed element $a \in A$ has a decomposition

$$a = b \cup c, \quad b \cap c = 0,$$

where the element $b \in A$ is perfect and the element $c \in A$ is scattered.

Proof. We put

$$(1) \qquad b = \mathsf{U}\mathsf{E}_x[a \cap \mathrm{D}x \geq x] \text{ and } c = a - b.$$

Hence obviously

$$(2) \qquad a = b \cup c \text{ and } b \cap c = 0.$$

D being an increasing function on A to A, the same clearly applies to the function D_a defined by the formula

$$\mathrm{D}_a x = a \cap \mathrm{D}x \text{ for every } x \in A.$$

Hence, by Theorem 1, we conclude from (1) that b is a fixpoint of D_a; that is,

$$(3) \qquad b = \mathrm{D}_a b = a \cap \mathrm{D}b.$$

Consequently $b \leq a$ and $\mathrm{D}b \leq \mathrm{D}a$; since the element a is closed, we have $\mathrm{D}a \leq a$, $\mathrm{D}b \leq a$ and therefore, by (3), $b = \mathrm{D}b$; that is, the element b is perfect. If an element $x \leq c$ is dense-in-itself, that is, $\mathrm{D}x \geq x$, we have, by (1),

$$a \cap \mathrm{D}x \geq x,$$

and hence $x \leq b$; therefore, by (2), $x = 0$. Thus the element c is scattered. This completes the proof.

It should be mentioned that the operation D in a derivative algebra

$$\mathfrak{A} = \langle A, \leq, \mathrm{D} \rangle$$

is assumed to be not only increasing, but distributive under finite joins, that is,

$$\mathrm{D}(x \cup y) = \mathrm{D}x \cup \mathrm{D}y \text{ for any } x, y \in A.$$

Under this assumption we can improve Theorem 14 by showing that every closed element $a \in A$ has a unique decomposition

$$a = b \cup c, \quad b \cap c = 0,$$

where b is perfect and c is scattered. In fact, let

$$a = b' \cup c', \quad b' \cap c' = 0$$

be another decomposition of this kind. We then have

$$b = \mathrm{D}b \leq \mathrm{D}(b \cup b') = \mathrm{D}((b - b') \cup b') = \mathrm{D}(b - b') \cup \mathrm{D}b' = \mathrm{D}(b - b') \cup b'.$$

Hence

$$b - b' \leq \mathrm{D}(b - b');$$

that is, $b - b'$ is dense-in-itself. Since, moreover, $b - b' \leq c'$, and c' is scattered, we conclude that $b - b' = 0$. Similarly we get $b' - b = 0$. Consequently $b = b'$, and hence also $c = c'$.

If, instead of Theorem 1, we apply Theorem 5, we obtain the following result (of which, however, no interesting topological consequences are known):

THEOREM 15. *$\mathfrak{A} = \langle A, \leq, \mathrm{D} \rangle$ being a complete derivative algebra, every closed element $a \in A$ has two decompositions*

$$a = b \cup c = b' \cup c', \quad b \cap c = b' \cap c' = 0,$$

where b, c, b', c' are elements of A such that

$$\mathrm{D}b' = b \text{ and } \mathrm{D}c' = c.$$

Proof. From Theorem 5 (with $a = b$) we conclude that there are two elements $c, b' \in A$ such that

(1) $$\mathrm{D}(a - c) = b' \text{ and } \mathrm{D}(a - b') = c.$$

By putting

(2) $$b = a - c \text{ and } c' = a - b'$$

we obtain, from (1),

(3) $$\mathrm{D}b = b' \text{ and } \mathrm{D}c' = c.$$

Since the function D is increasing and the element a is closed, (1) implies

$$c \leq \mathrm{D}a \leq a \text{ and } b' \leq \mathrm{D}a \leq a;$$

hence, by (2),

$$a = b \cup c = b' \cup c' \text{ and } b \cap c = b' \cap c' = 0. \tag{4}$$

By (3) and (4) the proof has been completed.

Theorems 1 and 5 can be applied not only to the operation D, but also to other topological operations which are defined in terms of D and, like the latter, are increasing; for instance, to the operation I defined by the formula

$$\mathrm{I}x = x - \mathrm{D}\overline{x};$$

$\mathrm{I}x$ referred to as the *interior* of the element x. Theorem 5 can of course be applied to two different topological operations, provided both are increasing.

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