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**Relationships between centrality
measures and VCG mechanism**

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Abstract

In this work we show that some recent centrality measures in network analysis are exactly an application of the principles underlying the Vickrey-Clarke-Groves (VCG) mechanism. We then present specific examples of completely different frameworks which highlights how these centrality measures à la VCG can indeed provide valuable information to fairly assess the importance of the analyzed network elements. However, by taking inspiration from the relevant literature on the VCG auction design, we verify that in general cases centrality measures à la VCG can determine a poor estimate of the actual significance of some network elements; therefore, we provide a general approach to effectively improve such estimates, based on applying the VCG rule to suitable groups of elements of the network.

Keywords: Network analysis, Centrality measures, VCG mechanism, Externalities

1. Introduction

In network analysis framework, several centrality measures have been defined in the literature with the aim of studying the structure of the network and, in particular, identifying the most important elements (e.g. nodes or links) of the network (see Koschützki et al. 2005, Newman 2010). These measures have been effectively applied in many different contexts, such as, for instance, in telecommunications, railways, air transport, postal services, data networks, social networks. Depending on the specific context they are referring to, they represent completely different meanings (e.g. independence, risk, power, consensus, value, brokerage).

Any proposed measure is based on some criterion aimed at answering what is centrality. For instance, Freeman (1977, 1979) observed that the properties of the center of a star-shaped network could be applied to define the characteristic of suitable centrality measures. Thus, he provided a formulation of the well-known betweenness centrality of a node, which gets its maximum value exactly for the center of a star-shaped network. In particular, betweenness centrality of a node is based on the assumption that shortest paths are the drivers of any consideration about the centrality of a node, since the resources of a network are most efficiently used when the content of the linkages (e.g. traffic, information) follows shortest paths; in fact, betweenness centrality measures the degree to which a node is on shortest paths connecting pairs of other nodes (it considers the number of shortest paths from any node to all others that pass through that node).¹

Stephenson and Zelen (1989) relaxed the assumption that the content of the linkages have to spread exclusively along shortest paths, while providing propagation models where arbitrary paths can play a role. Newman (2005) shared such a point of view and defined a version of node betweenness centrality which includes further paths between nodes, although the shortest ones are considered more crucial than the others (in particular, it computes how often any given node falls on a random walk between another pair of nodes). In Borgatti (2005), a dynamic view of the centrality concept is proposed, in the sense that the

¹ Unfortunately, the computational effort to exactly determine the betweenness centrality can exponentially grows with the size of the network; in fact, identifying algorithms to compute effective approximations of betweenness centrality is a significant research topic in the network analysis framework. However, in this work we are not interested to computational aspects of the centrality measures.

importance of a node in a network is based on how traffic/information actually flows through the network. A weighted version of betweenness centrality is then introduced in Borgatti and Everett (2006), where all shortest paths are weighted inversely proportional to their length, as the authors assumed the principle that the longer a path, the less significant it is to determine the centrality of the elements. Furthermore, Gómez, Figueira and Eusébio (2013) observed that single dimensional metrics are not effective for dealing with many real-world problems and thus they extend some classical centrality measures to take into account several dimensions (e.g. flows between pair of nodes and cost associated with communications).

In a recent paper, Everett and Borgatti (2010) propose a new approach to measure the centrality of the network elements, based on considering the direct contribution of the element to the overall network centrality and of an indirect contribution of the element to the centrality of all other elements. In this work, we first contribute to analyze and interpret some basics of centrality theory by showing how the new centrality measures introduced in Everett and Borgatti (2010) are exactly an application of the well-known Vickrey-Clarke-Groves (VCG) mechanism (Ausubel and Milgrom 2002, Pekeč and Rothkopf 2003, Avenali 2009) to the context of centrality measures.² Moreover, we present some examples of completely different frameworks where applying the principle underlying the VCG rule indeed provides valuable information for a fair assessment of the actual centrality of the analyzed network elements. Then, by taking inspiration from the relevant literature on auction design, which highlights some drawbacks of the VCG mechanism (Ausubel and Milgrom 2002, Hidvégi, Wang and Whinston 2007, Avenali 2009)³, we verify that also centrality measures à la VCG suffer from similar drawbacks. In particular, we show that in general cases centrality measures à la VCG can determine a poor estimate of the actual significance of some network element (analogously, VCG auctions can generate very low revenue for the auctioneer). Therefore, by taking as a starting point some results developed within the VCG auction design, we provide a general approach to refine such estimates, based on applying the VCG rule to suitable groups of elements of the network (correspondingly, in VCG

² Vickrey-Clarke-Groves (VCG) mechanism (Vickrey 1961, Clarke 1971, Groves 1973), also known as the generalized Vickrey auction, is the generalization of the second-price Vickrey rule for single-item case and can be applied in the context of combinatorial auctions. It is largely studied by the economics, computer science and operations research communities.

³ For instance, VCG payment scheme is highly affected by revenue failures and vulnerability to collusion and shill bidding.

auctions the revenue failure can be effectively hindered if the auctioneer may compel specific groups of players to take part to the auction as coalitions, instead of as separate single players).

This paper is organized as follows. Section 2 briefly illustrates those characteristics of the VCG mechanism which will be recalled later in the paper. Section 3 shows that the approach proposed in Everett and Borgatti (2010) is an application of the VCG mechanism, and then presents some examples to point out how centrality measures à la VCG can provide valuable information. Section 4 discusses how centrality measures à la VCG can poorly estimate the actual importance of some network elements, and thus provides a general approach to improve such estimates. Finally, Section 5 concludes.

2. The VCG mechanism

In this section we illustrate how the VCG mechanism works in a setting which is very general and effective to introduce the main ideas in next sections. In particular, let us consider an auction framework, where:

- The auction is direct, namely the auctioneer sells items while the participants offer to buy them.
- $n \geq 2$ rival participants with independent and private valuations and with no budget constraints take part in the auction. Let T be the set of the n players.⁴
- $m \geq 1$ items are simultaneously auctioned off. Although some of the items on sale could be identical, each item is uniquely determined (thus, every possible copy of an object is identified separately). Let H be the set of the m items.
- Any bidder is allowed to submit offers for any set of items on sale (bundle). Bidding for a bundle means that if the bid is selected as winning, then all the items in the bundle must be allocated with the player who submitted the offer.
- The auctioneer has to choose the winning bids taking into account that some pairs of bids are incompatible, i.e. they cannot be both simultaneously selected as winning. In particular, every player can transmit information to the auctioneer on which bids (among the ones he has announced) are

⁴ Assuming players with independent and private valuations means that no valuation changes across the auction course; therefore, for instance, a bidder does not alter his valuations even if he discovers opponents' valuations or in the case when some specific items are being won by other participants.

incompatible; moreover, the auctioneer himself consider as incompatible any pair of bids which share an item (both if submitted by the same player or by distinct players).

In the relevant literature, such a format is referred to as combinatorial auction format (Pekeč and Rothkopf 2003, Rothkopf, Pekeč and Harstad 1998, De Vries and Vohra 2003). Submitting bids on bundles and signaling incompatibilities among these bids to the auctioneer allow the players to model and manage possible complementarity/substitutability relationships among items⁵, and therefore to offer up to their valuations without running the risk of undergoing irrational allocations (Avenali and Bassanini 2007).

Under the so-called *first-price rule*, items are allocated to those players who offer for them at the most (winning bids are identified by maximizing the auctioneer's revenue⁶), and each player has to pay to the auctioneer exactly what he has offered in his winning bids.

As known, under the VCG rule, what players win depends on what they offer, while what they pay depends on what opponents offer. In particular, winning bids are identified by maximizing the auctioneer's revenue (as the first-price rule does), while any player $t \in T$ has to pay to the auctioneer an amount which reflects the *externality* generated on the other bidders by player t 's participation in the auction⁷. In fact, the VCG payment of bidder t (denoted by p_t^{VCG}) is equal to the summation of the externalities imposed on every player $j \in T - \{t\}$, each one equal to the difference between the value of the winning bids of player j when bidder t does not participate in the auction (let us denote it by $p_{j,-t}$) and the value of the winning bids of player j when player t takes part in the auction (denoted by p_j), that is, $p_t^{VCG} = \sum_{j \in T - \{t\}} (p_{j,-t} - p_j)$. By letting p_t be the value of the winning bids of player t when all players take part in the auction, and by setting

⁵ Complementarity occurs when a player values a bunch of items more than the sum of the values of every single items, while substitutability when he values the set less than the sum.

⁶ In general, alternative sets of winning offers that ensure the same (maximum) revenue can exist. If any, ties are broken randomly. In particular, the order in which these sets are found along the computation phase depends on an identifying label assigned to each submitted offer before starting the computation; such labels are randomly assigned.

⁷ In an economic system, a positive (negative) externality is a revenue (cost) which is imposed to an agent a by a decision/action of another agent b because of the absence of a market where b can sell to (buy from) a a specific item at a price that balances such a revenue (cost). For instance, let us consider a firm which pollutes a river by dumping waste material. All the houses in the neighborhood will lose value and thus every private house owner will be deemed to bear a negative externality (measured by the depreciation cost of his own house). By designing a specific market where the firm must acquire the right to pollute from the house owners, the externality turns into a fair compensation which balances the market value decrement of the houses. In the auction context, the externality is generated by the fact that there no exist a market where the players' participation in the auction can be negotiated and priced (obviously, if we does not consider collusion among participants).

$p_{-t} = \sum_{j \in T - \{t\}} p_{j,-t}$ and $p = p'_t + \sum_{j \in T - \{t\}} p'_j$, then we can rearrange the expression of p_t^{VCG} as difference between the total value of the winning bids of the other players when bidder t does not participate in the auction (i.e. p_{-t}) and the total value of the winning offers of the other players when player t takes part in the auction (which is equal to $p - p_t$), namely, $p_t^{VCG} = \sum_{j \in T - \{t\}} (p_{j,-t} - p_j) = p_{-t} - (p - p_t)$. Moreover, since $p_t^{VCG} = p_{-t} - (p - p_t) = p_t - (p - p_{-t})$ and p_t is the summation of the prices offered by player t for his winning bids, the VCG rule imposes a discount equal to $\dot{p}_t = p - p_{-t}$ on the overall offered price p_t , that is, $\dot{p}_t = p - p_{-t} = p_t - p_t^{VCG}$. By construction, p_t^{VCG} is nonnegative for any $t \in T$. Definitively, the revenue the auctioneer obtains by means of the VCG rule is $p^{VCG} = \sum_{t \in T} p_t^{VCG}$.⁸

Let us consider the following example with 3 bidders and 2 items; player a is interested in the pair of items Y and Z , and values them at 50, while bidder b values item Y at 60, and player c values item Z at 40. For simplicity, let us assume that they offer up to their valuations. It easy to verify what follows ($p = 100$ as b offers 60 for Y and c bids 40 for Z):

$$p_a^{VCG} = \sum_{j \in T - \{a\}} (p_{j,-a} - p_j) = (60 - 60) + (40 - 40) = p_{-a} - (p - p_a) = 100 - (100 - 0) = 0,$$

$$p_b^{VCG} = (50 - 0) + (0 - 40) = 50 - (100 - 60) = 10,$$

$$p_c^{VCG} = (0 - 0) + (60 - 60) = 60 - (100 - 40) = 0,$$

$$p^{VCG} = 0 + 10 + 0 = 10.$$

Thus, the discounts allowed with respect to the offered prices are respectively:

$$\dot{p}_a = (p - p_{-a}) = 100 - 100 = 0,$$

$$\dot{p}_b = 100 - 50 = 50,$$

$$\dot{p}_c = 100 - 60 = 40,$$

$$\dot{p} = 0 + 50 + 40 = 90.$$

⁸ Incidentally, let us recall that, assuming players with independent and private valuations and with no budget constraints, the VCG mechanism has the significant property of making truthful bidding a dominant strategy for every player (it is strategy-proof); this means that it is able to extract from the players all the information concerning their valuations and thus to induce maximum allocative efficiency (Milgrom 2004).

Note that bidder b generates a negative externality of 50 on player a and a positive externality of 40 on player c , therefore according to the VCG rule his overall payment is 10. On the contrary, payers a and c induce no externality on the other bidders and thus their VCG payments are both equal to 0.

It is important to remark that p_t^{VCG} represents an aggregated measure of the externalities generated on the other players by bidder t 's participation, in the sense that if ties among the winning bids of t 's opponents occur when t does not take part to competition, then there can be alternative scenarios in terms of generated externalities.⁹ For instance, let us extend the previous example by introducing one more player, say d ; player d is interested in item Y and values it at 10. For simplicity, let us assume again that all bidders offer up to their valuations. Now when bidder b does not take part in the auction there are two alternative scenarios: (i) both items are allocated to a and thus b generates a negative externality of 50 on player a and a positive externality of 40 on player c ($p_b^{VCG} = \sum_{j \in T - \{b\}} (p_{j,-b} - p_j) = (50 - 0) + (0 - 40) + (0 - 0) = 10$); (ii) else item Y and Z are allocated respectively to d and c , and therefore b generates only a negative externality of 10 on player d ($p_b^{VCG} = (0 - 0) + (40 - 40) + (10 - 0) = 10$). Summarizing, the overall externality generated by player b is equal to 10 ($p_b^{VCG} = p_{-b} - (p - p_b) = 50 - (100 - 60) = 10$), while the externality generated by player b on every opponent is not uniquely determined.

3. The relevance of externalities in centrality measures

In a recent paper Everett and Borgatti (2010) studied some theoretical aspects of centrality measures in network analysis.¹⁰ In particular, given any network W and set V of the analyzed network elements (e.g. nodes or arcs), they observed that by selecting any metric C to measure the centrality, and considering the sum of the centrality scores of the elements in V , it is possible to define the total centrality of any element $t \in V$ as $c_t^{tot} = \sum_{j \in V} c_j - \sum_{j \in V - \{t\}} c_{j,-t} = c - c_{-t}$, where c_j is the centrality score of element j given network W , $c_{j,-t}$ is the centrality measure of element j after removing t from network W , $c = \sum_{j \in V} c_j$

⁹ When $m = 1$ (just one item is auctioned off), the VCG mechanism collapses into the so-called *second-price rule* or Vickrey auction (Milgrom 2004), where the winning player generates only negative externalities on the rivals and his payment is always equal to second highest bid.

¹⁰ Since the structure of a network is usually modeled as a graph, in the following we will speak indifferently of networks and graphs.

represents the overall centrality of all elements, and $c_{-t} = \sum_{j \in V - \{t\}} c_{j,-t}$ is the overall centrality of the residual elements after removing t from W . Therefore, Everett and Borgatti (2010) remarked that the total centrality of any element reflects the element's direct contribution to the network overall centrality but also the indirect contribution of the element to the centrality of the other elements of the network. Moreover, they call c_t the endogenous centrality of the element, and $\sum_{j \in V - \{t\}} (c_j - c_{j,-t})$ the exogenous centrality of the element; thus, the total centrality c_t^{tot} of t is the sum of the endogenous and exogenous centralities of t , that is, $c_t^{tot} = c_t + \sum_{j \in V - \{t\}} (c_j - c_{j,-t})$.

Everett and Borgatti (2010) defined total, endogenous and exogenous centrality concepts by taking inspiration from an approach used in last decades to study the resilience or robustness of a network (Koschützki et al. 2005, Snediker, Murray and Matisziw 2008, Zobel 2011), which consists of measuring the degradation of the network performances (in terms of some specific properties) after the removal of nodes and/or arcs.¹¹

It is easy to verify that such centrality concepts are exactly an application of the generalized Vickrey principle. In fact, by substituting the metric of the offered price with the metric C underlying the centrality measure at issue, and by considering the network elements in V instead of the players in T , the exogenous centrality is the “payment” returned by the VCG rule (with the opposite sign) applied to the new metric (i.e., $\sum_{j \in V - \{t\}} c_j - c_{j,-t} = -c_t^{VCG}$ corresponds to $-p_t^{VCG} = -\sum_{j \in V - \{t\}} (p_{j,-t} - p_j)$), the endogenous centrality is the “value of the winning bids” (i.e. c_t matches with p_t), and the total centrality is the resulting “discount” under the VCG rule (i.e. c_t^{tot} corresponds to p_t). Therefore, by setting $\sum_{j \in V - \{t\}} c_j - c_{j,-t} = -c_t^{VCG}$ and $c_t^{tot} = \dot{c}_t$, we can formally write:

$$c_t^{tot} = c - c_{-t} = c_t + \sum_{j \in V - \{t\}} (c_j - c_{j,-t}) = c_t - c_t^{VCG} = \dot{c}_t$$

¹¹ In general, the removal of nodes and/or arcs disrupts the paths between the nodes and thus making the communication between nodes harder or impossible. There are several ways of measuring the degradation of the network performance after the removal (see Koschützki et al. 2005). For instance, a simple way to measure the performance degradation it is to calculate the decrease in size of the largest connected component in the network (a connected component is any set of nodes of the network such that a path exists between any two nodes of the set), where the size can be modelled, for example, in terms of cardinality of the connected component, or as weighted sum of the nodes of the connected component.

In other words, the exogenous centrality is the externality (with the opposite sign) generated by the presence of element t on the other elements of the network.¹² The main difference with respect to the auction framework (apart from the strategic interaction among the players, obviously) lies in the fact that the VCG payment under the metric of the offered price is always nonnegative, while under other metrics for the centrality measures the overall externality generated by an element can also be negative.

Moreover, centrality measures based on the application of the principle underlying the VCG rule have a natural interpretation; in fact, in the Vickrey's language, the total centrality of an element reflects the sum of its centrality and of the positive and negative externalities which it generates on all other elements (positive externalities when the centrality of other elements benefits from its "presence" in the network, while negative externalities in the case that its "presence" reduces the centrality of other elements). Since these centrality measures (each one associated with a different metric C) follow the VCG paradigm of analyzing the marginal contribute of an element per time in a given context, in the following we will refer to these centrality measures also as *VCG centralities*.¹³

As shown in following examples, there are cases where the centrality score of an element v of a network could be misleading if we apply centrality measures which do not take into account the potential role of the other elements in the network, namely, the contribution to the network of the other elements in the case that the network should operate without v . In other words, it can be useful to study the centrality of an element of a network by also investigating how much the other elements of the network would value the miss of v . In particular, if we look at the externalities which the elements generate on the other ones, some elements could be considered much more or less crucial than they appear at first sight. To better clarify such ideas, let us consider the following different cases.

¹² In the network context, the externality represents the sum of the negative and positive effects, as measured by some metric, which are imposed upon the other elements of the network.

¹³ In some cases it could be useful to define and apply centrality measures which reflects only negative (only positive) externalities generated by an element of the network on the other ones. Further research could be focused on this issue.

Case 1

As a first example, let us focus on the betweenness centrality of a node v in a network, which is normally defined as the share of shortest paths from any node of the network to all others that pass through node v (from now on denoted by bc_v)¹⁴. In particular, let us consider two transport networks represented by the weighted directed graphs \check{G} and \bar{G} in Figure 1.

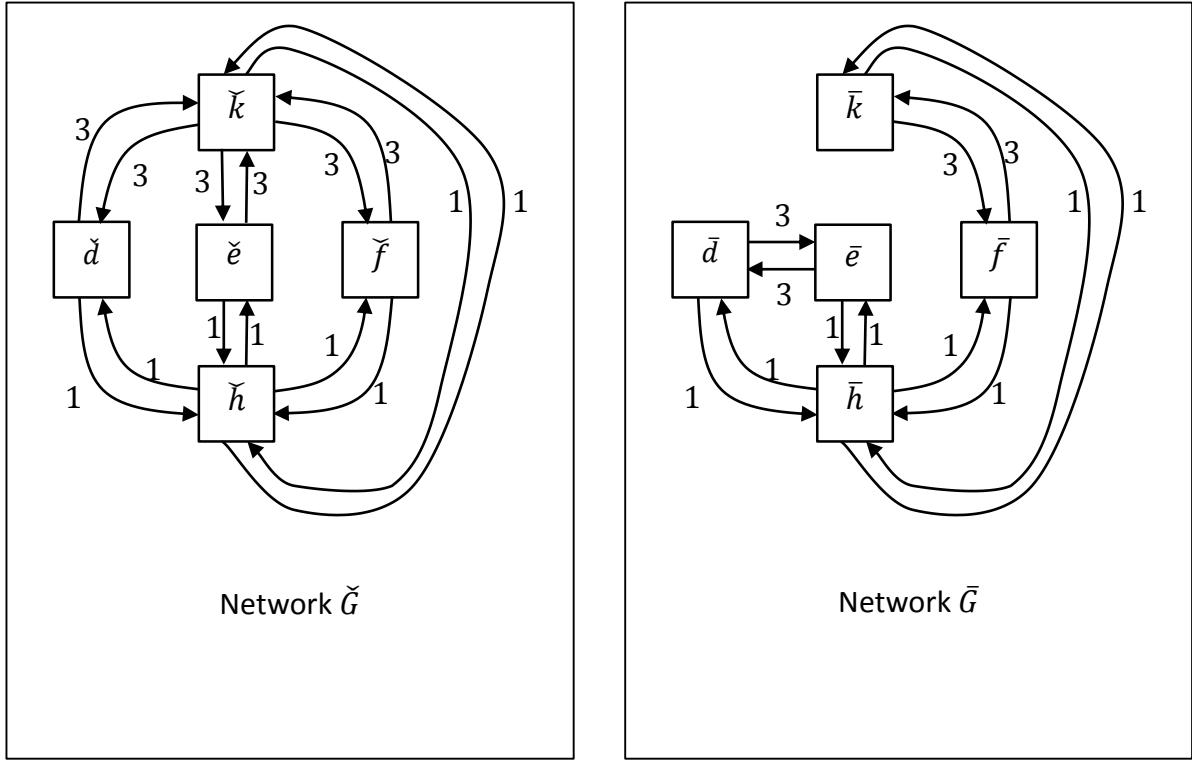


Figure 1

More formally, graphs $\check{G} = (\check{N}, \check{A}, \check{c})$ and $\bar{G} = (\bar{N}, \bar{A}, \bar{c})$, where $\check{N} = \{\check{d}, \check{e}, \check{f}, \check{h}, \check{k}\}$ and $\bar{N} = \{\bar{d}, \bar{e}, \bar{f}, \bar{h}, \bar{k}\}$ are the node sets, $\check{A} = \{\check{a}_1 = (\check{d}, \check{h}), \check{a}_2 = (\check{d}, \check{k}), \dots\}$ and $\bar{A} = \{\bar{a}_1 = (\bar{d}, \bar{h}), \bar{a}_2 = (\bar{d}, \bar{k}), \dots\}$ are the arc sets, $\check{c} = (\check{c}_1 = 1, \check{c}_2 = 3, \dots)^T \in \mathbb{R}_+^{|\check{A}|}$ and $\bar{c} = (\bar{c}_1 = 3, \bar{c}_2 = 1, \dots)^T \in \mathbb{R}_+^{|\bar{A}|}$ are the vectors of costs associated with the arcs¹⁵ (where \mathbb{R}_+ is the set of positive reals and zero).

¹⁴ Classical definition of betweenness centrality focuses only on shortest paths which pass through nodes and thus arcs are excluded from the set of shortest paths which determine betweenness centrality.

¹⁵ For instance, if nodes represent cities, a cost can represent the physical length of the corresponding link between a pair of cities, or the monetary cost that must be supported to travel along that route, or the required time to move from a city to another.

By direct inspection of \check{G} , it is easy to verify that totally there are 12 shortest paths and everyone enters and exits exclusively node \check{h} ; therefore, $bc_{\check{h}} = \frac{12}{12} = 1$ while $bc_{\check{d}} = bc_{\check{e}} = bc_{\check{f}} = bc_{\check{k}} = \frac{0}{12} = 0$. Equally, we see that $bc_{\bar{h}} = \frac{12}{12} = 1$ and $bc_{\bar{d}} = bc_{\bar{e}} = bc_{\bar{f}} = bc_{\bar{k}} = \frac{0}{12} = 0$. However, is indeed appropriate to state that nodes \check{h} and \bar{h} are equally crucial in their respective network? It is easy to verify that node \check{h} generates much more externalities on the other nodes of \check{G} than \bar{h} does on the other nodes of \bar{G} . In fact, node \check{h} prevents node \check{k} from being a crossroads of the shortest paths from any node of the network to all others, while node \bar{h} does not “subtract” shortest paths from any other node of \bar{G} . In particular, by removing node \check{h} from the network, we see that betweenness centrality of node \check{k} increases from 0 up to $\frac{6}{6} = 1$ (while the other ones are still 0); instead, if we remove \bar{h} from \bar{G} , the betweenness centrality of the remaining nodes does not change (they are still 0). Economically speaking, we can say that \check{h} generates a (negative) externality on node \check{k} , while \bar{h} induces no externality on any other node. In other words, node \check{h} can be substituted by its node competitors in terms of rearranging shortest paths, while the removal of node \bar{h} would be ruinous for network \bar{G} as several connections cannot be restored. For instance, in a transport network, node \check{h} can be somehow bypassed, while node \bar{h} is a bottleneck and thus pivotal, although both nodes have the same betweenness centrality.

To take into account such theoretical considerations, the centrality of these nodes could be represented by a measure of the marginal contribution of the node in terms of betweenness centrality, by subtracting to its betweenness centrality the generated externalities. In particular, the *VCG betweenness centrality* of nodes \check{h} and \bar{h} is, respectively, equal to $\dot{bc}_{\check{h}} = 1 - 1 = 0$ and $\dot{bc}_{\bar{h}} = 1 - 0 = 1 > \dot{bc}_{\check{h}}$; such centrality measures effectively reflects that node \check{h} seems to be less critical in the context of \check{G} than node \bar{h} is in the context of \bar{G} .

Case 2

Let us now focus on a different context, namely, a stylized scenario where economic agents (consumers and firms) interact with each other and aim at maximizing their own net surplus¹⁶. In such a case, economists are

¹⁶ The net surplus of a consumer is the difference between his willingness to pay for the purchased products and the prices paid to buy them, while the net surplus of a firm is the difference between revenues and costs (also referred to as profit or payoff).

usually interested in maximizing the welfare, that is, the summation of the net surplus of any economic agent; therefore, a measure of centrality of the economic agents is represented by their contributions to the welfare. In particular, let us consider a so-called linear city (Tirole 1988), where (i) consumers are uniformly distributed within the segment $[0,18]$ and have unit demand, and (ii) there are three firms a , b and c placed in 0, 4 and 18, respectively. Each firm is single-product and three products a , b , c are perceived by consumers as horizontally differentiated (i.e., even when they are offered at the same price, there can be consumers who purchase product a , other consumers who demand b and other consumers who buy c). Let the maximum willingness to pay of a consumer $z \in [0,18]$ for product a be equal to $8 - z$ (when it gets negative it means that consumer z would require a subside to buy the product); therefore, when z purchases from firm a , z 's net surplus is $ns_a(z, p_a) = 8 - z - p_a$, where $p_a \geq 0$ is the price required by firm a . Similarly, the net surplus of a consumer $z \in [0,4]$ who buys from firm b is $ns_b(z, p_b) = 6 + \frac{6}{5}(z - 4) - p_b$, where $p_b \geq 0$ is the price required by firm b , while net surplus of a consumer $z \in [4,18]$ who buys from firm b is $ns_b(z, p_b) = 6 + (4 - z) - p_b$. Finally, the net surplus of a consumer $z \in [0,18]$ who buys from firm b is $ns_c(z, p_c) = 5 + \frac{5}{8}(z - 18) - p_c$, where $p_c \geq 0$ is the price required by firm c . If a consumer z does not buy any product his net surplus is zero. Consumers want to maximize their net surplus.

Figure 2 shows the graphic representation of the willingness to pay of all consumers for any product (the dashed line represents the willingness to pay for product a , the double line represents the willingness to pay for product b , the dotted line represents the willingness to pay for product c). For any firm, the net surplus is equal to the multiplication of the share of consumers who buy its product and the required price (all costs are assumed to be zero).

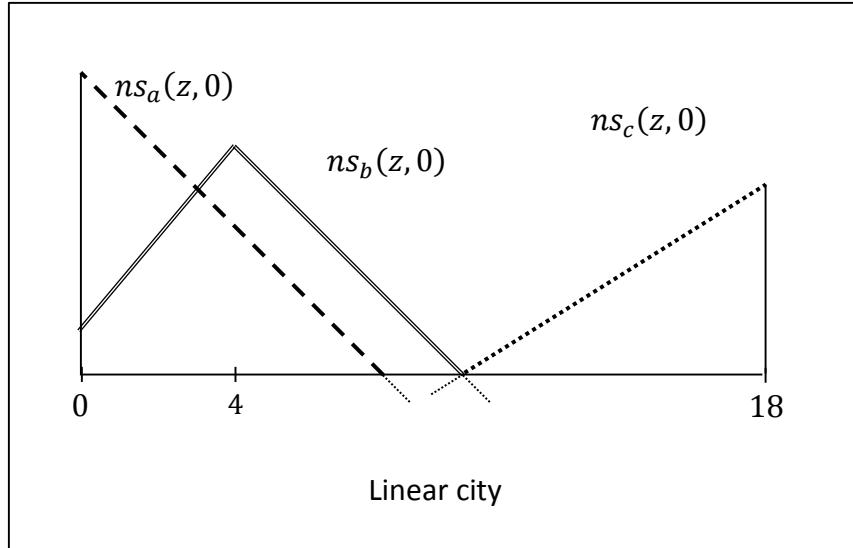


Figure 2

Incidentally, let us note that a naïve graph-based representation of the considered scenario is the weighted directed graph EG in Figure 3. Weights on arcs of graph EG represent capacities, some of which are definitively determined (the ones equal to infinity) while the others depends on the prices p_a , p_b and p_c ; in particular, $\pi_k(p_a, p_b, p_c)$ is firm k 's net surplus and $CS_k(p_a, p_b, p_c)$ is the net surplus of consumers buying from firm k , for $k = a, b, c$. It is easy to verify that, given the prices p_a , p_b and p_c and thus the arc capacities, the welfare induced in the scenario at issue is equal to determining the maximum flow between origin s and destination t of graph EG .

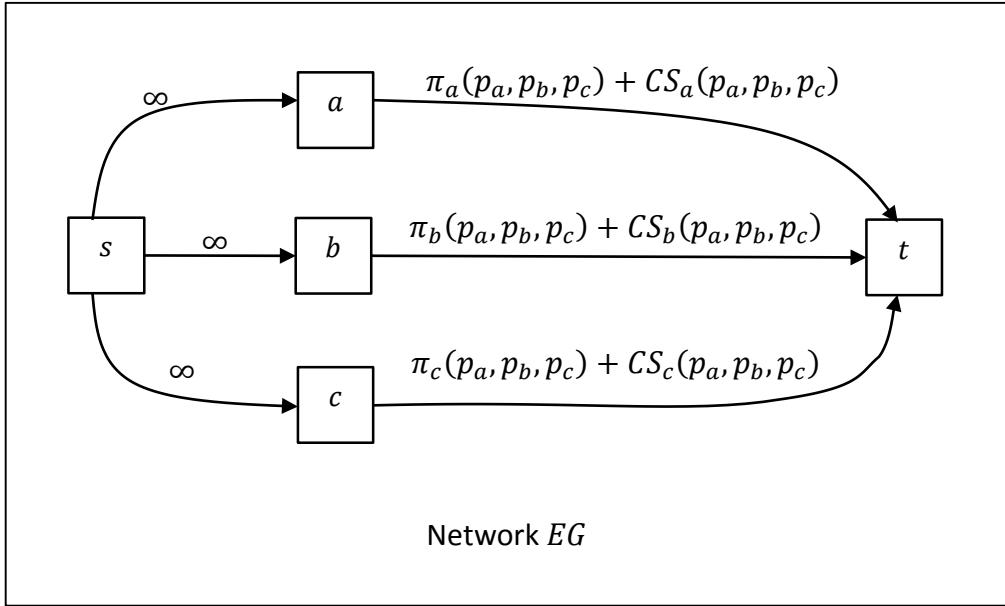


Figure 3

Note that, the maximum welfare which could be generated in the considered scenario would be equal to 74.9 (obtained by setting $p_a = p_b = p_c = 0$). Observe also that, the maximum contribution to the welfare which firms a and b could generate is equal to 32, while firm c 's potential contribution would be at the most 20 (such possible contributions to the welfare would be obtained by setting any product price at zero). However, any firm strategically sets the product price in order to maximize its net surplus. By applying Nash equilibrium methodology to determine the outcome of the strategic interaction among firms in the considered linear city (Tirole 1988), it is easy to verify that the net surplus of consumers buying from firms a , b , c is respectively equal to 4.3, 7.2, 5 (therefore, net surplus of all consumers $CS = CS_a + CS_b + CS_c$ is equal to 16.5), while the net surplus of firms a , b , c is respectively 11.3, 14.5, 10; therefore, the welfare is $W = 52.3$. In terms of contribution to the welfare, the price decided by firm a induce a firm a 's net surplus equal to $\pi_a = 11.3$ and a consumers' net surplus equal to $CS_a = 4.3$, and thus firm a 's impact on welfare is $W_a = 15.6$. Analogously, the contribution of firms b , c is respectively $W_b = \pi_b + CS_b = 14.5 + 7.2 = 21.7$, $W_c = \pi_c + CS_c = 10 + 5 = 15$. Therefore, should we conclude that firm b is the most important element in the given context? This analysis only restricts the attention on the absolute value of the impact of the firm, while it does not investigate on the marginal contribution of the firm to welfare, which in fact depends on the contribution which could be offered by the other firms. In order to analyze also this aspect, Table 1

reports the firms' contributions in different scenarios, namely, when all firms enter the market, and when one firm per time exits the market.

	firms a, b, c	firms b, c	firms a, c	firms a, b
W_a	$11.3 + 4.3 = 15.6$		$16 + 8 = 24$	$11.3 + 4.3 = 15.6$
W_b	$14.5 + 7.2 = 21.7$	$16.5 + 8.3 = 24.8$		$14.5 + 7.2 = 21.7$
W_c	$10 + 5 = 15$	$10 + 5 = 15$	$10 + 5 = 15$	
W	52.3	39.8	39	37.3

Table 1

By inspecting the table we can observe that firm a generates a negative externality of $3.1 = 24.8 - 21.7$ on firm b , firm b induces a negative externality of $8.4 = 24 - 15.6$ on firm a , and firm c generates no externality. Therefore, by taking into account both the absolute contribution to the welfare and the generated externality on the opponents, it results that the *VCG welfare centrality* of firms a, b, c is respectively equal to $\dot{W}_a = 15.6 - 3.1 = 12.5$, $\dot{W}_b = 21.7 - 8.4 = 13.3$, $\dot{W}_c = 15$. Now, the role of firm b does not seem so crucial anymore, since the strong competition with firm a induces a partial substitutability between a and b (namely, firm a 's contribution to the welfare could partially offset firm b 's one). On the other hand, firm c results the most important agent in the given context. For instance, let us assume that the illustrated case model the broadband retail market in a country, where segment $[0,10]$ represents the metropolitan areas and segment $[10,18]$ represents the rural areas. The proposed centrality analysis, based also on a measure of externalities, would suggest a public authority to have much more consideration for a monopoly firm in the rural areas instead of further promoting competition in the urban areas (in other words, the digital divide between rural and metropolitan areas, which the absence of monopolistic firm c would generate, would be worse than the lower level of competition in the metropolitan area which the absence of firm b , or else of firm a , would induce).

4. A general approach to better estimate the induced externalities

As known from the auction literature, the VCG approach can be affected by failure revenues (namely, the VCG payments can be extremely low or even zero although high bids for the items on sale have been submitted), since the externality induced by removing simultaneously some players can be largely different from the sum of the externalities generated by removing every single player per time (see Avenali 2009). In other words, some players can generate significant externalities on the others players only if assumed united in a same coalition. For instance, some offers submitted by distinct players are winning because they combine each other to defeat a single offer of a common opponent; more, an offer is selected as winning because it completes another offer by a different player which is unilaterally able to defeat the offers of a common opponent. Analogously, in the centrality measure framework, the centrality of an element could be poorly estimated if we look exclusively at the externality generated by its removal. For instance, some high network performances can be guaranteed by the joint contribution of a group of elements, while looking only at the contribution of a single element would not reflect the actual role of that element.

Before providing an example, let us introduce a few notation and definitions. Let us consider any network W and any metric C to measure the centrality. Let also V be the set of the analyzed network elements and $S \subseteq V$ be a nonempty set of network element. By applying the VCG mechanism to group S (Avenali 2009), the centrality measure à la VCG of S (denoted by \dot{c}_S) can be defined as follows:

$$\dot{c}_S = \sum_{j \in V} c_j - \sum_{j \in V - \{S\}} c_{j,-S} = \sum_{j \in S} c_j + \sum_{j \in V - \{S\}} (c_j - c_{j,-S}) = c_S - c_S^{VCG}$$

where c_j is the centrality score of element j given network W , $c_{j,-S}$ is the centrality measure of element j after removing each element of S from network W , $c = \sum_{j \in V} c_j$ represents the overall centrality of all elements, $c_{-S} = \sum_{j \in V - \{S\}} c_{j,-S}$ is the overall centrality of the residual elements after removing each element of S from W , $c_S^{VCG} = -\sum_{j \in V - \{S\}} (c_j - c_{j,-S})$ is the externality jointly generated by the presence of every element of S on the other elements of the network, $c_S = \sum_{j \in S} c_j$ is the overall centrality of the elements of S . In the following, we will refer to c_S^{VCG} as *joint externality* and to \dot{c}_S as *joint centrality*.

Let us now discuss an example. We consider a network $G = (N, A, u, uo, ut)$, where N is the node set, A is the arc set, u is the vector of the arc capacities, $uo \in \mathbb{R}_+^{|N|}$ is the vector of upper bounds on the flow that

can be generated by (or originated from) any node (in addition to the incoming flow), $ut \in \mathbb{R}_+^{|N|}$ is the vector of upper bounds on the flow that can be consumed by (or terminated to) each node (in addition to the outgoing flow).

Let the overall flow on the network be the selected metric, and the flow-based centrality of the arcs be the core of the network analysis. The flow maximization problem Υ can be formulated as follows (Bertsimas and Tsitsiklis 1997):

$$\begin{cases} \max \sum_{r \in A} f_r \\ -uo_k \leq \sum_{r \in A: r=(i,k)} f_r - \sum_{r \in A: r=(k,j)} f_r \leq ut_k \quad k \in N \\ 0 \leq f_r \leq u_r \quad r \in A \\ f_r \in \mathbb{R} \quad r \in A \end{cases}$$

where f_r is the flow on arc $r \in A$ in any feasible solution. Let $\bar{f} \in \mathbb{R}_+^{|A|}$ be an optimal solution to problem Υ ; then, the quantity $\sum_{r \in A} \bar{f}_r$ is the optimal flow of the problem.

First of all, let us observe that in such a framework the VCG flow-based centrality is always nonnegative.

Theorem. Given a network $G = (N, A, u, uo, ut)$ and the related flow maximization problem Υ . The VCG flow-based centrality of any arc is always nonnegative.

Proof. Removing an arc r from G is equivalent to adding the constraint $f_r = 0$ to the problem Υ (let us call Υ_{-r} the resulting flow maximization problem); thus, the optimal flow of problem Υ_{-r} can only decrease or remains unchanged. Let $\bar{f}^\Upsilon \in \mathbb{R}_+^{|A|}$ and $\bar{f}^{\Upsilon_{-r}} \in \mathbb{R}_+^{|A|}$ be optimal solutions to problem Υ and Υ_{-r} , respectively.

Therefore, for any arc $r \in A$ we have that $c = \sum_{j \in A} \bar{f}_j^\Upsilon \geq c_{-r} = \sum_{j \in A - \{r\}} \bar{f}_j^{\Upsilon_{-r}}$ and thus $c_r = c - c_{-r} \geq 0$.

In particular, let us consider the example in Figure 4, where $uo_{v_i} = ut_{v_i} = 0$ for $i = 1, \dots, 4$.

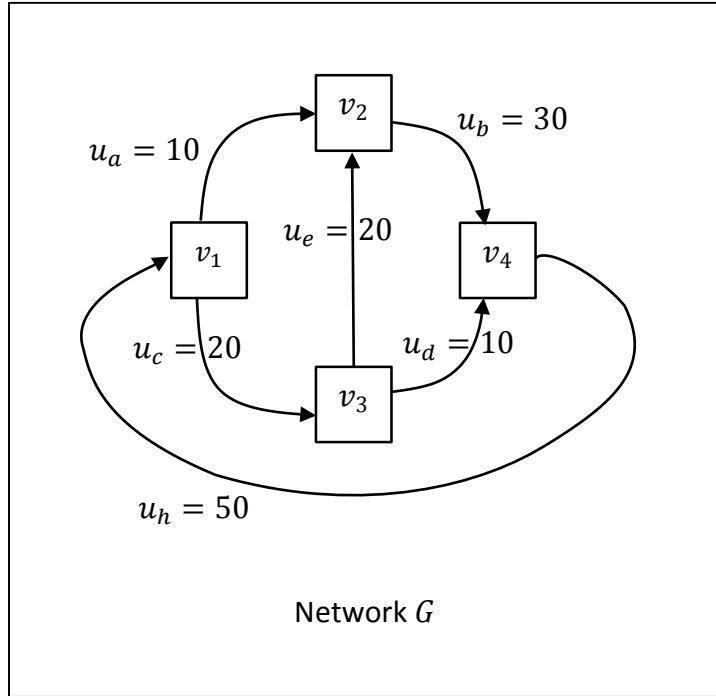


Figure 4

It is easy to verify (by solving the corresponding flow maximization problems) that the overall flow is equal to 110 (with $f_a = 10$, $f_b = 30$, $f_c = 20$, $f_d = 0$, $f_e = 20$, $f_h = 30$), and that the VCG flow-based centralities of the arcs are as follows:

$$\dot{c}_a = c - c_{-a} = 110 - 80 = c_a - c_a^{VCG} = 10 - ((20 - 30) + (20 - 20) + (0 - 0) + (20 - 20) + (20 - 30)) = 30$$

$$\dot{c}_b = c - c_{-b} = 110 - 30 = c_b - c_b^{VCG} = 30 - (-50) = 80$$

$$\dot{c}_c = c - c_{-c} = 110 - 30 = c_c - c_c^{VCG} = 20 - (-60) = 80$$

$$\dot{c}_d = c - c_{-d} = 110 - 110 = c_d - c_d^{VCG} = 0 - 0 = 0$$

$$\dot{c}_e = c - c_{-e} = 110 - 60 = c_e - c_e^{VCG} = 20 - (-30) = 50$$

$$\dot{c}_h = c - c_{-h} = 110 - 0 = c_h - c_h^{VCG} = 30 - (-80) = 110$$

Looking at these VCG flow-based centralities, link h appears to be the most crucial for the network, while link d seems of no significance. However, as regards link d , this is not a fair conclusion, as the following considerations show.

Let us compute the joint externality c_{cd}^{VCG} generated by the family of arcs c and d (and joint centrality \dot{c}_{cd}). It is easy to verify that this joint externality is exactly equal to the sum of the externalities generated separately by c and d (as well as the joint centrality is the sum of the VCG flow-based centralities of c and d):

$$c_{cd}^{VCG} = (10 - 10) + (10 - 30) + (0 - 20) + (10 - 30) = -60 = c_c^{VCG} + c_d^{VCG}$$

$$\dot{c}_{cd} = c_c - c_{-cd} = 110 - 30 = c_{cd} - c_{cd}^{VCG} = (20 + 0) - (-60) = 80 = \dot{c}_c + \dot{c}_d$$

Instead, let us now compute the joint externality c_{de}^{VCG} generated by the family of arcs d and e (and joint centrality \dot{c}_{de}). This joint externality is lower than the sum of the externalities generated separately by arcs d and e (while joint centrality \dot{c}_{de} is larger than the sum of \dot{c}_d and \dot{c}_e):

$$c_{de}^{VCG} = (10 - 10) + (10 - 30) + (0 - 20) + (10 - 30) = -60 < c_d^{VCG} + c_e^{VCG} = -30$$

$$\dot{c}_{de} = c_c - c_{-de} = 110 - 30 = c_{de} - c_{de}^{VCG} = (0 + 20) - (-60) = 80 > \dot{c}_d + \dot{c}_e = 50$$

This is because in the considered network the goal of maximizing the surplus is obtained by maximizing the flow from v_1 and v_4 , and there are three different paths from v_1 and v_4 (which we shortly denote by ordered strings of arcs): ab , cd , ceb . Paths cd and ceb have a first shared part, that is, arc c ; therefore, in a sense, arcs d and e compete (for a flow at the most equal to 10) to bring flow from v_1 and v_4 , and thus they can partially substitute each other in guaranteeing the performance of the network. Similarly, it is easy to verify that also d and b are partially in competition (up to a flow of ten), and that $c_{bd}^{VCG} = -80 < c_b^{VCG} + c_d^{VCG} = -50$ (as well as it results that $\dot{c}_{bd} = 110 > \dot{c}_b + \dot{c}_d = 80$).

Therefore, in the case where some fair criterion to share the identified joint externality c_{de}^{VCG} between arcs d and e is applied, we could claim that arc d can actually generate externality different from 0 (potentially, down to $-30 = c_{de}^{VCG} - (c_d^{VCG} + c_e^{VCG})$), and thus that d is of some importance.

A general approach to overcome this drawback of centrality measures à la VCG could be obtained by following what proposed in Avenali (2009) under a different context¹⁷. In that paper, a new issue is proposed, which is to find that partition of players into coalitions which maximize the auctioneer's revenue in the case whereby such coalitions take part to a VCG auction each one as a single agent (i.e. as a joint group). As shown

¹⁷ The assumed framework is auction design where players have independent and private valuations and no budget constraints.

in Avenali (2009), such a partition is the one that minimize the sum of the coalition discounts under a VCG auction where the participants are exactly the coalitions represented by the partition; moreover, determining such a partition is equivalent to finding a partition which maximizes the joint externalities generated by the coalitions represented by the partition (i.e. the joint VCG payments in the auction framework).

Therefore, putting apart the computational problem, we could better estimate the actual centrality of any network element by a two steps procedure: (i) identifying any partition of the elements which maximizes the sum of the joint centralities (in the following, referred to as *optimal partition*), (ii) for every identified optimal partition, sharing the joint centrality of any group of elements in the optimal partition among its members.

In particular, let us consider any network W and any metric to measure the centrality. Let V be the set of the analyzed network elements, and let Π be any partition of V in nonempty sets (obviously, $|\Pi| \leq |V|$); in addition, we denote by $\Psi(V)$ the set of all the partitions of V in nonempty sets. It is easy to verify that identifying a partition $\Pi \in \Psi(V)$ which maximizes the sum of the related joint centralities $\sum_{S \in \Pi} c_S$ is equivalent to selecting a partition $\Pi \in \Psi(V)$ which minimizes the sum of the related joint externalities $\sum_{S \in \Pi} c_S^{VCG}$.

Theorem. Determining a partition $\Pi \in \Psi(V)$ with maximum $\sum_{S \in \Pi} c_S$ is equivalent to finding a partition $\Pi \in \Psi(V)$ with minimum $\sum_{S \in \Pi} c_S^{VCG}$.

Proof. For any partition $\Pi \in \Psi(V)$, we have that $\sum_{S \in \Pi} c_S = \sum_{S \in \Pi} \sum_{t \in S} c_t = \sum_{t \in V} c_t$. Moreover, it results that $\sum_{S \in \Pi} c_S = \sum_{S \in \Pi} c_S - \sum_{S \in \Pi} c_S^{VCG} = \sum_{t \in V} c_t - \sum_{S \in \Pi} c_S^{VCG}$. Since $\max_{\Pi \in \Psi(V)} \sum_{S \in \Pi} c_S = \max_{\Pi \in \Psi(V)} [\sum_{t \in V} c_t - \sum_{S \in \Pi} c_S^{VCG}] = \sum_{t \in V} c_t - \min_{\Pi \in \Psi(V)} \sum_{S \in \Pi} c_S^{VCG}$, then determining a partition with maximum $\sum_{S \in \Pi} c_S$ is equivalent to finding a partition with minimum $\sum_{S \in \Pi} c_S^{VCG}$.

Coming back to the example in Figure 4, optimal partitions are $\{\{a\}, \{b\}, \{c\}, \{h\}, \{d, e\}\}$, $\{\{a\}, \{c\}, \{e\}, \{h\}, \{b, d\}\}$, $\{\{b\}, \{h\}, \{a, c\}, \{d, e\}\}$, $\{\{e\}, \{h\}, \{a, c\}, \{b, d\}\}$, $\{\{c\}, \{h\}, \{a, e\}, \{b, d\}\}$, where $c_a^{VCG} = -20$, $c_b^{VCG} = -50$, $c_c^{VCG} = -60$, $c_h^{VCG} = -80$, $c_{de}^{VCG} = -60$, $c_e^{VCG} = -30$, $c_{bd}^{VCG} = -80$, $c_{ac}^{VCG} =$

-80 , $c_{ae}^{VCG} = -50$, (and where $\dot{c}_a = 30$, $\dot{c}_b = 80$, $\dot{c}_c = 80$, $\dot{c}_h = 110$, $\dot{c}_{de} = 80$, $\dot{c}_e = 50$, $\dot{c}_{bd} = 110$, $\dot{c}_{ac} = 110$, $\dot{c}_{ae} = 80$). The sum of the joint externalities (joint centralities) for any optimal partition is -270 (380).

Given an optimal partition, the joint externality (joint centrality) of any identified group has to be divided somehow among the element of the coalition. This is not a main goal of this work and future research could investigate about this issue. In this work we apply a naïve criterion to split the joint externality of each group, which goes as follows. A share is allocated to any element of the group which is equal to the sum of the externality of the single element and of a slice of the difference between the joint externality and the externalities of the single members of the group; in particular, we equally share this difference among all members of the group.

Finally, after splitting the joint externality (joint centrality) among the members of each group for every optimal partition, the amount of externality (centrality) which is associated with any element is averaged by the number of the optimal partitions. We define this resulting quantity as *full externality (full centrality)*.¹⁸ From now on, we denote by $\overline{\dot{c}_t}$ and $\overline{c_t^{VCG}}$, respectively, the full centrality and the full externality of any element $t \in V$.

Let us now come back to the example in Figure 4. Let us consider, for instance, the joint externality $c_{de}^{VCG} = -60 < c_d^{VCG} + c_e^{VCG} = 0 - 30 = -30$ (the joint centrality $\dot{c}_{de} = 80 > \dot{c}_d + \dot{c}_e = 0 + 50 = 50$); by applying the described sharing criterion, we allocate to arcs d and e , respectively, an externality share equal to $0 + \frac{-60 - (-30)}{2} = -15$ and $-30 + \frac{-60 - (-30)}{2} = -45$ (a centrality share equal to $0 + \frac{80 - (50)}{2} = 15$ and $50 + \frac{80 - (50)}{2} = 65$). Furthermore, by taking into account all the computed optimal partitions, the full centralities and the full externalities of the arcs in A are as follows:

$$\overline{\dot{c}_a} = \frac{(30+30+30+30+30)}{5} = c_a - \overline{c_a^{VCG}} = 10 - \frac{(-20-20-20-20-20)}{5} = 30$$

$$\overline{\dot{c}_b} = \frac{(80+95+80+95+95)}{5} = 30 - \frac{(-50-65-50-65-65)}{5} = 89$$

$$\overline{\dot{c}_c} = \frac{(80+80+80+80+80)}{5} = 20 - \frac{(-60-60-60-60-60)}{5} = 80$$

¹⁸ The introduced names recall the full cost accounting, which is based on direct costs and a fair share of indirect costs.

$$\overline{\dot{c}_d} = \frac{(15+15+15+15+15)}{5} = 0 - \frac{(-15-15-15-15-15)}{5} = 15$$

$$\overline{\dot{c}_e} = \frac{(65+50+65+50+50)}{5} = 20 - \frac{(-45-30-45-30-30)}{5} = 56$$

$$\overline{\dot{c}_h} = \frac{(110+110+110+110+110)}{5} = 30 - \frac{(-80-80-80-80-80)}{5} = 110$$

Thus, by considering the externalities generated by groups of elements it clearly emerges that arc d has some importance in the given context ($\overline{\dot{c}_d} = 15 > \dot{c}_d = 0$). Moreover, let us observe that in this particular case every optimal partition highlights a positive role of d for the network; therefore, even by randomly choosing only one optimal partition to define the full externality and centrality (instead of considering all of them), arc d would never be of no significance.

Although the computational aspects of the centrality measures are not a goal of this work, we observe that, unfortunately, finding all optimal partitions requires in general a lot of computational effort. Thus, future research could be focused on designing and developing heuristics to quickly verify whether it is possible to identify a set of elements whose joint externalities are different from the sum of the single externalities. In such a case, after identifying one or more sub-optimal partitions by means of some heuristic, an approximation of the full externality could be promptly provided as shown above.

5. Conclusion

In this work we have shown how recent proposals in the literature related to the centrality measures are an application of the well-known generalized Vickrey mechanism. Moreover, we have provided examples to show how a proper measure of the centrality of an element should take into account the marginal contribution of the element to the network (for instance, in terms of connectivity or welfare generated).

Then, we have shown that centrality measures à la VCG inherit some drawbacks by the VCG mechanism, and can thus in general provide a poor estimate of the actual importance of some network elements. Moreover, by exploiting some results provided in the literature within the VCG auction design, we have proposed an approach to refine such estimates, based on considering also the joint externalities generated by groups of elements of the network and on sharing these externalities among the respective elements. We have referred to such new estimates as full centralities.

Our approach is general in the sense that it does not depend on a particular selected metric; therefore, it could be applied in several real-word networks, with respect to different metrics, to provide effective and fair estimates of the actual centrality of any element of the network.

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