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Abstract

This paper demonstrates that standard central limit theorem (CLT) results do not hold for means of nonparametric conditional efficiency estimators, and provides new CLTs that do hold, permitting applied researchers to estimate confidence intervals for mean conditional efficiency or to compare mean efficiency across groups of producers along the lines of the test developed by Kneip et al. (*JBES,* 2015b). The new CLTs are used to develop a test of the "separability" condition that is necessary for second-stage regressions of efficiency estimates on environmental variables. We show that if this condition is violated, not only are second-stage regressions meaningless, but also first-stage, unconditional efficiency estimates are without meaning. As such, the test developed here is of fundamental importance to applied researchers using nonparametric methods for efficiency estimation. Our simulation results indicate that our tests perform well both in terms of size and power. We present a real-world empirical example by updating the analysis performed by Aly et al. (*R. E. Stat.,* 1990) on U.S. commercial banks; our tests easily reject the assumption required for two-stage estimation, calling into question results that appear in *hundreds* of papers that have been published in recent years.

Keywords: technical efficiency, conditional efficiency, two-stage estimation, separability, data envelopment analysis (DEA), free-disposal hull (FDH).
JEL classification codes: C12, C14, C44, C46.

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1 Introduction

Nonparametric efficiency estimators are widely used to benchmark the performance of firms and other decision-making units. *Unconditional* versions of these estimators measure distance from a particular point in input-output space to an estimate of the boundary of the attainable set, i.e., the set of feasible combinations of inputs and outputs. Farrell (1957) is the first empirical example of such estimators, and relies on the convex hull of a set of observed input-output combinations to estimate the attainable set. This method has been popularized by Charnes et al. (1978) and is known in the literature as data envelopment analysis (DEA).[1] Deprins et al. (1984) relaxed the convexity assumption in the DEA estimator by using the free-disposal hull (FDH) of a set of observed input-output combinations to estimate the attainable set. More recently, Daraio and Simar (2005) have developed *conditional* measures of efficiency, allowing nonparametric estimation of technical efficiency conditional on some explanatory, contextual, "environmental" variables that are neither inputs nor outputs in the production process. Recent surveys of both the unconditional and conditional estimators are provided by Simar and Wilson (2013, 2015).[2]

The presence of environmental variables raises important questions for practitioners, such as the question of precisely how the environmental variables might affect the production process. Conceivably, the environmental variables might affect only the distribution of efficiency among firms. On the other hand, environmental variables might affect the production possibilities of firms. Or, environmental variables might affect *both* the distribution of efficiency as well as production possibilities.

Although there are numerous examples in the literature where the conditional efficiency estimators have been used, two-stage estimation procedures wherein technical efficiency is estimated by (unconditional) DEA or FDH estimators in the first stage, and the resulting efficiency estimates are regressed on some environmental variables in a second stage, remain very popular in the literature. Simar and Wilson (2007) cite 48 published papers that em-

[1] Banker et al. (1984) modified the Farrell (1957) estimator by using the conical hull of a set of observed input-output combinations to estimate the attainable set, thereby imposing an assumption of constant returns to scale.

[2] Examples of applications of conditional efficiency estimators include Halkos and Tzeremes, (2010, 2011a, 2011b, 2014), Verschelde and Rogge (2012), Zschille (2014, 2015), Rogge and De Jaeger (2013), Bădin et al. (2014), De Witte and Van Klaveren (2014), Tzeremes (2014, 2015), Mastromarco and Simar (2015), Cordero et al. (2015) and D'Alfonso et al. (2015).

ploy this approach and comment that "as far as we have been able to determine, none of the studies that employ this two-stage approach have described the underlying data-generating process." Simar and Wilson go on to (i) define a statistical model where truncated (but not censored, i.e., tobit, nor ordinary least squares) regression yields consistent estimation of model features, (ii) demonstrate that conventional, likelihood-based approaches to inference are invalid, and (iii) develop a bootstrap approach that yields valid inference in the second-stage regression. The model defined by Simar and Wilson rationalizes second-stage regressions of estimated efficiency on environmental variables in the sense that such a regression estimates a feature of the model described by Simar and Wilson. However, as noted by Simar and Wilson, the model contains a crucial feature—and a strong restriction—in the form of a "separability condition" that appears below as Assumption 2.1. Without this condition, second-stage regressions of estimated efficiency do not estimate any meaningful model feature; as Simar and Wilson (2007) note, this condition should be tested before estimating a second-stage regression, but until now no test has been available.

This paper presents a carefully-developed framework—i.e., a pair of statistical models—in order to make clear how environmental variables might be relevant. We develop a test of the separability condition described by Simar and Wilson. As will be seen below, this test is of fundamental importance whenever environmental variables are present. If the separability condition does not hold, unconditional DEA and FDH estimators have no useful interpretation; i.e., not only are second-stage regressions meaningless when the separability condition is violated, but the (unconditional) first-stage efficiency estimates are also meaningless. We also show that standard central limit theorem (CLT) results do not hold for means of conditional efficiency estimators, and extend the results of Kneip et al. (2015a) to prove new CLTs for means of conditional efficiency estimators. We use these new CLTs to develop our test of the separability condition. However, the new CLTs are useful beyond the test of separability since they allow one to estimate confidence intervals for mean conditional efficiency or to compare mean conditional efficiency across different groups along the lines of the test developed by Kneip et al. (2015b) for unconditional efficiency measures.

A number of papers have appeared in recent years using the model and approach for inference suggested by Simar and Wilson (2007). However, papers that estimate technical efficiency in the first stage and then regress these estimates on some environmental variables in a second-stage tobit model continue to appear. As far as we know, none of these papers present a statistical model in which second-stage tobit estimation would consistently estimate features of the model; the approach is ad hoc in each case. Moreover, regardless of how the second-stage regression is specified, any results from such regressions are meaningless for reasons given below when the separability condition is violated.[3] The statistical model in Simar and Wilson rationalizes second-stage regression of efficiency estimates on some environmental variables, but does not allow for the possibility that environmental variables might affect the production possibilities. If they do, then a different model is needed, and second-stage regression is not appropriate.

In the next section, we develop the statistical model. Estimators are discussed in Section 3, and the tests are developed in Section 5. Section 6 describes Monte Carlo experiments used to assess the size and power of our tests as well as results. In Section 7 we provide a real-world example by revisiting the work of Aly et al. (1990) and testing whether the assumptions given by Simar and Wilson (2007) that are required for the two-stage approach used by Aly et al. to be meaningful are satisfied. Conclusions are given in the final section. Appendix A gives technical assumptions used to derive results in Section 5, proofs of lemmas and theorems are given in Appendix B and Appendix C discusses how one can handle discrete environmental variables. Supplementary material mentioned in Sections 5.4 and 6 appears in separate Appendices D and E.

2 The Production Process in the Presence of Environmental Factors

In this section we formalize a statistical model of the production process along the lines of the probability framework of Cazals et al. (2002). The production process generates random variables (X, Y, Z) in an appropriate probability space, where $X \in \mathbb{R}_+^p$ is the vector of input

[3] A search on Google Scholar on 21 June 2016 using the keywords "dea," "efficiency," "tobit," and "two stage" returned 2,370 papers with dates between 2008 and 2016. As far as we know, none of these papers present a statistical model in which second-stage tobit estimation would consistently estimate features of the model; the approach is ad hoc in each case. Repeating the search after dropping the keyword "tobit" returned 11,100 papers over the same years. Even if only half of these hits are relevant, the searches indicate that the practice of regressing nonparametric efficiency estimates on some environmental variables in a second-stage regression is widespread, although perhaps many of these exercises yield meaningless results if the separability condition is frequently violated. Apparently, the warnings of Simar and Wilson (2007) have not been heeded.

quantities, $Y \in \mathbb{R}_{+}^{q}$ is the vector of output quantities and $Z \in \mathbb{R}^{r}$ is a vector of variables describing environmental factors. These factors Z are neither inputs nor outputs and are typically not under the control of the manager, but they may influence the production process in different ways as explained below. Let $f_{XYZ}(x, y, z)$ denote the joint density of (X, Y, Z) which has support $\mathcal{P} \subset \mathbb{R}_{+}^{p} \times \mathbb{R}_{+}^{q} \times \mathbb{R}^{r}$. This joint density can always be decomposed as

$$f_{XYZ}(x, y, z) = f_{XY|Z}(x, y \mid z) f_{Z}(z). \tag{2.1}$$

Let Ψ^{z} denote the conditional support of $f_{XY|Z}(x, y \mid z)$, i.e., the support of (x, y) given $Z = z$, and let $\mathcal{Z}$ be the support of $f_{Z}(z)$. Then Ψ^{z} is the set of feasible combinations of inputs and outputs for a firm facing the environmental conditions $Z = z$; i.e.,

$$\Psi^{z} = \{(X, Y) \mid X \text{ can produce } Y \text{ when } Z = z\}. \tag{2.2}$$

The environmental variables in Z can affect the production process either (i) only through Ψ^{z}, the support of (X, Y), or (ii) only through the density $f_{XY|Z}(x, y \mid z)$, thereby affecting the probability for a firm to be near its optimal boundary, or (iii) through both Ψ^{z} and $f_{XY|Z}(x, y \mid z)$. Let

$$\Psi = \bigcup_{z \in \mathcal{Z}} \Psi^{z}. \tag{2.3}$$

By construction, $\Psi^{z} \subseteq \Psi \; \forall \; z \in \mathcal{Z}$, and clearly $\Psi \subset \mathbb{R}_{+}^{p+q}$. However, whether Ψ is useful for benchmarking the performance of a firm producing output levels y from input levels x while facing levels z of the environmental variables depends on whether the "separability" condition described by Simar and Wilson (2007) is satisfied. This condition requires that Z affect production *only* through the conditional density $f_{XY|Z}(x, y \mid z)$ without affecting its support Ψ^{z}, and is stated explicitly in Assumption 2.1.

Assumption 2.1. *(Separability Condition): $\Psi^{z} = \Psi$ for all $z \in \mathcal{Z}$.*

Clearly, when Assumption 2.1 holds the joint support of (X, Y, Z) can be factorized as

$$\mathcal{P} = \Psi \times \mathcal{Z}, \tag{2.4}$$

and Ψ can be interpreted as the unconditional attainable set

$$\Psi = \{(X, Y) \mid X \text{ can produce } Y\}. \tag{2.5}$$

However, Ψ has the interpretation in (2.5) *if and only if* (iff) Assumption 2.1 holds. The separability condition is very strong and restrictive. Under Assumption 2.1, the environmental factors influence neither the *shape* nor the *level* of the boundary of the attainable set, and the potential effect of Z on the production process is only through the distribution of the inefficiencies. If the separability condition holds, it is meaningful to measure the efficiency of a particular production plan (x, y) by its distance to the boundary of Ψ. For example, under separability, the output-oriented Farrell efficiency score is given by

$$\lambda(x, y) = \sup\{\lambda > 0 \mid (x, \lambda y) \in \Psi\}. \tag{2.6}$$

In this case, it is meaningful to analyze the behavior of $\lambda(x, y)$ as a function of Z by using an appropriate regression model (see Simar and Wilson, 2007, 2011 for details).[4]

Alternatively, if the separability condition does not hold, then we have a more general situation where the factor Z may influence the level and the shape of the boundary of the attainable sets (and may also influence the conditional density $f_{XY|Z}(x, y \mid z)$). The following assumption characterizes this situation explicitly.

Assumption 2.2. *(Non Separability Assumption): $\Psi^z \neq \Psi$ for some $z \in \mathcal{Z}$, i.e., for some $z, \tilde{z} \in \mathcal{Z}$, $\Psi^z \neq \Psi^{\tilde{z}}$.*

Note that Assumptions 2.1 and 2.2 are mutually exclusive; one and only one holds in a given situation.

Under Assumption 2.2, the efficiency measure in (2.6) is difficult to interpret; in fact, it is economically meaningless because it does not measure the distance to the appropriate boundary. If Assumption 2.2 holds, the set Ψ can still be defined as in (2.3), but for benchmarking production units, the boundary of Ψ has little interest in this case because it may be unattainable for some firms faced with unfavorable conditions represented described by z. In such cases, the conditional measure

$$\lambda(x, y \mid z) = \sup\{\lambda > 0 \mid (x, \lambda y) \in \Psi^z\} \tag{2.7}$$

[4] We focus the presentation in this paper using output-oriented measures of efficiency such as the one in (2.6), but of course efficiency can be measured in other directions as desired. See the recent surveys by Simar and Wilson (2013, 2015) and the references cited therein for details. All of the results here are easily generalized to input, hyperbolic, and directional distance functions after straight-forward (but perhaps tedious) changes in notation.

introduced by Cazals et al. (2002) and Daraio and Simar (2005) gives a measure of distance to the appropriate, relevant boundary (i.e., the boundary that is attainable by firms operating under conditions described by z).

The distinction between Assumptions 2.1 and 2.2, and their implications for how environmental variables in Z affect the production process, has often been neglected in the literature where researchers analyze the effect of Z on $\lambda(X, Y)$ by estimating some regression of $\lambda(X, Y)$ on Z. Typically, starting with a sample of observations (X_i, Y_i, Z_i), DEA or FDH estimators $\widehat{\lambda}(X_i, Y_i)$ computed in a first stage are regressed on Z_i in a second-stage analysis. Even if Assumption 2.1 holds, additional problems described in Simar and Wilson (2007) remain to be solved in the second stage to obtain sensible inference. Theoretical results on how to make inference in a second stage linear regression, when appropriate, is described in detail by Kneip et al. (2015a). However, if Assumption 2.2 holds, the two-stage approach is almost certain to lead to incorrect results and inferences about the effect of Z on the production process. This explains why it is important, as noted by Simar and Wilson (2007)—indeed, essential—to test Assumption 2.1 against Assumption 2.2. If the test rejects separability in favor of Assumption 2.2, then only a second-stage regression of the conditional measure $\lambda(X, Y \mid Z)$ on Z can be meaningful, as described for example in Bădin et al. (2012).

In order to derive results below, the efficiency measures in (2.6) and (2.7) must be defined in terms of components of our probability model. Cazals et al. (2002) show that under free disposability (see Assumption 4.2 below) the output-oriented efficiency measure in (2.6) can be written as

$$\lambda(x, y) = \sup\{\lambda > 0 \mid H_{XY}(x, \lambda y) > 0\}, \tag{2.8}$$

where $H_{XY}(x, y) = \Pr(X \leq x,\ Y \geq y)$ is the probability of finding a firm dominating the production unit operating at the level (x, y).[5] This can be factored as $\Pr(X \leq x) \Pr(Y \geq y \mid X \leq x) = F_X(x) S_{Y|X}(y \mid X \leq x)$, where the latter conditional survival function is nonstandard due to the the condition $X \leq x$. For (x, y) such that x is in the interior of its support (i.e., $F_X(x) > 0$), the efficiency score can be written equivalently as

$$\lambda(x, y) = \sup\{\lambda > 0 \mid S_{Y|X}(\lambda y \mid X \leq x) > 0\}. \tag{2.9}$$

[5] Note that as usual, inequalities involving vectors are defined on an element-by-element basis.

Along the same lines, the conditional efficiency score can be expressed as

$$\lambda(x, y \mid z) = \sup\{\lambda > 0 \mid H_{XY|Z}(x, \lambda y \mid z) > 0\}, \tag{2.10}$$

where $H_{XY|Z}(x, y \mid z) = \Pr(X \leq x,\ Y \geq y \mid Z = z)$ is the probability of finding a firm dominating the production unit operating at the level (x, y) and facing environmental conditions z and is the distribution function corresponding to the conditional density $f_{XY|Z}(x, y \mid z)$ introduced earlier. Analogous to (2.9), the conditional efficiency measure can also be written as

$$\lambda(x, y \mid z) = \sup\{\lambda > 0 \mid S_{Y|X,Z}(\lambda y \mid X \leq x, Z = z) > 0\} \tag{2.11}$$

while noting the different roles of X and Z in the conditioning of the conditional survival function $S_{Y|X,Z}(y \mid X \leq x, Z = z) = \Pr(Y \geq y \mid X \leq x, Z = z)$.

3 Non-parametric Efficiency Estimators

The literature on nonparametric statistical inference for efficiency scores is by now well-developed. Here, we summarize the definitions and properties needed to test Assumption 2.1 versus Assumption 2.2. Consider a sample of identically, independently (iid) observations $\mathcal{S}_n = \{(X_i, Y_i, Z_i) \mid i = 1,\ \ldots,\ n\}$. Following Deprins et al. (1984), the FDH of the sample $\mathcal{S}_n$ is the set

$$\widehat{\Psi}_{\text{FDH}}(\mathcal{S}_n) = \bigcup_{(X_i,Y_i)\in\mathcal{S}_n} \left\{(x, y) \in \mathbb{R}_+^{p+q} \mid y \leq Y_i,\ x \geq X_i\right\}. \tag{3.1}$$

The convex hull of $\widehat{\Psi}_{\text{FDH}}(\mathcal{S}_n)$ given by

$$\begin{aligned}\widehat{\Psi}_{\text{DEA}}(\mathcal{S}_n) = \Big\{(x, y) \in \mathbb{R}_+^{p+q}\ |y \leq \sum_{i=1}^{n} \omega_i Y_i,\ x \geq \sum_{i=1}^{n} \omega_i X_i,& \\ \sum_{i=1}^{n} \omega_i = 1,\ \omega_i \geq 0\ \forall\ i = 1,\ \ldots,\ n\Big\}&\end{aligned} \tag{3.2}$$

provides the DEA estimator proposed by Farrell (1957) and popularized by Charnes et al. (1978).[6]

[6] Note that in (3.1)–(3.2), the data on Z_i are ignored; only the first $(p + q)$ components of the ordered $(p + q + r)$-tuples in $\mathcal{S}_n$ are used.

The corresponding efficiency estimators are obtained by plugging these estimators into the definition of $\lambda(x, y)$ in (2.6). Using $\widehat{\Psi}_{\text{FDH}}(\mathcal{S}_n)$ in the FDH case leads to

$$\widehat{\lambda}_{\text{FDH}}(x, y \mid \mathcal{S}_n) = \max_{i=1,\ldots,\, n \mid X_i \leq x} \left(\min_{j=1,\,\ldots,\,p} \left(\frac{Y_i^j}{y^j} \right) \right), \tag{3.3}$$

where y^j, Y_i^j denote the jth elements of y (i.e., the input vector corresponding to the fixed point of interest) and Y_i (i.e., the output vector corresponding to the ith observation in $\mathcal{S}_n$). This is simply the plug-in version of (2.8), where $H_{XY}(x, y)$ is replaced by its empirical version

$$\widehat{H}_{XY}(x, y) = n^{-1} \sum_{i=1}^{n} I(X_i \leq x, Y_i \geq y), \tag{3.4}$$

where $I(A)$ is the indicator function equal 1 if A is true and 0 otherwise. In the DEA case, replacing Ψ in (2.6) with $\widehat{\Psi}_{\text{DEA}}(\mathcal{S}_n)$ from (3.2) gives the DEA efficiency estimator

$$\begin{aligned}\widehat{\lambda}_{\text{DEA}}(x, y \mid \mathcal{S}_n) = \max_{\lambda,\omega_1,\,\ldots,\,\omega_n} \Big\{ \lambda > 0 \mid \lambda y \leq \sum_{i=1}^{n} \omega_i Y_i,\ x \geq \sum_{i=1}^{n} \omega_i X_i,& \\ \sum_{i=1}^{n} \omega_i = 1,\ \omega_i \geq 0\ \forall\, i = 1,\ \ldots,\ n \Big\}.&\end{aligned} \tag{3.5}$$

For the conditional efficiency scores we need to use a smoothed estimator of $H_{XY|Z}(x, y \mid z)$ to plug in (2.10), which requires a vector of bandwidths for Z. Denoting this r-vector of bandwidths by h, the conditional distribution function $H_{XY|Z}(x, y \mid z)$ is replaced by the estimator

$$\widehat{H}_{XY|Z}(x, y \mid z) = \frac{\sum_{i=1}^{n} I(X_i \leq x, Y_i \geq y) K_h(Z_i - z)}{\sum_{i=1}^{n} K_h(Z_i - z)}, \tag{3.6}$$

where $K_h(\cdot) = (h_1\ \ldots\ h_r)^{-1} K\left((Z_i - z)/h\right)$ and the division between vectors is understood to be component-wise. As explained in the literature (e.g., see Daraio and Simar, 2007b), the kernel function $K(\cdot)$ must have bounded support (e.g., the Epanechnikov kernel).[7] This provides the estimator

$$\widehat{\lambda}_{\text{FDH}}(x, y \mid z, \mathcal{S}_n) = \max_{i \in \mathcal{I}(z,h)} \left(\min_{j=1,\,\ldots,\,p} \left(\frac{Y_i^j}{y^j} \right) \right), \tag{3.7}$$

[7] An alternative would be, following Bădin et al. (2010), to plug a smoothed estimator of $S_{Y|X,Z}(y \mid X \leq x, Z = z)$ into (2.11), but as shown in Simar et al. (2015), if the two methods are asymptotically equivalent, the latter provides a bandwidth for z that depends on x and the resulting efficiency estimate may not be monotone decreasing in x in finite samples, as the target $\lambda(x, y \mid z)$ is.

where $\mathcal{I}(z,h) = \{i \mid z-h \leq Z_i \leq z+h\}$.

Alternatively, where one is willing to assume that the conditional attainable sets are convex, Daraio and Simar (2007b) suggest a conditional DEA estimator of $\lambda(x,y \mid z)$, namely

$$\widehat{\lambda}_{\text{DEA}}(x,y \mid z,\mathcal{S}_n) = \max_{\lambda,\omega_1,\,\ldots,\,\omega_n} \Big\{\lambda > 0 \mid \lambda y \leq \sum_{i\in\mathcal{I}(z,h)} \omega_i Y_i,\ x \geq \sum_{i\in\mathcal{I}(z,h)} \omega_i X_i,$$
$$\text{for some } \omega_i \geq 0 \text{ such that } \sum_{i\in\mathcal{I}(z,h)} \omega_i = 1,\Big\}. \tag{3.8}$$

Note that the conditional estimators in (3.7) and (3.8) are just localized version of the unconditional FDH and DEA efficiency estimators given in (3.3) and (3.5), where the degree of localization is controlled by the bandwidth in h. Practical aspects for choosing bandwidths are discussed below in Section 5.3.

The properties of nonparametric efficiency estimators have been examined in a number of papers in recent years. Park et al. (2000) and Daouia et al. (2015) derive the rate of convergence and limiting distribution of the FDH efficiency estimator. Kneip et al. (1998) derived the rate of convergence of the DEA estimator in (3.5), while Kneip et al. (2008) derived its limiting distribution. Kneip et al. (2015a) provide results on the moments of both FDH and DEA estimators. See Simar and Wilson (2013, 2015) for comprehensive surveys of the literature. To summarize relevant results for the unconditional efficiency estimators, under Assumptions 2.1, 4.1, 4.2 and some additional, appropriate regularity conditions (e.g., monotonicity, smoothness of the frontier and smoothness of the density of (X,Y)), for a fixed point (x,y) in the interior of Ψ, as $n \to \infty$,

$$n^{\kappa}\left(\widehat{\lambda}(x,y \mid \mathcal{S}_n) - \lambda(x,y)\right) \xrightarrow{\mathcal{L}} Q_{xy}(\cdot) \tag{3.9}$$

where $Q_{xy}(\cdot)$ is a regular, non-degenerate distribution with parameters depending on the characteristics of the DGP and on (x,y), and κ determines the rate of convergence.[8] For the FDH estimator, $\kappa = 1/(p+q)$ while for the DEA estimator, $\kappa = 2/(p+q+1)$. For the FDH case, the limiting distribution belongs to the Weibull family, but with parameters that are difficult to estimate. For the DEA case, the limiting distribution does not have a closed form. Hence in either case, inference on individual efficiency scores requires bootstrap

[8] Here and in the exposition that follows, we omit the subscripts "FDH" and "DEA" from the efficiency estimator in order to describe results in a generic fashion, thereby conserving space.

techniques. In the DEA case, Kneip et al. (2008) provide theoretical results for both a smoothed bootstrap and for subsampling, while Kneip et al. (2011) and Simar and Wilson (2011) provide details and methods for practical implementation. Subsampling can also be used for inference in the FDH case; see Jeong and Simar (2006) and Simar and Wilson (2011).

Jeong et al. (2010) show that the conditional version of the FDH and DEA efficiency estimators share properties similar to their unconditional counterparts whenever the elements of Z are continuous.[9] The sample size n is replaced by the effective sample size used to build the estimates, which is of order $nh_1 \ldots h_r$, which we write hereafter as nh^r for simplicity (hoping the reader will indulge the abuse of notation, since the individual bandwidths may differ). For a fixed point (x, y) in the interior of Ψ^z, as $n \to \infty$,

$$(nh^r)^\kappa \left(\widehat{\lambda}(x, y \mid z, \mathcal{S}_n) - \lambda(x, y \mid z) \right) \xrightarrow{\mathcal{L}} Q_{xy|z}(\cdot) \tag{3.10}$$

where again $Q_{xy|z}(\cdot)$ is a regular, non-degenerate limiting distribution analogous to the corresponding one for the unconditional case. The main argument in Jeong et al. (2010) relies on regularity conditions discussed in the next section, but also on the property that there are enough points in a neighborhood of z, which is obtained with the additional assumption that $f_Z(z)$ is bounded away from zero at z and that if the bandwidth is going to zero, it should not go too fast (see Jeong et al., 2010, Proposition 1; if $h \to 0$, h should be of order $n^{-\alpha}$ with $\alpha < 1/r$). We will return to this point in the discussion following Lemma 4.1 below.

4 New Results on Conditional Efficiency Estimators

4.1 Asymptotic Moments of Conditional Efficiency Estimators

As noted by Kneip et al. (2015a), availability of the asymptotic results for efficiency estimated at a fixed point (x, y) is useful, but not sufficient for analyzing the behavior of statistics that are function of FDH or DEA estimators evaluated at random points (X_i, Y_i). Kneip et al. (2015a) provide results on moments of *unconditional* efficiency estimators evaluated at random points, as well as central limit theorems for means of such estimators. However, similar results for *conditional* efficiency estimators have been unavailable until now. Such

[9] We discuss below in Appendix C how discrete "environmental" variables can be handled. Otherwise, except in Appendix C, we assume throughout that all elements of Z are continuous.

results are important. As noted in the discussion in Section 2, whenever Assumption 2.2 (as opposed to Assumption 2.1) holds, unconditional efficiency estimators have no useful interpretation and unconditional efficiency estimators must be used. In situations where 2.2 holds, the results obtained below will facilitate the use of conditional efficiency estimators, allowing researchers to make inference regarding mean efficiency, or to test other hypotheses regarding model structure analogous to the tests developed Kneip et al. (2015b) for unconditional efficiency estimators. Moreover, the results obtained in this Section will be used later in Section 5 to develop tests of separability versus non-separability, i.e., tests of Assumption 2.1 versus Assumption 2.2.

In the discussion below, we denote the FDH and DEA efficiency estimators by $\widehat{\lambda}(X_i, Y_i \mid \mathcal{S}_n)$ to stress the fact that the estimator is to be evaluated at a random point (X_i, Y_i).

Kneip et al. (2015a) prove that for the unconditional FDH and DEA estimators, under some regularity conditions (see Kneip et al., 2015a for details) and as $n \to \infty$,

$$E\left(\widehat{\lambda}(X_i, Y_i \mid \mathcal{S}_n) - \lambda(X_i, Y_i)\right) = Cn^{-\kappa} + R_{n,\kappa} \tag{4.1}$$

$$E\left(\left(\widehat{\lambda}(X_i, Y_i \mid \mathcal{S}_n) - \lambda(X_i, Y_i)\right)^2\right) = o\big(n^{-\kappa}\big), \tag{4.2}$$

and

$$\left|\mathrm{COV}\left(\widehat{\lambda}(X_i, Y_i \mid \mathcal{S}_n) - \lambda(X_i, Y_i),\ \widehat{\lambda}(X_j, Y_j \mid \mathcal{S}_n) - \lambda(X_j, Y_j)\right)\right| = o\big(n^{-1}\big) \tag{4.3}$$

for all $i, j \in \{1,\ \ldots,\ n\}$, $i \neq j$ and where $R_{n,k} = o\left(n^{-\kappa}\right)$. The values of the constant C, the rate κ, and the remainder term $R_{n,\kappa}$ depends on which estimator is used. For the DEA estimator, $\kappa = 2/(p+q+1)$ and $R_{n,\kappa} = O(n^{-3\kappa/2}(\log n)^{\alpha_1})$; for the FDH estimator, $\kappa = 1/(p+q)$ and $R_{n,\kappa} = O(n^{-2\kappa}(\log n)^{\alpha_2})$. The values of $\alpha_j > 1$, $j = 1, 2$ are given in Kneip et al. (2015a). For purposes of the results needed here, the $\log n$ factor contained in $R_{n,\kappa}$ does not play a role and can be ignored. The results outlined here are valid under a set of corresponding regularity assumptions (see Theorems 3.1 and 3.3 in Kneip et al., 2015a).

Similar results are needed for the asymptotic moments of the conditional efficiency estimators. To achieve this we follow the arguments of Jeong et al. (2010), who note that for a given h, the conditional FDH and DEA estimators in (3.7) and (3.8) do not target $\lambda(x, y \mid z)$, but instead estimate

$$\lambda^h(x, y \mid z) = \sup\{\lambda > 0 \mid (x, y) \in \Psi^{z,h}\}, \tag{4.4}$$

with the *conditional* attainable set given by

$$\begin{aligned}\Psi^{z,h} &= \{(X,Y) \mid X \text{ can produce } Y, \text{ when } |Z-z| \le h\} \\ &= \left\{(x,y) \in \mathbb{R}_+^{p+q} \mid H^h_{XY|Z}(x,y \mid z) > 0\right\} \\ &= \left\{(x,y) \in \mathbb{R}_+^{p+q} \mid f^h_{XY|Z}(\cdot,\cdot \mid z) > 0\right\} \end{aligned} \tag{4.5}$$

where $H^h_{XY|Z}(x,y \mid z) = \Pr(X \le x, Y \ge y \mid z-h \le Z \le z+h)$ gives the probability of finding a firm dominating the production unit operating at the level (x,y) and facing environmental conditions Z in an h-neighborhood of z and $f^h_{XY|Z}(\cdot,\cdot \mid z)$ is the corresponding conditional density of (X,Y) given $|Z-z| \le h$. Alternatively, (4.4) can be written as

$$\lambda^h(x,y \mid z) = \sup\left\{\lambda > 0 \mid H^h_{XY|Z}(x,\lambda y \mid z) > 0\right\}. \tag{4.6}$$

Moreover, it is clear that $\Psi^{z,h} = \bigcup_{|\widetilde{z}-z|\le h} \Psi^{\widetilde{z}}$.

Consequently, for all points (x,y) in the support of $f_{XY|Z}(x,y \mid z)$, the error of estimation can be decomposed as

$$\widehat{\lambda}(x,y \mid z) - \lambda(x,y \mid z) = \underbrace{\widehat{\lambda}(x,y \mid z) - \lambda^h(x,y \mid z)}_{=\Delta_1} + \underbrace{\lambda^h(x,y \mid z) - \lambda(x,y \mid z)}_{=\Delta_2}, \tag{4.7}$$

where the first difference (Δ_1) is due to the estimation error in the localized problem and the second difference (Δ_2) is the non-random bias (≤ 0) introduced by the localization.

Some assumptions are needed to define a statistical model. The next three assumptions are conditional analogs of standard assumptions made by Shephard (1970), Färe (1988), Kneip et al. (2015a) and others.

Assumption 4.1. *For all $z \in \mathcal{Z}$, Ψ^z and $\Psi^{z,h}$ are closed.*

Assumption 4.2. *For all $z \in \mathcal{Z}$, both inputs and outputs are strongly disposable; i.e., for any $z \in \mathcal{Z}$, $\widetilde{x} \ge x$ and $0 \le \widetilde{y} \le y$, if $(x,y) \in \Psi^z$ then $(\widetilde{x},y) \in \Psi^z$ and $(x,\widetilde{y}) \in \Psi^z$. Similarly, if $(x,y) \in \Psi^{z,h}$ then $(\widetilde{x},y) \in \Psi^{z,h}$ and $(x,\widetilde{y}) \in \Psi^{z,h}$.*

Assumption 4.2 corresponds to Assumption 1F in Jeong et al. (2010), and amounts to a regularity condition on the conditional attainable sets justifying the use of the localized versions of the FDH and DEA estimators. The assumption imposes weak monotonicity on the frontier in the space of inputs and outputs for a given $z \in \mathcal{Z}$, and is standard in micro-economic theory of the firm.

When the DEA estimators are used, the following assumption (corresponding to Assumption 1D in Jeong et al., 2010) is also needed.

Assumption 4.3. *For all $z \in \mathcal{Z}$, Ψ^z and $\Psi^{z,h}$ are convex in $\mathbb{R}_+^{p+q}$.*

The next assumption concerns the regularity of the density of Z and of the conditional density of (X, Y) given $Z = z$, as a function of z in particular near the efficient boundary of Ψ^z (see Assumption 6 in Jeong et al., 2010).

Assumption 4.4. *Z has a continuous density $f_Z(\cdot)$ such that for all $z \in \mathcal{Z}$ $f_Z(z)$ is bounded away from zero. Moreover the conditional density $f_{XY|Z}(\cdot, \cdot \mid z)$ is continuous in z and is strictly positive in a neighborhood of the boundary points.*

A number of additional assumptions are needed to complete the statistical model and to permit statistical analysis of the conditional estimators that have been introduced above as well as the test statistics introduced below. These assumptions are given in Appenidx A. Depending on the estimators that are used in a particular situation (i.e., either DEA or FDH), only a subset of the assumptions listed in Appendix A are required.

Our first result establishes smoothness of the potential influence of z on the frontier of Ψ^z. The result is needed in order to control the bias due to the localization, and is expressed in terms of a continuity condition of $\lambda(\cdot, \cdot \mid z)$ as a function of z.

Lemma 4.1. *Under either Assumption A.5 (for FDH case) or under Assumption A.6 (for the DEA case), For all (x, y) in the support of (X, Y),*

$$\lambda^h(x, y \mid z) - \lambda(x, y \mid z) = O(h) \tag{4.8}$$

as $h \to 0$.

Note that if Z is separable and has no effect on the frontier and (4.8) is trivially satisfied for all h. As noted in Bădin et al. (2015), it is easy to show that if $h \propto n^{-\gamma}$ with $1/r > \gamma > 1/(r + \kappa^{-1})$, the difference in (4.8) will be $o\left((nh^r)^{-\kappa}\right)$. We need $\gamma < r^{-1}$ to ensure there are enough observations in the h-neighborhood of z (see Proposition 1 in Jeong et al., 2010).

Since we cannot find an explicit expression for the second component Δ_2 in (4.7), and since the Weibull distribution linked to the first component Δ_1 contains unknown parameters, the best that can be done is to determine the order of an optimal bandwidth by balancing the order of the two error terms which leads to $h \propto n^{-1/(r+\kappa^{-1})}$, and then to take, as usual in nonparametric smoothing techniques, a smaller bandwidth to eliminate the bias term due to the localization as suggested in Jeong et al. (2010, Assumption 2). As expected, the order of the optimal bandwidth depends on the dimensions of Z as well as of X and Y. Below, in Section 5.3, we show how to select bandwidths h of appropriate order in applied work (see also the discussions in Bădin et al., 2015).

The following result provides moments for the conditional efficiency estimators.

Theorem 4.1. *Let $n_h = \min(n, nh^r)$. Suppose Assumptions 4.1, 4.2, 4.4, A.1, A.2, A.3 and A.4 hold. Then under Assumption A.5 for FDH case, or under Assumptions 4.3 and A.6 for the DEA case, as $n \to \infty$,*

$$E\left(\widehat{\lambda}(X_i, Y_i \mid Z_i, \mathcal{S}_n) - \lambda^h(X_i, Y_i \mid Z_i)\right) = C_c n_h^{-\kappa} + R_{c,n_h,\kappa}, \tag{4.9}$$

where $R_{c,n_h,\kappa} = o\left(n_h^{-\kappa}\right)$,

$$E\left(\left(\widehat{\lambda}(X_i, Y_i \mid Z_i, \mathcal{S}_n) - \lambda^h(X_i, Y_i \mid Z_i)\right)^2\right) = o\big(n_h^{-\kappa}\big), \tag{4.10}$$

and

$$\left|COV\left(\widehat{\lambda}(X_i, Y_i \mid Z_i, \mathcal{S}_n) - \lambda^h(X_i, Y_i \mid Z_i),\ \widehat{\lambda}(X_j, Y_j \mid Z_j, \mathcal{S}_n) - \lambda^h(X_j, Y_j \mid Z_j)\right)\right| = o\big(n_h^{-1}\big) \tag{4.11}$$

for all $i, j \in \{1, \ \ldots, \ n\}$, $i \neq j$. In addition, for the conditional DEA estimator $R_{c,n_h,\kappa} = O(n_h^{-3\kappa/2}(\log n_h)^{\alpha_1})$ and for the conditional FDH estimator $R_{c,n_h,\kappa} = O(n_h^{-2\kappa}(\log n_h)^{\alpha_2})$.

As will be seen, the $\log(n_h)$ factors appearing in the expressions for $R_{c,n_h,\kappa}$ do not play a role in the results that are derived below. The results here should not be surprising since the number of observations used to estimate the moments is reduced by the bandwidths; e.g., the rates n^κ for the unconditional estimators are reduced to n_h^κ for the conditional estimators.

4.2 Central Limit Theorems for Conditional Efficiency Estimators

Consider the sample means

$$\widehat{\mu}_n = n^{-1} \sum_{i=1}^{n} \widehat{\lambda}(X_i, Y_i \mid \mathcal{S}_n) \tag{4.12}$$

and

$$\widehat{\mu}_{c,n} = n^{-1} \sum_{i=1}^{n} \widehat{\lambda}(X_i, Y_i \mid Z_i, \mathcal{S}_n) \tag{4.13}$$

of unconditional and conditional efficiency estimators. The efficiency estimators in (4.12) and (4.13) could be either FDH or DEA estimators; differences between the two are noted below when relevant. In this subsection, we use the properties of moments of the conditional efficiency estimators derived in Section 4.1 to develop CLTs for means of conditional efficiency estimators.

For the case of means of *un*conditional efficiency estimators, Theorem 4.1 of Kneip et al. (2015a) establishes that

$$\sqrt{n}\left(\widehat{\mu}_n - \mu - Cn^{-\kappa} - R_{n,\kappa}\right) \xrightarrow{\mathcal{L}} N(0, \sigma^2) \tag{4.14}$$

as $n \to \infty$, where $\mu = E\left(\lambda(X,Y)\right)$ and $\sigma^2 = \mathrm{VAR}\left(\lambda(X,Y)\right)$. The theorem also establishes that $\widehat{\sigma}^2 = n^{-1}\sum_{i=1}^{n}\left(\widehat{\lambda}(X_i, Y_i \mid \mathcal{S}_n) - \widehat{\mu}_n\right)^2$ is a consistent estimator of σ^2. Conventional CLTs (e.g., the Lindeberg-Feller CLT) do not account for the bias term $Cn^{-\kappa}$, and hence are invalid for means of unconditional efficiency estimators unless unless $\kappa > 1/2$. In the case of FDH estimators, $\kappa > 1/2$ iff $(p+q) \leq 1$; in the case of DEA estimators, $\kappa > 1/2$ iff $(p+q) \leq 2$. If $\kappa = 1/2$, the bias is stable as $n \to \infty$, but if $\kappa < 1/2$, the bias explodes asymptotically. Kneip et al. (2015a) solve this problem by incorporating a generalized jackknife estimate of the bias and considering, when needed, test statistics based on averages over a subsample of observations. We use a similar approach below, although with the conditional efficiency estimators, the problem is rather more complicated than the one in Kneip et al. (2015a) due to the localization in the conditional efficiency estimators.

Define

$$\mu_c^h = E\left(\lambda^h(X, Y \mid Z)\right) = \int_{\mathcal{P}} \lambda^h(x, y \mid z) f_{XYZ}(x, y, z)\, dx\, dy\, dz \tag{4.15}$$

and

$$\sigma_c^{2,h} = \mathrm{VAR}\left(\lambda^h(X, Y \mid Z)\right) = \int_{\mathcal{P}} \left(\lambda^h(x, y \mid z) - \mu_c^h\right)^2 f_{XYZ}(x, y, z)\, dx\, dy\, dz. \tag{4.16}$$

These are the localized analogs of μ and σ^2. Next, let $\overline{\mu}_{c,n} = n^{-1}\sum_{i=1}^{n} \lambda^h(X_i, Y_i \mid Z_i)$. Although $\overline{\mu}_{c,n}$ is not observed, by the Lindeberg-Feller CLT

$$\sqrt{n}\left(\overline{\mu}_{c,n} - \mu_c^h\right) \xrightarrow{\mathcal{L}} N(0, \sigma_c^{2,h}) \tag{4.17}$$

under mild assumptions.

An obvious solution might be to estimate μ_c^h by $\widehat{\mu}_{c,n}$, but this proves problematic. To see this, define $\zeta_n = \widehat{\mu}_{c,n} - \overline{\mu}_{c,n}$. It is clear that $E(\zeta_n) = C_c n_h^{-\kappa} + R_{c,n_h,\kappa}$ by (4.9), and $\text{VAR}(\zeta_n) = o\left(n_h^{-1}\right)$ due to (4.10) and (4.11). It follows that $\zeta_n - E(\zeta_n) = o_p\left(n_h^{-1/2}\right)$. Now define $\widetilde{\mu}_{c,n} = E\left(\widehat{\mu}_{c,n}\right)$. Then

$$\widetilde{\mu}_{c,n} = \mu_c^h + C_c n_h^{-\kappa} + R_{c,n_h,\kappa}, \tag{4.18}$$

and it follows that

$$\begin{aligned}\widehat{\mu}_{c,n} - \widetilde{\mu}_{c,n} &= \overline{\mu}_{c,n} - \mu_c^h + \zeta_n - E(\zeta_n), \\ &= \overline{\mu}_{c,n} - \mu_c^h + o_p\left(n_h^{-1/2}\right).\end{aligned} \tag{4.19}$$

Clearly $\sqrt{n}(\widehat{\mu}_{c,n} - \widetilde{\mu}_{c,n})$ diverges as $n \to \infty$ since although $\sqrt{n}(\overline{\mu}_{c,n} - \mu_c^h) \xrightarrow{\mathcal{L}} N(0, \sigma_c^{2,h})$, $n^{1/2} o_p\left(n_h^{-1/2}\right)$ diverges if $n_h < n$ since $n_h = nh^r = n^{1-\gamma r}$ with $1/(r + \kappa^{-1}) < \gamma < 1/r$. Moreover, unless Z is irrelevant, $n_h < n$ for an optimal choice of h. Changing the scaling and considering $n^a(\widehat{\mu}_{c,n} - \widetilde{\mu}_{c,n})$ for some a such that $0 < a < (1 - \gamma r)/2 < 1/2$ does not work because the limiting distribution collapses to a point mass at zero in this case. Consequently, it seems there is no way to develop a CLT for means of conditional efficiency estimators analogous to the one in (4.14) for means of unconditional efficiency estimators.

The following result will be useful for the results developed below.

Lemma 4.2. *Under the assumptions Theorem 4.1, for $\kappa = 1/(p+q)$ in the case of the FDH estimator and for $\kappa = 2/(p+q+2)$ in the case of the DEA estimator,*

$$E\left(\widehat{\lambda}(X_i, Y_i \mid Z_i, \mathcal{S}_n)\right) = \mu_c^h + C_c n_h^{-\kappa} + R_{c,n_h,\kappa} \tag{4.20}$$

and

$$VAR\left(\widehat{\lambda}(X_i, Y_i \mid Z_i, \mathcal{S}_n)\right) = \sigma_c^{2,h} + o\left(n_h^{-\kappa/2}\right), \tag{4.21}$$

where $R_{c,n_h,\kappa} = o(n_h^{-\kappa})$.

Next, suppose $n_h < n$ (i.e., Z is relevant), and consider a random subsample $\mathcal{S}_{n_h}^*$ from $\mathcal{S}_n$ of size n_h where for simplicity we use the optimal rates for the bandwidths so that $n_h = \lfloor n^{1/(\kappa r+1)} \rfloor$ where $\lfloor a \rfloor$ denotes floor(a), i.e., the integer part of a. Define

$$\widehat{\mu}_{c,n_h} = \frac{1}{n_h} \sum_{\{(X_i,Y_i,Z_i) \in \mathcal{S}_{n_h}^*\}} \widehat{\lambda}(X_i, Y_i \mid Z_i, \mathcal{S}_n), \tag{4.22}$$

and let $\widetilde{\mu}_{c,n_h} = E(\widehat{\mu}_{c,n_h})$. Note that the estimators on the right-hand side of (4.22) are computed relative to the full sample $\mathcal{S}_n$, but the summation is over elements of the sub-sample $\mathcal{S}^*_{n_h}$.

The next result provides our first CLT for means of conditional efficiency estimators.

Theorem 4.2. *Under the assumptions of Theorem 4.1, the following conditions hold as $n \to \infty$ with $\kappa = 1/(p+q)$ for the FDH case and $\kappa = 2/(p+q+1)$ for the DEA case: (i) $\widetilde{\mu}_{c,n_h} = \mu_c^h + C_c n_h^{-\kappa} + R_{c,n_h,\kappa}$; (ii) $\widehat{\mu}_{c,n_h} - \widetilde{\mu}_{c,n_h} = \overline{\mu}_{c,n_h} - \mu_c^h + o\left(n_h^{-1/2}\right)$; (iii) $\sqrt{n_h}\left(\widehat{\mu}_{c,n_h} - \mu_c^h - C_c n_h^{-\kappa} - R_{c,n_h,\kappa}\right) \xrightarrow{\mathcal{L}} N(0, \sigma_c^{2,h})$; and (iv) $\widehat{\sigma}^{2,h}_{c,n} = n^{-1}\sum_{i=1}^n \left[\widehat{\lambda}(X_i, Y_i \mid Z_i, \mathcal{S}_n) - \widehat{\mu}_{c,n}\right]^2 \xrightarrow{p} \sigma_c^{2,h}$.*

There are no cases where standard CLTs with rate $\sqrt{n}$ may be used with means of conditional efficiency estimators, unless Z is irrelevant with respect to the support of (X, Y) (i.e., unless Assumption 2.1 holds). Theorem 4.2 provides a CLT for means of conditional efficiency estimators, but the convergence rate is $\sqrt{n_h}$ as opposed to $\sqrt{n}$, and the result is of practical use only if $\kappa > 1/2$. If $\kappa = 1/2$, the bias term $C_c n_h^{-\kappa}$ does not vanish, and if $\kappa < 1/2$, the bias term explodes as $n \to \infty$. These cases are addressed below.

4.3 Bias corrections and subsample averaging

For the unconditional case, all necessary details can be found in Kneip et al. (2015a, Theorems 4.3 and 4.4). Here, we derive corresponding results for conditional efficiency estimators. Assume the observations in $\mathcal{S}_n$ are randomly ordered, and to simplify notation, assume n is even. Let $\mathcal{S}^{(1)}_{n/2}$ denote the set of the first $n/2$ observations from $\mathcal{S}_n$, and let $\mathcal{S}^{(2)}_{n/2}$ denote the set of remaining $n/2$ observations from $\mathcal{S}_n$.[10] Next, for $j \in \{1,\ 2\}$ define

$$\widehat{\mu}^j_{c,n/2} = (n/2)^{-1} \sum_{(X_i,Y_i,Z_i)\in\mathcal{S}^{(j)}_{n/2}} \widehat{\lambda}(X_i, Y_i \mid Z_i, \mathcal{S}^{(j)}_{n/2}). \tag{4.23}$$

Let $\widetilde{\mu}_{c,n/2} = E\left(\widehat{\mu}^1_{c,n/2}\right) = E\left(\widehat{\mu}^2_{c,n/2}\right)$ and define

$$\overline{\mu}^j_{c,n/2} = \frac{2}{n} \sum_{(X_i,Y_i,Z_i)\in\mathcal{S}^{(j)}_{n/2}} \lambda^h(X_i, Y_i \mid Z_i). \tag{4.24}$$

[10] If n is odd, $\mathcal{S}^{(1)}_{n/2}$ can contain the first $\lfloor n/2 \rfloor$ observations and $\mathcal{S}^{(2)}_{n/2}$ can contain remaining $n - \lfloor n/2 \rfloor$ observations from $\mathcal{S}_n$. The fact that $\mathcal{S}^{(2)}_{n/2}$ contains one more observation than $\mathcal{S}^{(1)}_{n/2}$ makes no difference asymptotically.

By (4.19),

$$\widehat{\mu}^{j}_{c,n/2} - \widetilde{\mu}_{c,n/2} = \overline{\mu}^{j}_{c,n/2} - \mu^h_c + o_p(n_h^{-1/2}) \tag{4.25}$$

for $j \in \{1,\ 2\}$. Now define $\widehat{\mu}^*_{c,n/2} = \left(\widehat{\mu}^1_{c,n/2} + \widehat{\mu}^2_{c,n/2}\right)/2$. Clearly,

$$\widehat{\mu}^*_{c,n/2} - \widetilde{\mu}_{c,n/2} = \overline{\mu}_{c,n} - \mu^h_c + o_p(n_h^{-1/2}). \tag{4.26}$$

Subtracting (4.19) from (4.26) and re-arranging terms yields

$$\widehat{\mu}^*_{c,n/2} - \widehat{\mu}_{c,n} = \widetilde{\mu}_{c,n/2} - \widetilde{\mu}_{c,n} + o_p(n_h^{-1/2}). \tag{4.27}$$

Since $\widetilde{\mu}_{c,n/2} - \widetilde{\mu}_{c,n} = C_c(2^\kappa - 1)n_h^{-\kappa} + R_{c,n_h,\kappa}$ we obtain an estimator

$$\widetilde{B}^c_{\kappa,n_h} = (2^\kappa - 1)^{-1}\left(\widehat{\mu}^*_{c,n/2} - \widehat{\mu}_{c,n}\right) = C_c n_h^{-\kappa} + R_{c,n_h,\kappa} + o_p(n_h^{-1/2}), \tag{4.28}$$

of the leading bias term $C_c n_h^{-\kappa}$ in Theorem 4.2, part (iii), noting that the remainder term $R_{c,n_h,\kappa} = o(n_h^{-\kappa})$ can be neglected.

Of course, for n even there are $\binom{n}{n/2}$ possible splits of the sample $\mathcal{S}_n$. As noted by Kneip et al. (2015b), the variation in $\widetilde{B}^c_{\kappa,n_h}$ can be reduced by repeating the above steps $K \ll \binom{n}{n/2}$ times, shuffling the observations before each split of $\mathcal{S}_n$, and then averaging the bias estimates. This yields a generalized jackknife estimate

$$\widehat{B}^c_{\kappa,n_h} = K^{-1}\sum_{k=1}^{k} \widetilde{B}^c_{\kappa,n_h,k}, \tag{4.29}$$

where $\widetilde{B}^c_{\kappa,n_h,k}$ represents the value computed from (4.28) using the kth sample split.

Combining results yields the following:

Theorem 4.3. *Under the Assumptions of Theorem 4.1, with $\kappa = 1/(p+q) \geq 1/3$ in the FDH case or $\kappa = 2/(p+q+1) \geq 2/5$ in the DEA case,*

$$\sqrt{n_h}\left(\widehat{\mu}_{c,n_h} - \mu^h_c - \widehat{B}^c_{\kappa,n_h} - R_{c,n_h,\kappa}\right) \xrightarrow{\mathcal{L}} N(0, \sigma^{2,h}_c) \tag{4.30}$$

as $n \to \infty$.

If κ is smaller than 1/3 in the FDH case, or 2/5 in the DEA case, then the remainder term does not vanish fast enough and $\sqrt{n_h}R_{c,n_h,\kappa} \to \infty$ as $n \to \infty$. In such cases, the approach of averaging efficiency scores over a subsample of smaller size as in Kneip et al. (2015a) must be employed.

Define $n_{h,\kappa} = \lfloor n_h^{2\kappa} \rfloor$ so that $\sqrt{n_{h,\kappa}} < n_h^{1/2}$ when $\kappa < 1/2$. Then define

$$\widehat{\mu}_{c,n_{h,\kappa}} = \frac{1}{n_{h,\kappa}} \sum_{(X_i,Y_i,Z_i)\in\mathcal{S}^{**}_{n_{h,\kappa}}} \widehat{\lambda}(X_i, Y_i \mid Z_i, \mathcal{S}_n) \tag{4.31}$$

where $\mathcal{S}^{**}_{n_{h,\kappa}}$ is a random subsample of size $n_{h,\kappa}$ from $\mathcal{S}_n$.

Theorem 4.4. *Under the Assumptions of Theorem 4.1, with $\kappa = 1/(p+q)$ in the FDH case or $\kappa = 2/(p+q+1)$ in the DEA case,*

$$\sqrt{n_{h,\kappa}}\left(\widehat{\mu}_{c,n_{h,\kappa}} - \mu_c^h - \widehat{B}^c_{\kappa,n_h} - R_{c,n_h,\kappa}\right) \xrightarrow{\mathcal{L}} N(0, \sigma_c^{2,h}), \tag{4.32}$$

as $n \to \infty$ whenever $\kappa < 1/2$.

Remark 4.1. *Kneip et al. (2015a) note that for selected values of $p+q$, two different CLTs are available for means of unconditional efficiency estimators. The same is true for the conditional cases. With the DEA estimator when $p+q=4$ (so that $\kappa = 2/5$), using Theorem 4.3 neglects a term $\sqrt{n_h} R_{c,n_h,\kappa} = O\big(n_h^{-1/10}\big)$, whereas using Theorem 4.4, and an average over a subsample we neglect a term $\sqrt{n_{h,\kappa}} R_{c,n_h,\kappa} = O\left(n_h^{-1/5}\right)$ and we might expect a better approximation. For the conditional FDH estimator when $p+q=3$ (and hence $\kappa = 1/3$), using Theorem 4.3 implies an error of order $O\left(n_h^{-1/6}\right)$, and using an average over a subsample implies, by Theorem 4.4, an error of the smaller order $O\left(n_h^{-1/3}\right)$.*

5 Testing Separability

5.1 Basic Ideas

The goal is to test the null hypothesis of separability (Assumption 2.1) against its complement (Assumption 2.2). The idea for building a test statistics is to compare the conditional and unconditional efficiency scores using relevant statistics that are functions of $\widehat{\lambda}(X_i, Y_i \mid \mathcal{S}_n)$ and $\widehat{\lambda}(X_i, Y_i \mid Z_i, \mathcal{S}_n)$ for $i = 1, \ldots, n$. Note that under Assumption 2.1, $\lambda(X,Y) = \lambda(X, Y \mid Z)$ with probability one, even if Z may influence the distribution of the inefficiencies inside the attainable set, and the two estimators converge to the same object. But under Assumption 2.2, the conditional attainable sets Ψ^z are different and the two estimators converge to different objects. Moreover, under Assumption 2.2, $\lambda(X, Y) \geq \lambda(X, Y \mid Z)$ with strict inequality holding for some $(X, Y, Z) \in \mathcal{P}$.

The approach developed here is similar to those developed in Kneip et al. (2015b) for testing constant versus variable returns to scale or for testing convexity versus non-convexity of the attainable set. Recall the sample means in (4.12) and (4.13), where the efficiency estimators on the right-hand sides of (4.12) and (4.13) could be either FDH or DEA estimators. For purposes of the following discussion, suppose the same type of estimators (FDH or DEA) are used in both (4.12) and (4.13). By construction $(\widehat{\mu}_n - \widehat{\mu}_{c,n}) \geq 0$, and the null hypothesis of separability should be rejected if this difference is "too big". However, several problems remain to be solved, requiring some preliminary steps to adapt the existing results to the setup here. We demonstrate below in Section 6 that the procedure works well in practice with finite sample sizes.

5.2 Test Statistics

As noted above, in order to test the hypothesis that Z is separable, i.e., to test H_0: Assumption 2.1 holds versus H_1: Assumption 2.2 holds, one might consider the difference between estimators of $\mu = E(\lambda(X, Y))$ and $\mu_c^h = E(\lambda^h(X, Y \mid Z))$, which under the null estimate the same quantity. When the null is true, $\lambda(X, Y) \equiv \lambda^h(X, Y|Z)$ with probability one, for all values of h. Under the null, the two estimators $\widehat{\mu}_n$ and $\widehat{\mu}_{c,n_h}$ have (when appropriately rescaled, depending on the value of κ), an asymptotic normal distribution with mean $\mu = \mu_c^h$ and variance $\sigma^2 = \sigma_c^{2,h}$ for all h, and so both are consistent estimators of the common μ. As explained in the preceding section, we can also, in both cases, correct for the inherent bias of the estimators.

However, the properties of $(\widehat{\mu}_n - \widehat{\mu}_{c,n_h})$ (and their bias-corrected versions) are complicated due to the covariance between the two estimators, and this covariance is hard to estimate. Even in the limiting case where h is big enough so that $n_h = n$, it is clear that under the null, the asymptotic distribution of $(\widehat{\mu}_n - \widehat{\mu}_{c,n_h})$ will be degenerate with mass one at zero.[11]

The solution used here is analogous to the method used in the test for convexity of Ψ described by Kneip et al. (2015b). In particular, the sample $\mathcal{S}_n$ can be split into two

[11] As observed by Hall et al. (2004), if Z is irrelevant in the production process (independent of (X, Y)), the optimal value of the bandwidth is infinity. This limiting case is more restrictive that the hypothesis to be tested here, but may arise in practice.

independent, parts $\mathcal{S}_{1,n_1}$, $\mathcal{S}_{2,n_2}$ such that $n_1 = \lfloor n/2 \rfloor$, $n_2 = n - n_1$, $\mathcal{S}_{1,n_1} \bigcup \mathcal{S}_{2,n_2} = \mathcal{S}_n$, and $\mathcal{S}_{1,n_1} \bigcap \mathcal{S}_{2,n_2} = \emptyset$. The n_1 observations in $\mathcal{S}_{1,n_1}$ are used for the unconditional estimates, while the n_2 observations in $\mathcal{S}_{2,n_2}$ are used for the conditional estimates.[12]

After splitting the sample, compute the estimators

$$\widehat{\mu}_{n_1} = n_1^{-1} \sum_{(X_i,Y_i)\in\mathcal{S}_{1,n_1}} \widehat{\lambda}(X_i, Y_i \mid \mathcal{S}_{1,n_1}) \tag{5.1}$$

and

$$\widehat{\mu}_{c,n_{2,h}} = n_{2,h}^{-1} \sum_{(X_i,Y_i,Z_i)\in\mathcal{S}^*_{2,n_{2,h}}} \widehat{\lambda}(X_i, Y_i \mid Z_i, \mathcal{S}_{2,n_2}), \tag{5.2}$$

where as above in Section 4.2, $\mathcal{S}^*_{2,n_{2,h}}$ in (5.2), is a random subsample from $\mathcal{S}_{2,n_2}$ of size $n_{2,h} = \min(n_2, n_2 h^r)$. Consistent estimators of the variances are given in the two independent samples by

$$\widehat{\sigma}^2_{n_1} = n_1^{-1} \sum_{(X_i,Y_i)\in\mathcal{S}_{1,n_1}} \left(\widehat{\lambda}(X_i, Y_i \mid \mathcal{S}_{1,n_1}) - \widehat{\mu}_{n_1}\right)^2 \tag{5.3}$$

and

$$\widehat{\sigma}^{2,h}_{c,n_2} = n_2^{-1} \sum_{(X_i,Y_i,Z_i)\in\mathcal{S}_{2,n_2}} \left(\widehat{\lambda}(X_i, Y_i \mid Z_i, \mathcal{S}_{2,n_2}) - \widehat{\mu}_{c,n_2}\right)^2 \tag{5.4}$$

(respectively), where the full (sub)samples are used to estimate the variances.

The estimators of bias corresponding to (4.28) for a single split of each subsample for the unconditional and conditional cases are given by

$$\widetilde{B}_{\kappa,n_1} = (2^\kappa - 1)^{-1} \left(\widehat{\mu}^*_{n_1/2} - \widehat{\mu}_{n_1}\right) \tag{5.5}$$

and

$$\widetilde{B}^c_{\kappa,n_{2,h}} = (2^\kappa - 1)^{-1} \left(\widehat{\mu}^*_{c,n_2/2} - \widehat{\mu}_{c,n_2}\right). \tag{5.6}$$

[12] Kneip et al. (2015b) proposed splitting the sample unevenly to account for the difference in the convergence rates between the (unconditional) DEA and FDH estimators used in their convexity test, giving more observations to the subsample used to compute FDH estimates than to the subsample used to compute DEA estimates. Recall that the unconditional efficiency estimators converge at rate n^κ, while the conditional efficiency estimators converge at rate $(nh^r)^\kappa$. The optimal bandwidths are of order $n^{-\kappa/(r\kappa+1)}$, giving a rate of $n^{\kappa/(r\kappa+1)}$ for the conditional efficiency estimators. Using the logic of Kneip et al. (2015b), The full sample $\mathcal{S}_n$ can be split so that the estimators in the two subsamples achieve the same rate of convergence by setting $n_1^\kappa = n_2^{\kappa/(r\kappa+1)}$. This gives $n_1 = n_2^{1/(r\kappa+1)}$. Values of n_1, n_2 are obtained by finding the root η_0 in $n - \eta - \eta^{1/(r\kappa+1)} = 0$ and setting $n_2 = \lfloor \eta_0 \rfloor$ and $n_1 = n - n_2$. However, this will often result in too few observations in the first subsample to obtain meaningful results. For example, if $p = q = r = 1$ and $n = 200$, following the reasoning above would lead to $n_1 = 22$ and $n_2 = 178$.

For the unconditional case in (5.5), $\widehat{\mu}^*_{n_1/2} = \left(\widehat{\mu}^1_{n_1/2} + \widehat{\mu}^2_{n_1/2}\right)/2$, and for $j \in \{1,\ 2\}$, $\widehat{\mu}^j_{n_1/2} = (n_1/2)^{-1} \sum_{(X_i,Y_i,Z_i)\in\mathcal{S}^{(j)}_{n_1/2}} \widehat{\lambda}(X_i, Y_i \mid \mathcal{S}^{(j)}_{n_1/2})$, where $\mathcal{S}^{(j)}_{n_1/2}$ is the jth part of a random split of the full (sub)sample $\mathcal{S}_{n_1}$. Details are given in Kneip et al. (2015a). For the conditional case in (5.6), $\widehat{\mu}^*_{c,n_2/2} = \left(\widehat{\mu}^1_{c,n_2/2} + \widehat{\mu}^2_{c,n_2/2}\right)/2$, and for $j \in \{1,\ 2\}$, $\widehat{\mu}^j_{c,n_2/2} = (n_2/2)^{-1} \sum_{(X_i,Y_i,Z_i)\in\mathcal{S}^{(j)}_{n_2/2}} \widehat{\lambda}(X_i, Y_i \mid Z_i, \mathcal{S}^{(j)}_{n_2/2})$, where $\mathcal{S}^{(j)}_{n_2/2}$ is the jth part of a random split of the full (sub)sample $\mathcal{S}_{n_2}$. The bias estimates in (5.5)–(5.6) can then be averaged over K random splits of the two subsamples $\mathcal{S}_{n_1}$ and $\mathcal{S}_{n_2}$ to obtain bias estimates $\widehat{B}_{\kappa,n_1}$ for the unconditional case and $\widehat{B}^c_{\kappa,n_{2,h}}$ for the conditional case.

For small values of $(p+q)$ such that $\kappa \geq 1/3$ in the FDH case or $\kappa \geq 2/5$ when DEA estimators are used, Theorem 4.3 and Kneip et al. (2015a, Theorem 4.3) can be used to construct an asymptotically normal test statistic for testing the null hypothesis of separability. In particular, since our bias-corrected sample means are independent due to splitting the original sample into independent parts, and since two sequences of independent variables each with normal limiting distributions have a joint bivariate normal limiting distribution with independent marginals, if follows that for the values of $(p+q)$ given above

$$T_{1,n} = \frac{\left(\widehat{\mu}_{n_1} - \widehat{\mu}_{c,n_{2,h}}\right) - \left(\widehat{B}_{\kappa,n_1} - \widehat{B}^c_{\kappa,n_{2,h}}\right)}{\sqrt{\frac{\widehat{\sigma}^2_{n_1}}{n_1} + \frac{\widehat{\sigma}^{2,h}_{c,n_2}}{n_{2,h}}}} \xrightarrow{\mathcal{L}} N(0,1) \tag{5.7}$$

under the null. Alternatively, for $\kappa < 1/2$, similar reasoning with Theorem 4.4 and Kneip et al. (2015a, Theorem 4.4) leads to

$$T_{2,n} = \frac{\left(\widehat{\mu}_{n_{1,\kappa}} - \widehat{\mu}_{c,n_{2,h,\kappa}}\right) - \left(\widehat{B}_{\kappa,n_1} - \widehat{B}^c_{\kappa,n_{2,h}}\right)}{\sqrt{\frac{\widehat{\sigma}^2_{n_1}}{n_{1,\kappa}} + \frac{\widehat{\sigma}^{2,h}_{c,n_2}}{n_{2,h,\kappa}}}} \xrightarrow{\mathcal{L}} N(0,1) \tag{5.8}$$

under the null, where $n_{1,\kappa} = \lfloor n_1^{2\kappa} \rfloor$ with $\widehat{\mu}_{n_{1,\kappa}} = n_{1,\kappa}^{-1} \sum_{(X_i,Y_i)\in\mathcal{S}^*_{n_{1,\kappa}}} \widehat{\lambda}(X_i, Y_i \mid \mathcal{S}_{n_1})$, and $\mathcal{S}^*_{n_{1,\kappa}}$ is a random subsample of size $n_{1,\kappa}$ taken from $\mathcal{S}_{n_1}$ (see Kneip et al., 2015a for details). For the conditional part, we have similarly and as described in the preceding section, $n_{2,h,\kappa} = [n_{2,h}^{2\kappa}]$, with $\widehat{\mu}_{c,n_{2,h,\kappa}} = n_{2,h,\kappa}^{-1} \sum_{(X_i,Y_i,Z_i)\in\mathcal{S}^*_{n_{2,h,\kappa}}} \widehat{\lambda}(X_i, Y_i \mid Z_i, \mathcal{S}_{n_2})$ where $\mathcal{S}^*_{n_{2,h,\kappa}}$ is a random subsample of size $n_{2,h,\kappa}$ from $\mathcal{S}_{n_2}$.

Given a random sample $\mathcal{S}_n$, one can compute values $\widehat{T}_{1,n}$ or $\widehat{T}_{2,n}$ depending on the value of $(p+q)$. From the discussion in Section 5.1, it is clear that a one-sided test is appropriate;

hence the null should be rejected whenever null whenever $1-\Phi(\widehat{T}_{1,n})$ or $1-\Phi(\widehat{T}_{2,n})$ is less than the desired test size, e.g., .1, .05, or .01, where $\Phi(\cdot)$ denotes the standard normal distribution function.

5.3 Bandwidth Optimization

As noted above, explicit expressions for the two components Δ_1 and Δ_2 of the estimation error in (4.7) are not available. Consequently, the best that can be done is to determine the order of optimal bandwidths by balancing the order of the two error terms yielding $h \propto n^{-1/(r+\kappa^{-1})}$ as explained earlier. Although the order by itself is of little help in applications, following the suggestion of Jeong et al. (2010) one can select optimal bandwidths for estimating the conditional distribution $H_{XY|Z}(x, y \mid z)$ by $\widehat{H}_{XY|Z}(x, y \mid z)$ given in (3.6). This can be accomplished using the least-squares cross-validation (LSCV) procedure described by Li et al. (2013), smoothing only on the r conditioning variables in Z, and not the dependent variables (X, Y). Note that, as proved by Hall et al. (2004), if one component of Z is irrelevant, then the corresponding bandwidth obtained by LSCV will converge to infinity as $n \to \infty$; but for relevant components of Z, LSCV gives a bandwidth with optimal rate $h \propto n^{-1/(r+4)}$ for estimating $H_{XY|Z}(x, y \mid z)$.

Recall that if Z is relevant, the optimal bandwidths for estimating $\lambda(x, y \mid z)$ have a different order ($h \propto n^{-1/(r+\kappa^{-1})}$, as opposed to $h \propto n^{-1/(r+4)}$) due to the presence of the localizing bias. In practice, one can optimize bandwidths using LSCV, and then correct the resulting bandwidths by multiplying by the scaling factor $n^{1/(r+4)}n^{-1/(r+\kappa^{-1})} = n^{(\kappa^{-1}-4)/((r+4)(r+\kappa^{-1})}$ to obtain bandwidths h with optimal order for estimating $\lambda(x, y \mid z)$. To avoid numerical difficulties, for the jth element Z_i^j of Z_i, $j = 1, \ldots, r$, $i = 1, \ldots, n$, one should in practice bound the LSCV search between a small factor, say 0.01, times the normal reference rule bandwidth (i.e., $0.01 \times 1.06\widehat{\sigma}_j n^{1/5}$, where $\widehat{\sigma}_j$ is the sample standard deviation of the observations Z_i^j, $j = 1, \ldots, n$) and 2 times the difference $(\max_i(Z_i^j) - \min_i(Z_i^j))$. If Z_i^j is irrelevant, LSCV will drive the jth element h_j of h to its upper bound; using a bounded kernel (e.g., the Epanechnikov kernel), no smoothing will be done in the jth dimension of Z when this happens. In such cases, there is no need to apply the scaling factor above to h_j.

5.4 Replicability

It is important to note that tests based on the statistics defined in (5.7) and (5.8) are valid for *any* split of a given sample of size n into mutually exclusive, collectively exhaustive subsamples of sizes n_1 and n_2. However, there are $n!/((n_1!)(n_2!))$ possible splits (e.g., for $n = 100$ and $n_1 = \lfloor n_1 \rfloor$, $n_2 = n - n_2$ there are more than 10^{25} possible splits), and results may vary over these splits. This means that two researchers using the same data might reach different results by using different splits of the sample. Worse, a naive or dishonest researcher might be tempted to split the sample repeatedly until the desired result is obtained.

It does not appear possible to combine information across many splits of a given sample and to obtain meaningful results. One might split the sample randomly, say 100 or 1,000 times, and then average the resulting values of the test statistic from (5.7) or (5.8), but the values are not independent across the different sample splits, and the covariance is of complicated and unknown form.

In order to make results of our tests repeatable and verifiable, we propose a deterministic rule to randomly split n observations on $(p + q + r)$ variables. Our rule is expressed as an algorithm, consisting of the following steps.

1. Arrange data in an n by $(p + q + r)$ matrix; the ordering of the rows and columns is not relevant. Divide the values in each column by the standard deviation of all the values in the column.
2. Compute the sum of values in each column. Create character strings by writing the sums in format E18.10 with UTF-8 encoding.
3. Use Secure Hash Algorithm-2 to create an SHA-256 hash of the character strings created in step 2; each hash is a string of characters 0–9, A–H with UTF-8 encoding.
4. Sort the hash strings created in step 3 in ascending order, and use this sorting to sort the columns of the data matrix.
5. For each row in the column-sorted matrix, create a character string of the row's $(p + q + r)$ values by writing in format aE18.10 where a is the value of $(p + q + r)$.
6. Use Secure Hash Algorithm-2 to create an SHA-256 hash with UTF-8 encoding of the character strings created in step 5.
7. Sort the hash strings created in step 6 in ascending order, and use this sorting to sort the rows of the column-sorted data matrix obtained in step 4.

8. Restore the original ordering of the columns using information saved from step 4.
9. Multiply each column of the row-sorted data matrix obtained in step 8 by the corresponding standard deviations used in step 1.
10. Use the first n_1 rows of the row-sorted data matrix obtained in step 9 to form subsample $\mathcal{S}_{1,n_1}$, and use the remaining rows to form subsample $\mathcal{S}_{2,n_2}$.

Following this algorithm ensures that the final ordering of the observations does not depend on the initial ordering, nor on any choice made by the researcher provided values in the data are not tampered with. Using UTF-8 encoding ensures that character data will be encoded the same way by researchers working anywhere in the world. Secure Hash Algorithm-2 is described by National Institute of Standards and Technology (2015) and is based on character representations of the data, and hence creates orderings that can be regarded as pseudo-random, unique, and independent of numerical values in the data matrix. The widely used *R* programming language (R Development Core Team, 2008) with the 'digest' package (Eddelbuettel, 2016) can be used to compute the SHA-256 hashes. *R* code implementing the algorithm as a function is given in Appendix D, along with some examples illustrating usage of the code.

6 Monte Carlo Evidence

We perform Monte Carlo experiments to gauge the performance of the separability test described in Section 5. In each experiment, we simulate $n \in \{100,\ 200,\ 1000\}$ observations with $r \in \{1,\ 2,\ 3\}$ and $(p, q) \in \{(1,1),\ (2,1),\ (2,2),\ (3,2),\ (3,3)\}$ so that $(p+q) \in \{2,\ 3,\ 4,\ 5,\ 6\}$. To generate an observation $(X_i, Y_,, Z_i)$, we first generate a $(p+q)$-tuple $v = \left[v_p', v_q'\right]'$ uniformly distributed on a unit sphere centered at the origin in $\mathbb{R}^{p+q}$, where v_p and v_q are column vectors of length p and q, respectively. We then set $X = (1 - |v_p|)$ and $Y^{\text{eff}} = |v_q|$ to obtain *fully efficient* levels of inputs and outputs. Next, we simulate an $(r \times 1)$ vector Z of independent draws from the uniform distribution on $(0, 2)$, and a draw u from the half-normal distribution $N^+(0,1)$. Then we computed "observed" output levels $Y = Y^{\text{eff}} \left[1 + \delta(Z'\beta)\right] / (1+u)$ where β is an $(r \times 1)$ vector of ones, $\delta \in \{0,\ 0.1,\ \ldots,\ 0.9,\ 1.0,\ 1.5,\ 2.0\}$ and $(1+u) \geq 1$ is the random inefficiency. Repeating this for $i = 1,\ 2,\ \ldots,\ n$ results in a simulated random sample $\mathcal{S}_n = \{(X_i,\ Y_i,\ Z_i)\}_{i=1}^{n}$. By construction, when $\delta = 0$, Z plays no role and Assumption 2.1 (separability of Z) holds.

Otherwise, when $\delta > 0$, separability does not hold and instead Assumption 2.2 holds.[13]

The results of our experiments using DEA estimators with $r \in \{1,\ 2\}$ are shown in Tables 1–2. Table 1 gives results for tests for separability with $r = 1$, while Table E.3 gives results for the corresponding experiments where $r = 2$. Both tables contain 3 groups of results corresponding to 100, 200, or 1,000 observations. Within each of these groups, we show, for various values of δ, rejection rates for the separability tests for nominal test sizes of .10, .05, and .01 with $(p + q)$ ranging from 2 to 6. The first row in each group corresponds to $\delta = 0$, where the null hypothesis is true; the remaining rows give rejection rates with increasing departures from the null, corresponding to increasing values of δ. Additional results from experiments with the DEA estimator and $r = 3$ are presented in Table E.1 in Appendix E. Results from the same experiments but using the FDH estimator, for $r \in \{1,\ 2,\ 3\}$ are given in Tables E.2–E.4 in Appendix E.

Overall, the results in Tables 1–2 (and in Table E.1) confirm that the tests tend to reject the null hypothesis of separability at increasing rates both (i) with increasing departure from the null and (ii) as sample size increases. For each $r \in \{1,\ 2,\ 3\}$, rejection rates when $\delta = 0$ (i.e., the realized sizes of the tests) are larger than the nominal sizes (.1, .05, and .01) when $n = 100$. With $n = 200$, the realized test sizes are smaller than with $n = 100$, and with $n = 1,000$ the realized sizes are much closer to the corresponding nominal sizes. Note also that realized sizes tend to increase (for constant r and holding n fixed at 100 or 200) as the number of input-output dimensions $(p + q)$ increases from 2 to 4, but then improves when $(p + q)$ increases from 4 to 5. and $T_{1,n}$ is replaced by $T_{2,n}$. Recalling the discussion in Sections 4 and 5, the statistic $T_{2,n}$ defined in (5.8) uses a subsample of efficiency estimates to compute the sample mean of unconditional efficiency estimates, and a smaller subsample of estimates to compute the sample mean of conditional efficiency estimates than is used for

[13] Given a $(p + q)$-vector v of iid draws from the $N(0, 1)$ distribution, $u = v(v'v)^{-1/2}$ is a vector of coordinates from a uniform distribution on the unit sphere in $\mathbb{R}^{p+q}$ (Muller, 1959; Marsaglia, 1972). Setting $Y = |u_q|$ amounts to reflecting any point that lies below one or more of the u_p axes around those axes. Similarly, $-|u_p|$ reflects around the u_q axes, but in negative directions; adding 1 shifts the resulting points to the positive orthant in $\mathbb{R}^{p+q}$. This amounts to generating uniform points on a unit sphere centered at $\left[\mathbf{1}'_p, \mathbf{0}'_q\right]'$, reflecting the points so that all lie on the part of the sphere in the unit hypercube with in the positive orthant with a corner at the origin. We then projecting points away from this "frontier" in the output directions. We use the massively parallel Palmetto Cluster at Clemson University for our experiments, generating pseudo-random uniform deviates using independent Mersenne Twister generators on each processor; see Matsumoto and Nishimura (2000) for details. Standard normal deviates are generated from uniform $(0, 1)$ deviates using the transformation method.

$T_{1,n}$ defined in (5.7). A similar effect is seen in the tests involving unconditional efficiency estimators developed by Kneip et al. (2015b).

Comparing corresponding sets of results for given values of n, δ, and $(p+q)$ across the different values $r \in \{1,\ 2,\ 3\}$ in Tables 1, 2 and E.1), it is evident that there is a price to pay in terms of power for increasing values of r. This is particularly true for the larger values of $(p+q)$. When $n = 1,000$, $(p+q) = 6$ and $\delta = 2.0$, the achieved rejection rate is 0.923 at nominal test size .1 when $r = 1$, but only 0.551 when $r = 2$ and only 0.266 when $r = 3$. DEA estimators (both the unconditional and conditional versions) suffer from the well-known curse of dimensionality, so these results are not surprising. The effect is much less pronounced when $(p+q) = 2$, 3, or even 4, suggesting that one might want to use dimension-reduction methods similar to the method used below for the empirical example appearing in Section 7 in applications where there are more than 3–4 dimensions in input-output space.

As noted above, corresponding results obtained with FDH estimators are given in Appendix E. Inspection of those results reveal the same overall patterns seen with the DEA estimators, with the exception that the realized sizes are larger and the power of the tests is smaller with FDH estimators than with DEA estimators. This is due to the slower convergence rate of FDH estimators as opposed to DEA estimators.

Results from a second and third set of experiments are given in Appendix E. In the second set, data are simulated from a DGP where the environmental variables Z affect only the distribution of inefficiency, but not the frontier, so that Assumption 2.1 holds for all values of δ (see Appendix E for details). Results from the second set of experiments appear in Tables E.5–E.7 for DEA estimators, and in Tables E.8–E.10 for FDH estimators. As δ increases, Z has an increasing impact on the dispersion of inefficiency, but rejection rates increase only slightly. Some increase is to be expected, perhaps, since increased dispersion of the inefficiency process means that there are fewer observations near the frontier, making the frontier harder to estimate.

In the third set of experiments, the environmental variables affect both the frontier and the dispersion of inefficiency, with both effects increasing as δ increases (again, see Appendix E for details). Results obtained with DEA estimators are shown in Tables E.11–E.13, while results obtained with FDH estimators are given in Tables E.14–E.16. Relative to corresponding results from the first set of experiments described above where Z affects only the frontier when $\delta > 0$, test power is lower in the third set of experiments. As in the second set of experiments, there is a price to pay for increasing dispersion of the inefficiency process, which makes estimation of the frontier increasingly difficult. Nonetheless, the decline in power relative to that in the first set of experiments leaves intact the qualitative statements made earlier. For $n = 200$, $(p+q) = 6$, and $r = 1$, realized sizes are identical in the first and third set of experiments when $\delta = 0.0$. When $\delta > 0$, the corresponding estimated rejection rates in Tables 1 and E.11 are very close and significantly different in perhaps only a few cases.

7 Empirical Illustration using Bank Data

As a final exercise, we revisit the empirical examples provided by Simar and Wilson (2007), where estimated efficiency of U.S. Banks is regressed on some explanatory variables in a second-stage analysis. We start with the same data used by Simar and Wilson (2007), and consider both the subsample of 322 banks as well as the full sample of 6,955 banks examined by Simar and Wilson. The data include observations on 3 inputs (purchased funds, core deposits, and labor) and 4 outputs (consumer loans, business loans, real estate loans, and securities held). The data also include observations for two continuous explanatory variables used by Simar and Wilson (2007), namely *SIZE* (i.e., the log of total assets, reflecting banks' sizes) and *DIVERSE* (i.e., a measure of diversity of banks' loan portfolios). Specific definitions of variables and other data details are given in Simar and Wilson (2007).

Our empirical examples here and in Simar and Wilson (2007) are motivated by Aly et al. (1990), who similarly estimate efficiency for a sample of 322 U.S. banks operating during the fourth quarter of 1986, and then attempt to explain variation in the first-stage efficiency estimates in a second-stage regression by regressing estimated efficiency on continuous variables reflecting bank size and loan-type diversity, as well as binary dummy variables reflecting membership in a multi-bank holding company and presence in a metropolitan statistical area. Whereas Aly et al. used the second-stage regression in an attempt to better understand the performance of U.S. banks' operations, Simar and Wilson carefully note that their second-stage regressions are only for purposes of illustrating the bootstrap methods for inference developed in their paper. As discussed above, and as noted by Simar and Wilson, such second-stage regressions can only be meaningful if the separability condition in Assumption

2.1 holds. Simar and Wilson also noted that this condition should be tested before employing a second-stage regression, but until now no such test has been available.

It is well-known that the distribution of U.S. bank sizes is heavily skewed to the right; in fact, the distribution of total assets of U.S. banks is roughly log-log-normal (e.g., see Wheelock and Wilson, 2001 for discussion). In order to use global bandwidths, as opposed to adaptive bandwidths (which would increase computational burden), we first eliminate very large banks and other outliers from the sub-sample of 322 observations as described by Florens et al. (2014) (who used the same data in an empirical illustration), leaving 303 observations for analysis. Similarly, we omit the largest 5-percent of banks from the full sample of 6,955 observations, leaving 6,607 observations. To further reduce computational burden, we exploit multicollinearity among the input and output variables by aggregating inputs into a single measure and also aggregating outputs into a single measure using eigensystem techniques employed by Florens et al. (2014) in their analysis of the subsample of our data and as described by Daraio and Simar (2007a, pp. 148–150). Due to the high degrees of correlation among the original input and output variables, little information is lost by this aggregation, while dimensionality is reduced from $(p + q) = 7$ to 2.

We test the separability condition (Assumption 2.1) using both the subsample of 303 observations and the "full" sample of 6,607 observations using DEA estimators in both input and output directions, with bandwidths optimized by least-squares cross-validation and then adjusted to obtain the optimal order as discussed above. We first test separability marginally by considering only *SIZE*, and then by considering only *DIVERSE* so that $r = 1$. We also perform joint tests ($r = 2$) considering both *SIZE* and *DIVERSE*.

Results for the tests for both samples are shown in Table 3. In all cases, we reject the null hypothesis (i.e., Assumption 2.1) in favor of the alternative hypothesis (i.e., Assumption 2.2) with p-values well less than 0.00005. In the individual tests where $r = 1$, we reject with *SIZE* more strongly than with *DIVERSE*. With the joint tests where $r = 2$, the values of the test statistics are between those where we test only with *SIZE* and only with *DIVERSE*, as one would expect.

The rejection of separability with respect to *SIZE* is hardly surprising given that larger banks necessarily can produce more output than smaller banks. Of course, *SIZE* is highly correlated with banks' inputs and outputs. Nonetheless, this variable is used by by Aly et al. (1990) in their second-stage regression, and one must assume separability in order to believe the second-stage estimation makes any sense at all. Moreover, Aly et al. are not the only ones to use such variables in second-stage regressions. The rejection with respect to *DIVERSE* is less obvious a priori, and suggests that conditional efficiency estimators should be used to analyze efficiency among banks.[14]

8 Conclusions

We have provided CLTs for conditional efficiency estimators, allowing researchers to estimate confidence intervals for mean conditional efficiency or to compare mean conditional efficiency across groups of producers analogous to the test of equivalent mean unconditional efficiency developed in Kneip et al. (2015b). We have also provided a test of the separability condition described by Simar and Wilson (2007) on which many papers that regress estimated efficiency scores on some environmental variables depend. The condition is a restrictive, but can now be tested empirically. In our empirical example in Section 7, patterned after the application by Aly et al. (1990), we easily reject separability. This suggests that results of the second-stage regression in Aly et al. (1990) are meaningless, or at best very difficult to interpret. Furthermore, it raises the question of whether separability would similarly be rejected in the hundreds or thousands of papers that have regressed estimated efficiencies on environmental variables in a second stage regression. It is perhaps too much to expect that all of these studies be re-examined, but now that an easily-implemented test of separability has been made available, researchers should employ the test before proceeding to a second-stage regression. Moreover, whenever the test rejects separability, the researcher should use conditional efficiency estimators instead of unconditional estimators in order to estimate distance to the relevant frontier (i.e., to the frontier of Ψ^z instead of the frontier of Ψ which has no particular economic meaning when separability does not hold). Whenever separability is rejected, the new CLT results will be useful tools for empirical researchers.

Of course, failure to reject the null hypothesis of separability does not by itself imply that separability holds. As is always the case, our test can do only one of two things: it can

[14] Note that the second-stage regression in Simar and Wilson (2007) was used only to illustrate how one might apply the bootstrap methods proposed there. But, results from the second-stage regression in Aly et al. (1990), and those from similar exercises in other papers that have regressed estimates of bank efficiency on total assets, are rendered dubious and likely meaningless by the results obtained here.

either reject, or fail to reject the null hypothesis. Failure to reject might be due to other factors, such as insufficient data, or too many dimensions. In the later case, we have shown in our empirical example how dimensionality can be reduced before testing separability.

It should be remembered, as noted in Section 3, that the conditional efficiency estimators provide consistent estimates regardless of whether separability holds, but the unconditional efficiency estimators provide meaningfully consistent estimates if and only if separability holds. Of course, if separability holds, the unconditional estimators converge faster than their conditional counterparts. But when testing separability, these points argue in favor of a *conservative* test. Whereas one might ordinarily test a null hypothesis at the 10, 5, or 1-percent level, here one might want to test at a 20, 30, 40, or even larger percentage level. The cost of a type-I error is slower convergence due to subsequent use of the conditional efficiency estimators, whereas the cost of a type-II error is loss of any statistical or economic meaning due to subsequent inappropriate use of unconditional efficiency estimators. The cost of a type-II error here is arguably greater than the cost of a type-I error, which is the reverse of the usual situation in hypothesis testing. Here, however, reversing things by testing a null hypothesis of non-separability versus an alternative hypothesis of separability would result in a test with poor size and power properties, as separability is a much more restrictive condition than non-separability.

Appendices

A Technical Details

The assumptions listed here impose regularity conditions on the data-generating process. The first assumption appears as Assumption 4 in Jeong et al. (2010).

Assumption A.1. *The joint density $f_{XYZ}(\cdot,\cdot,\cdot)$ of (X,Y,Z) is continuous on its support.*

The next assumptions are needed to establish results for the moments of the conditional FDH and DEA estimators in Section 4.1. The assumptions here are conditional analogs of Assumptions 3.1–3.4 and 3.6 (respectively) in Kneip et al. (2015a). Assumption A.2, part (iii) and Assumption A.3, part(iii) appear as Assumption 5 in Jeong et al. (2010).

Assumption A.2. *For all $z \in \mathcal{Z}$, (i) the conditional density $f_{XY|Z}(\cdot, \cdot \mid z)$ of $(X, Y) \mid Z = z$ exists and has support $\mathcal{D}^z \subset \Psi^z$; (ii) $f_{XY|Z}(\cdot, \cdot \mid z)$ is continuously differentiable on $\mathcal{D}^z$; and (iii) $f^h_{XY|Z}(\cdot, \cdot \mid z)$ converges to $f_{XY|Z}(\cdot, \cdot \mid z)$ as $h \to 0$.*

Assumption A.3. *(i) $\mathcal{D}^{z*} := \{(x, \lambda(x, y \mid z)y) \mid (x, y) \in \mathcal{D}^z\} \subset \mathcal{D}^z$; (ii) $\mathcal{D}^{z*}$ is compact; and (iii) $f_{XY|Z}(x, \lambda(x, y \mid z)y \mid z) > 0$ for all $(x, y) \in \mathcal{D}^z$.*

Assumption A.4. *For any $z \in \mathcal{Z}$, $\mathcal{D}^z$ is almost strictly convex; i.e., for any (x, y), $(\widetilde{x}, \widetilde{y}) \in \mathcal{D}^z$ with $\left(\frac{x}{||x||}, y\right) \neq \left(\frac{\widetilde{x}}{||\widetilde{x}||}, \widetilde{y}\right)$, the set $\{(x^*, y^*) \mid (x^*, y^*) = (x, y) + \alpha((\widetilde{x}, \widetilde{y}))$ for some $\alpha \in (0, 1)\}$ is a subset of the interior of $\mathcal{D}^z$.*

Assumption A.5. *For all $z \in \mathcal{Z}$, (i) $\lambda(x, y \mid z)$ is twice continuously differentiable on $\mathcal{D}^z$; and (ii) all the first-order partial derivatives of $\lambda(x, y \mid z)$ with respect to x and y are nonzero at any point $(x, y) \in \mathcal{D}^z$.*

Assumption A.6. *For any $z \in \mathcal{Z}$, $\lambda(x, y \mid z)$ is three times continuously differentiable with respect to x and y on $\mathcal{D}^z$.*

When the conditional FDH estimator is used, Assumption A.5 is needed; when the conditional DEA estimator is used, this is replaced by the stronger Assumption A.6.

Note that under the separability condition in Assumption 2.1, the assumptions here reduce to the corresponding assumptions in Kneip et al. (2015a) due to the discussion in Section 2.

B Proofs of Lemmas and Theorems

Proof of Lemma 4.1. Either assumption A.5 or A.6 is sufficient to establish Lipschitz continuity of $\lambda(x, y \mid z)$ as a function of z. The result follows immediately. ■

Proof of Theorem 4.1. Under (i) Assumptions 4.1, 4.2, 4.4, A.1, A.2 and two-times differentiability (due to Assumption A.5) of $\lambda(x, y \mid z)$ with respect to x and y for the FDH case, or under (ii) Assumptions 4.1, 4.2, 4.3, 4.4, A.1, A.2 and three-times differentiability (due to Assumption A.6) of $\lambda(x, y \mid z)$ with respect to x and y for the DEA case, Jeong et al. (2010) prove, using the result in Lemma 4.1 and $h = O\left((nh^r)^{-\kappa}\right)$, that the asymptotic behavior of $(nh^r)^\kappa \left(\widehat{\lambda}(x, y \mid z, \mathcal{S}_n) - \lambda(x, y \mid z)\right)$ is the same as the asymptotic behavior of

$(nh^r)^\kappa \left(\widehat{\lambda}(x, y \mid z, \mathcal{S}_n) - \lambda^h(x, y \mid z)\right)$, which leads to the result in (3.10). For any given h, we are in a localized version of the framework of Kneip et al. (2015a) for unconditional efficiencies, except that here $\lambda^h(X_i, Y_i \mid Z_i)$ is the object of interest.

If Z is irrelevant, i.e. if Assumption 2.1 holds, then the optimal $h \to \infty$ and $n_h = n$. Otherwise Assumption 2.2 holds and $h \to 0$ as $n \to \infty$, and the order of the number of observations affecting the estimator is $n_h = nh^r$. Moreover, this is the order of the cardinality of $\mathcal{I}(z, h)$ for all z. Then for the FDH case, the results follow directly from the proof of Theorem 3.3 in Kneip et al. (2015a) after changing notation there to reflect the different number of observations. Similarly for the DEA case, the results follow directly from the proof of Theorem 3.1 in Kneip et al. (2015a). ■

Proof of Lemma 4.2. The result in (4.20) follows directly from Theorem 4.1. In addition,

$$\begin{aligned}
\mathrm{VAR}(\widehat{\lambda}(X_i, Y_i \mid Z_i, \mathcal{S}_n)) =& E\left(\left(\widehat{\lambda}(X_i, Y_i \mid Z_i, \mathcal{S}_n) - \lambda^h(X_i, Y_i \mid Z_i)\right)^2\right) \\
&+ E\left(\left(\lambda^h(X_i, Y_i \mid Z_i) - E\left(\widehat{\lambda}(X_i, Y_i \mid Z_i, \mathcal{S}_n)\right)\right)^2\right) \\
&+ 2E\Big(\left(\lambda^h(X_i, Y_i \mid Z_i) - E\left(\widehat{\lambda}(X_i, Y_i \mid Z_i, \mathcal{S}_n)\right)\right) \\
&\quad \left(\widehat{\lambda}(X_i, Y_i \mid Z_i, \mathcal{S}_n) - \lambda^h(X_i, Y_i \mid Z_i)\right)\Big). \qquad (2.1)
\end{aligned}$$

Using the result in (4.9) from Theorem 4.1,

$$\begin{aligned}
E\left(\left[\lambda^h(X_i, Y_i \mid Z_i) - E\left(\widehat{\lambda}(X_i, Y_i \mid Z_i, \mathcal{S}_n)\right)\right]^2\right) &= \sigma_c^{2,h} + \left[E\left(\widehat{\lambda}(X_i, Y_i \mid Z_i, \mathcal{S}_n) - \lambda^h(X_i, Y_i \mid Z_i)\right)\right]^2 \\
&= \sigma_c^{2,h} + C_c^2 n_h^{-2\kappa} + o\left(n_h^{-2\kappa}\right). \qquad (2.2)
\end{aligned}$$

Applying the Cauchy-Schwartz inequality, the result in (4.21) in Theorem 4.1 and (2.2), the last term in (2.1) is bounded by $o\left(n_h^{\kappa/2}\right)$, establishing the result in (4.21). ■

Proof of Theorem 4.2. Let

$$\overline{\mu}_{c,n_h} = \frac{1}{n_h} \sum_{(X_i, Y_i, Z_i) \in \mathcal{S}^*_{n_h}} \lambda^h(X_i, Y_i \mid Z_i). \qquad (2.3)$$

By the Lindeberg-Feller CLT, $\sqrt{n_h}(\overline{\mu}_{c,n_h} - \mu_c^h) \xrightarrow{\mathcal{L}} N(0, \sigma_c^{2,h})$. Define $\zeta_{n_h} = \widehat{\mu}_{c,n_h} - \overline{\mu}_{c,n_h}$. Using Lemma 4.2, we have $E(\zeta_{n_h}) = C_c n_h^{-\kappa} + R_{c,n_h,\kappa}$, $\mathrm{VAR}(\zeta_{n_h}) = o\left(n_h^{-1}\right)$ and $\zeta_{n_h} - E(\zeta_{n_h}) = o_p\left(n_h^{-1/2}\right)$.

It can be shown that $\widetilde{\mu}_{c,n_h} = \mu_c^h + E(\zeta_{n_h})$, and part (i) of the results is obtained by substitution for $E(\zeta_{n_h})$. Next, note that $\widehat{\mu}_{c,n_h} - \widetilde{\mu}_{c,n_h} = \left(\zeta_{n_h} + \overline{\mu}_{c,n_h}\right) - \left(\mu_c^h - E(\zeta_{n_h})\right) = \overline{\mu}_{c,n_h} - \left(\mu_c^h + (\zeta_{n_h} + E(\zeta_{n_h})\right)$. The last term in parentheses is $o_p\left(n_h^{-1/2}\right)$, establishing the result in (ii). Part (iii) follows directly from part (ii). Finally,

$$\begin{aligned}
\widehat{\sigma}_{c,n}^{2,h} &= n^{-1}\sum_{i=1}^{n}(\widehat{\lambda}(X_i, Y_i \mid Z_i, \mathcal{S}_n))^2 - \widehat{\mu}_{c,n_h}^2 \\
&\xrightarrow{p} E[(\widehat{\lambda}(X_i, Y_i \mid Z_i, \mathcal{S}_n))^2] - (\mu_c^h)^2 \\
&= \mathrm{VAR}(\widehat{\lambda}(X_i, Y_i \mid Z_i, \mathcal{S}_n)) + \left[E\left(\widehat{\lambda}(X_i, Y_i \mid Z_i, \mathcal{S}_n)\right)\right]^2 - (\mu_c^h)^2.
\end{aligned}$$

The result obtains after applying the results of Lemma 4.2. ∎

Proof of Theorem 4.3. The result follows by substituting (4.29) in Theorem 4.2, part (iii), and noting that for the indicated ranges of values for κ, $\sqrt{n_h}R_{c,n_h,\kappa} = o(1)$. ∎

Proof of Theorem 4.4. Let

$$\overline{\mu}_{c,n_{h,\kappa}} = \frac{1}{n_{h,\kappa}} \sum_{(X_i,Y_i,Z_i)\in\mathcal{S}^{**}_{n_{h,\kappa}}} \lambda^h(X_i, Y_i \mid Z_i). \tag{2.4}$$

Clearly,

$$\widehat{\mu}_{c,n_{h,\kappa}} - \mu_c^h = \overline{\mu}_{c,n_{h,\kappa}} - \mu_c^h + \frac{1}{n_{h,\kappa}} \sum_{(X_i,Y_i,Z_i)\in\mathcal{S}^{**}_{n_{h,\kappa}}} \left(\widehat{\lambda}(X_i, Y_i \mid Z_i, \mathcal{S}_n) - \lambda^h(X_i, Y_i \mid Z_i)\right). \tag{2.5}$$

Since $n_{h,\kappa} \to \infty$ as $n \to \infty$, $\sqrt{n_{h,\kappa}}\left(\overline{\mu}_{c,n_{h,\kappa}} - \mu_c^h\right) \xrightarrow{\mathcal{L}} N(0, \sigma_c^{2,h})$. By Lemma 4.2, the third term on the right-hand side of (2.5) has expectation $\mu_c^h + C_c n_h^{-\kappa} + R_{c,n_h,\kappa}$ and variance $\sigma_c^{2,h} + o(n_h^{-\kappa/2})$. Replacing $C_c n_h^{-\kappa}$ with $\widehat{B}^c_{\kappa,n_h}$ and then multiplying both sides by $\sqrt{n_{h,\kappa}}$ yields the result. ∎

C Discrete Environmental Variables

In applied work, it is often the case that researchers include binary or categorical variables in second-stage regressions of estimated efficiency on environmental variables. All of the results obtained in the main part of this paper assume Z is continuous. However, in order for second-stage regressions to estimate any useful, meaningful feature, the separability condition in Assumption 2.1 must also hold with respect to discrete environmental variables.

Testing the separability condition in the case of discrete variables can be done using results and ideas from Kneip et al. (2015b), where a test of equivalent mean efficiency across two groups of producers is developed. To illustrate, suppose $r = 1$ and Z is a binary dummy variable. To test separability, first shuffle the observations, and then divide into two groups of size $n_1 = \lfloor n/2 \rfloor$ and $n_2 = n - n_1$. Apply the unconditional efficiency estimator to group 1. For group 2, a conditional efficiency estimator is needed, but since Z is discrete, there is no smoothing to be done.[15] Since Z is binary, there are only two sets Ψ^z. Hence, in the second group, divide observations into two sub-groups according to whether $Z = 0$ or $Z = 1$; observations in each sub-group, estimate efficiency using the same *unconditional* efficiency estimator used with group 1, ignoring observations in the other group. This will yield a set of n_2 *conditional* efficiency estimates since the n_2 observations have been divided into sub-groups.

Note that the conditional estimates from group 2 have the usual convergence rate of the unconditional efficiency estimator since no bandwidth is involved since Z is discrete. One can now apply the difference-in-means test as described in Kneip et al. (2015b), taking care to compute the bias-correction terms for group 2 separately and independently for observations in the subgroup (of group 2) where $Z = 0$ and the subgroup where $Z = 1$. This will necessitate splitting each sub-group (of group 2) to compute the generalized jackknife estimates of bias for observations in each sub-group. See Kneip et al. (2015b) for details.

[15] The problem here is rather different from the problem of nonparametric estimation of regressions or densities, where one can smooth across discrete categories of data using the methods discussed by Li and Racine (2007). Here, we are interested in boundaries of support, as opposed to densities or conditional mean functions.

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Table 1: Rejection Rates for Separability Test using DEA ($r = 1$)

		$T_{1,n}$									$T_{2,n}$					
		$p=1,\ q=1$			$p=2,\ q=1$			$p=2,\ q=2$			$p=3,\ q=2$			$p=3,\ q=3$		
n	δ	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
100	0.00	0.165	0.107	0.034	0.182	0.117	0.050	0.221	0.161	0.074	0.174	0.113	0.043	0.188	0.119	0.047
	0.10	0.196	0.114	0.031	0.193	0.115	0.045	0.244	0.184	0.092	0.194	0.123	0.054	0.177	0.101	0.047
	0.20	0.282	0.202	0.091	0.232	0.157	0.065	0.247	0.186	0.113	0.195	0.127	0.060	0.190	0.127	0.057
	0.30	0.377	0.292	0.148	0.278	0.204	0.112	0.288	0.219	0.127	0.189	0.120	0.047	0.195	0.124	0.054
	0.40	0.486	0.377	0.225	0.288	0.218	0.117	0.308	0.248	0.150	0.200	0.134	0.061	0.203	0.137	0.062
	0.50	0.597	0.501	0.325	0.353	0.269	0.152	0.320	0.259	0.159	0.176	0.115	0.056	0.173	0.117	0.052
	0.60	0.656	0.558	0.395	0.392	0.306	0.184	0.330	0.256	0.173	0.198	0.143	0.073	0.169	0.114	0.045
	0.70	0.719	0.628	0.457	0.450	0.371	0.225	0.401	0.342	0.219	0.197	0.131	0.053	0.183	0.126	0.052
	0.80	0.757	0.670	0.499	0.458	0.380	0.261	0.398	0.339	0.222	0.195	0.131	0.048	0.182	0.117	0.057
	0.90	0.827	0.763	0.601	0.518	0.431	0.285	0.432	0.358	0.246	0.186	0.124	0.054	0.205	0.147	0.079
	1.00	0.857	0.800	0.657	0.515	0.444	0.316	0.463	0.398	0.276	0.228	0.157	0.069	0.219	0.148	0.074
	1.50	0.934	0.890	0.773	0.660	0.589	0.436	0.570	0.501	0.370	0.203	0.139	0.070	0.205	0.129	0.056
	2.00	0.939	0.906	0.825	0.744	0.670	0.520	0.595	0.550	0.410	0.233	0.167	0.081	0.182	0.118	0.057
200	0.00	0.163	0.093	0.025	0.153	0.092	0.030	0.191	0.135	0.060	0.149	0.091	0.028	0.130	0.073	0.025
	0.10	0.186	0.111	0.037	0.181	0.128	0.055	0.207	0.147	0.057	0.136	0.083	0.028	0.138	0.071	0.024
	0.20	0.385	0.275	0.115	0.278	0.189	0.086	0.253	0.190	0.091	0.156	0.095	0.026	0.119	0.076	0.026
	0.30	0.567	0.458	0.264	0.391	0.300	0.161	0.320	0.250	0.138	0.157	0.088	0.030	0.154	0.091	0.032
	0.40	0.726	0.636	0.451	0.528	0.435	0.265	0.412	0.333	0.201	0.172	0.114	0.045	0.168	0.097	0.028
	0.50	0.822	0.744	0.568	0.650	0.567	0.400	0.542	0.461	0.311	0.188	0.116	0.047	0.187	0.106	0.032
	0.60	0.896	0.843	0.709	0.715	0.617	0.464	0.597	0.513	0.378	0.206	0.137	0.055	0.164	0.102	0.030
	0.70	0.939	0.897	0.780	0.789	0.730	0.570	0.660	0.589	0.446	0.208	0.144	0.055	0.170	0.105	0.028
	0.80	0.959	0.931	0.851	0.836	0.767	0.605	0.716	0.654	0.518	0.215	0.155	0.061	0.199	0.124	0.038
	0.90	0.972	0.948	0.883	0.844	0.790	0.670	0.793	0.736	0.613	0.246	0.166	0.076	0.193	0.127	0.038
	1.00	0.983	0.968	0.909	0.903	0.842	0.736	0.828	0.778	0.660	0.268	0.192	0.082	0.208	0.132	0.052
	1.50	0.997	0.993	0.978	0.965	0.942	0.877	0.918	0.882	0.786	0.351	0.268	0.130	0.250	0.187	0.067
	2.00	0.999	0.999	0.993	0.981	0.967	0.924	0.956	0.928	0.864	0.384	0.275	0.158	0.296	0.220	0.105

Table 1: Rejection Rates for Separability Test using DEA ($r = 1$, continued)

		$T_{1,n}$									$T_{2,n}$					
n	δ	$p=1,\ q=1$			$p=2,\ q=1$			$p=2,\ q=2$			$p=3,\ q=2$			$p=3,\ q=3$		
		.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
1000	0.00	0.119	0.055	0.013	0.154	0.086	0.023	0.141	0.087	0.024	0.115	0.059	0.010	0.114	0.053	0.009
	0.10	0.420	0.290	0.122	0.320	0.216	0.083	0.289	0.198	0.071	0.130	0.063	0.013	0.117	0.055	0.012
	0.20	0.834	0.751	0.520	0.786	0.687	0.465	0.715	0.633	0.427	0.247	0.151	0.045	0.136	0.080	0.014
	0.30	0.978	0.949	0.849	0.962	0.931	0.836	0.922	0.887	0.752	0.399	0.298	0.118	0.250	0.158	0.042
	0.40	0.996	0.993	0.974	0.990	0.985	0.962	0.985	0.975	0.942	0.562	0.428	0.233	0.363	0.237	0.099
	0.50	1.000	1.000	0.999	1.000	0.999	0.993	0.994	0.993	0.984	0.671	0.560	0.329	0.486	0.334	0.136
	0.60	1.000	1.000	0.998	1.000	1.000	1.000	1.000	1.000	0.998	0.791	0.679	0.447	0.540	0.433	0.210
	0.70	1.000	1.000	1.000	1.000	1.000	0.999	1.000	1.000	0.998	0.841	0.753	0.555	0.624	0.485	0.279
	0.80	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.887	0.817	0.636	0.666	0.539	0.320
	0.90	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.917	0.866	0.697	0.726	0.628	0.402
	1.00	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.944	0.911	0.778	0.781	0.672	0.428
	1.50	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.990	0.976	0.912	0.902	0.815	0.644
	2.00	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.989	0.983	0.950	0.923	0.863	0.680

Table 2: Rejection Rates for Separability Test using DEA ($r = 2$)

		$T_{1,n}$									$T_{2,n}$					
n	δ	$p=1,\ q=1$			$p=2,\ q=1$			$p=2,\ q=2$			$p=3,\ q=2$			$p=3,\ q=3$		
		.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
100	0.00	0.161	0.090	0.020	0.152	0.104	0.036	0.163	0.101	0.040	0.157	0.092	0.027	0.187	0.117	0.048
	0.10	0.184	0.115	0.045	0.199	0.133	0.052	0.188	0.112	0.038	0.175	0.100	0.034	0.152	0.106	0.042
	0.20	0.228	0.151	0.059	0.202	0.134	0.052	0.216	0.155	0.060	0.167	0.108	0.041	0.184	0.110	0.044
	0.30	0.280	0.205	0.111	0.238	0.160	0.071	0.229	0.158	0.066	0.182	0.127	0.038	0.181	0.125	0.054
	0.40	0.333	0.242	0.126	0.240	0.154	0.062	0.238	0.175	0.088	0.173	0.119	0.043	0.185	0.119	0.049
	0.50	0.384	0.285	0.138	0.235	0.177	0.083	0.235	0.169	0.090	0.181	0.111	0.048	0.191	0.123	0.046
	0.60	0.419	0.328	0.177	0.263	0.185	0.094	0.266	0.201	0.102	0.188	0.118	0.034	0.186	0.123	0.044
	0.70	0.423	0.337	0.183	0.286	0.221	0.116	0.266	0.199	0.102	0.170	0.101	0.037	0.183	0.113	0.045
	0.80	0.491	0.406	0.244	0.299	0.215	0.107	0.270	0.195	0.101	0.193	0.137	0.055	0.190	0.123	0.043
	0.90	0.494	0.392	0.256	0.287	0.218	0.098	0.271	0.213	0.126	0.174	0.114	0.040	0.183	0.120	0.047
	1.00	0.535	0.420	0.244	0.328	0.242	0.121	0.289	0.221	0.118	0.214	0.136	0.053	0.170	0.121	0.048
	1.50	0.588	0.483	0.295	0.316	0.232	0.127	0.308	0.219	0.135	0.191	0.117	0.042	0.186	0.122	0.063
	2.00	0.633	0.522	0.359	0.313	0.237	0.117	0.312	0.251	0.153	0.186	0.114	0.047	0.161	0.093	0.036
200	0.00	0.120	0.058	0.009	0.142	0.085	0.019	0.150	0.093	0.029	0.146	0.074	0.020	0.113	0.062	0.013
	0.10	0.190	0.102	0.024	0.163	0.090	0.020	0.205	0.129	0.031	0.119	0.069	0.017	0.143	0.094	0.026
	0.20	0.283	0.199	0.075	0.203	0.135	0.055	0.219	0.159	0.062	0.149	0.082	0.027	0.148	0.085	0.022
	0.30	0.407	0.308	0.153	0.274	0.183	0.075	0.243	0.169	0.077	0.128	0.076	0.021	0.136	0.083	0.022
	0.40	0.501	0.378	0.210	0.314	0.222	0.094	0.289	0.217	0.103	0.157	0.082	0.021	0.145	0.083	0.028
	0.50	0.590	0.486	0.293	0.391	0.284	0.141	0.361	0.258	0.138	0.155	0.095	0.031	0.143	0.085	0.023
	0.60	0.633	0.536	0.357	0.420	0.329	0.166	0.355	0.272	0.144	0.188	0.119	0.033	0.155	0.095	0.021
	0.70	0.722	0.623	0.408	0.431	0.336	0.164	0.382	0.276	0.155	0.187	0.108	0.031	0.158	0.089	0.026
	0.80	0.726	0.648	0.467	0.515	0.399	0.256	0.413	0.317	0.188	0.195	0.114	0.032	0.148	0.087	0.024
	0.90	0.766	0.675	0.478	0.511	0.412	0.244	0.424	0.346	0.204	0.187	0.118	0.039	0.137	0.076	0.027
	1.00	0.795	0.714	0.532	0.518	0.411	0.271	0.453	0.365	0.235	0.188	0.103	0.039	0.139	0.082	0.028
	1.50	0.875	0.818	0.660	0.598	0.491	0.322	0.462	0.379	0.246	0.206	0.129	0.051	0.163	0.090	0.036
	2.00	0.875	0.816	0.684	0.639	0.544	0.369	0.520	0.443	0.291	0.186	0.125	0.037	0.157	0.101	0.033

Table 2: Rejection Rates for Separability Test using DEA ($r = 2$, continued)

		$T_{1,n}$									$T_{2,n}$					
n	δ	$p=1,\ q=1$			$p=2,\ q=1$			$p=2,\ q=2$			$p=3,\ q=2$			$p=3,\ q=3$		
		.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
1000	0.00	0.099	0.041	0.005	0.137	0.071	0.015	0.117	0.058	0.015	0.113	0.049	0.007	0.090	0.039	0.002
	0.10	0.327	0.231	0.074	0.237	0.144	0.046	0.205	0.141	0.042	0.138	0.061	0.008	0.112	0.058	0.006
	0.20	0.639	0.505	0.295	0.606	0.465	0.238	0.462	0.357	0.181	0.165	0.088	0.012	0.134	0.068	0.011
	0.30	0.879	0.797	0.614	0.785	0.700	0.453	0.742	0.637	0.421	0.253	0.155	0.043	0.164	0.088	0.020
	0.40	0.951	0.917	0.790	0.902	0.857	0.683	0.885	0.805	0.649	0.334	0.230	0.074	0.211	0.128	0.031
	0.50	0.981	0.959	0.893	0.962	0.942	0.844	0.928	0.894	0.770	0.431	0.314	0.140	0.312	0.195	0.052
	0.60	0.987	0.976	0.937	0.985	0.970	0.904	0.958	0.934	0.847	0.487	0.376	0.166	0.335	0.219	0.093
	0.70	0.993	0.989	0.968	0.991	0.987	0.944	0.978	0.963	0.907	0.524	0.398	0.212	0.336	0.252	0.098
	0.80	0.997	0.997	0.978	0.994	0.988	0.953	0.990	0.983	0.939	0.568	0.448	0.226	0.395	0.266	0.112
	0.90	0.998	0.998	0.991	0.996	0.992	0.967	0.990	0.980	0.949	0.618	0.510	0.280	0.421	0.317	0.128
	1.00	0.998	0.996	0.992	0.999	0.996	0.982	0.996	0.986	0.966	0.617	0.504	0.293	0.429	0.312	0.120
	1.50	1.000	1.000	0.999	1.000	0.999	0.993	0.999	0.995	0.984	0.726	0.599	0.384	0.513	0.382	0.193
	2.00	1.000	0.999	0.999	1.000	0.999	0.995	1.000	0.999	0.995	0.739	0.640	0.420	0.551	0.412	0.210

Table 3: Tests of Separability on Banking Data

	Input		Output	
	$T_{1,n}$	p-value	$T_{1,n}$	p-value
$n = 303$				
SIZE	13.9836	0.0000	8.3160	0.0000
DIVERSE	6.6719	0.0000	6.9408	0.0000
joint test	10.1990	0.0000	7.6620	0.0000
$n = 6,607$				
SIZE	41.5341	0.0000	36.8514	0.0000
DIVERSE	14.4167	0.0000	16.2464	0.0000
joint test	24.4306	0.0000	32.8130	0.0000

Nonparametric Estimation of Efficiency in the Presence of Environmental Variables: Appendices D–E

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D Randomization Algorithm

The following R code implements the algorithm described in Section 5.4 for randomizing data. The user passes to the function an n by $(p+q+r)$ data matrix where the columns and rows can be in any order, and an n by $(p+q+r)$ matrix with randomized observations or rows is returned. Note that the code makes use of the "digest" package (Eddelbuettel, 2016) to compute SHA-256 hashes. The package is not a part of the basic R distribution, but can be downloaded from the Comprehensive R Archive Network at https://cran.r-project.org.

```
randomize <- function (xyz) {
   if (is.data.frame(xyz)) xyz=as.matrix(xyz)
   if (length(colnames(xyz))>0) colnames(xyz)<-NULL
   if (length(rownames(xyz))>0) rownames(xyz)<=NULL
   n=nrow(xyz)
   k=ncol(xyz)
   #
   # first, standardize so units do not matter:
   std=apply(xyz,2,sd)
   for (j in 1:k) {
      xyz[,j]=xyz[,j]/std[j]
      }
   #
   # then order the columns:
   key1=vector(length=k)
   totals=apply(xyz,2,sum)
   require(digest)
   for (j in 1:k) {
      t1=enc2utf8(formatC(totals[j],format="E",width=18,digits=10))
      key1[j]=enc2utf8(digest(t1,algo="sha256",serialize=FALSE,ascii=TRUE))
      }
   jj=sort(key1,index.return=TRUE)$ix
   xyz=xyz[,jj]
   #
   # now order the rows:
   key2=vector(length=n)
   for (i in 1:n) {
      t1=enc2utf8(paste(formatC(xyz[i,],format="E",width=18,digits=10),
                        collapse=""))
      key2[i]=enc2utf8(digest(t1,algo="sha256",serialize=FALSE,ascii=TRUE))
      }
   ii=sort(key2,index.return=TRUE)$ix
   xyz=xyz[ii,]
   #
   # restore order of the columns:
   xyz[,jj]=xyz
   #
   # un-do the standardizations:
   for (j in 1:k) {
      xyz[,j]=xyz[,j]*std[j]
      }
   return(xyz)
   }
```

The following shows an *R* session where the above code is used to randomize data from Charnes et al. (1978). The data are loaded from the FEAR package by Wilson (2008). The data are first loaded, stored in a matrix `d`, and then printed. Next, the data are randomized, with the result stored in `d2` and printed. Then the original data in `d` are copied to a new matrix `dnew`, and 0.0001 is added to the element in row 36, column 6. The resulting matrix is then randomized with the result stored in d3, which is then printed. Note that the ordering of the rows in d3 is very different from the ordering in d2, illustrating that even a small change in the data causes a very different ordering of the observations. This is characteristic of well-designed hash algorithms, and is known as the "avalanche effect" (Feistel, 1973). Finally, the columns of the original matrix d are permuted randomly and stored in the matrix dnew. The randomization function is then applied to this matrix, with the result stored in d4. The matrices d2 and d4 are compared and found to be the same, illustrating that the randomization algorithm is not affected by the ordering of variables in the data.

```
R version 3.1.0 (2014-04-10) -- "Spring Dance"
Copyright (C) 2014 The R Foundation for Statistical Computing
Platform: x86_64-apple-darwin13.1.0 (64-bit)

R is free software and comes with ABSOLUTELY NO WARRANTY.
You are welcome to redistribute it under certain conditions.
Type 'license()' or 'licence()' for distribution details.

  Natural language support but running in an English locale

R is a collaborative project with many contributors.
Type 'contributors()' for more information and
'citation()' on how to cite R or R packages in publications.

Type 'demo()' for some demos, 'help()' for on-line help, or
'help.start()' for an HTML browser interface to help.
Type 'q()' to quit R.

> wdir=Sys.getenv("PWD")
> setwd(wdir)
> source('randomize.R')
> #
> #
> require(FEAR)
Loading required package: FEAR
FEAR (Frontier Efficiency Analysis with R) version 2.0.1 installed
Copyright Paul W. Wilson 2014
  Type "fear.license()" to view the software license for FEAR
  Type "fear.cite()" to view the proper citation for FEAR
Read 2 items
> data(ccr)
> ccr=as.matrix(ccr)[,2:9]
> colnames(ccr) <- NULL
> d=ccr
> #
> # d is the data matrix.
> print(d)
        [,1]  [,2]   [,3]   [,4] [,5]   [,6]   [,7]  [,8]
```

[1,]	86.13	16.24	48.21	49.69	9	54.53	58.98	38.16
[2,]	29.26	10.24	41.96	40.65	5	24.69	33.89	26.02
[3,]	43.12	11.31	38.19	35.03	9	36.41	40.62	28.51
[4,]	24.96	6.14	24.81	25.15	7	14.94	17.58	16.19
[5,]	11.62	2.21	6.85	6.37	4	7.81	6.94	5.37
[6,]	11.88	4.97	18.73	18.04	4	12.59	16.85	12.84
[7,]	32.64	6.88	28.10	25.45	7	17.06	16.99	17.82
[8,]	20.79	12.97	54.85	52.07	8	20.29	30.64	33.16
[9,]	34.40	11.04	38.16	42.40	8	26.13	29.80	26.29
[10,]	61.74	14.50	49.09	42.92	9	46.42	51.59	35.20
[11,]	52.92	11.67	39.48	39.64	5	39.80	37.73	30.29
[12,]	36.00	10.15	37.80	39.52	5	37.84	47.85	25.35
[13,]	39.20	10.80	41.04	41.12	7	26.48	31.36	26.54
[14,]	14.60	2.88	9.64	11.14	3	10.31	10.86	7.47
[15,]	4.29	5.42	21.45	17.27	5	14.39	18.30	14.33
[16,]	27.25	14.17	56.46	55.26	9	32.94	36.03	38.19
[17,]	22.63	4.43	15.40	15.00	2	17.25	20.80	12.07
[18,]	28.00	7.61	28.73	27.04	9	27.55	38.19	20.44
[19,]	53.56	13.70	53.04	49.85	7	41.12	43.80	36.54
[20,]	25.42	9.05	29.69	31.74	4	29.43	42.63	23.34
[21,]	31.57	10.08	39.34	40.57	6	37.46	51.02	27.44
[22,]	16.34	5.84	20.89	22.10	4	19.40	25.18	16.52
[23,]	44.28	14.14	56.70	52.27	11	39.88	47.72	38.97
[24,]	19.74	6.43	24.20	25.66	3	25.72	30.81	16.54
[25,]	24.40	8.05	33.42	31.29	7	24.88	25.27	22.43
[26,]	41.40	11.70	44.01	46.35	7	31.62	40.78	31.16
[27,]	27.20	9.38	37.80	31.55	4	31.31	38.32	25.03
[28,]	23.92	7.12	25.58	29.01	3	21.00	21.30	18.30
[29,]	10.62	2.55	10.10	9.09	4	6.51	7.02	6.16
[30,]	12.48	6.14	23.13	22.46	6	11.64	15.26	15.68
[31,]	19.32	5.89	24.01	24.74	6	12.58	15.90	14.42
[32,]	6.30	1.93	7.11	7.68	4	4.59	6.16	4.99
[33,]	46.62	14.65	65.71	57.49	10	43.76	46.64	39.10
[34,]	38.95	12.82	47.02	48.92	9	32.38	38.55	31.05
[35,]	61.60	15.56	53.98	50.29	6	34.64	45.46	39.22
[36,]	31.08	6.26	22.18	21.96	4	11.52	15.14	13.91
[37,]	19.35	6.68	22.61	23.31	4	15.96	19.21	15.30
[38,]	11.20	3.08	9.90	10.06	2	9.91	12.30	7.22
[39,]	34.40	11.61	41.79	41.79	5	30.44	33.53	29.80
[40,]	35.55	6.48	21.69	21.69	6	22.63	25.24	17.15
[41,]	30.53	9.30	35.50	35.14	8	24.41	27.16	25.30
[42,]	25.44	7.10	26.81	26.23	3	23.11	22.67	17.56
[43,]	26.66	11.43	41.36	44.63	6	21.82	31.45	27.54
[44,]	39.79	22.49	84.77	76.12	11	63.92	79.67	63.11
[45,]	8.32	3.64	12.92	13.13	2	9.47	11.92	8.85
[46,]	59.78	13.52	48.80	49.69	15	33.94	39.18	34.61
[47,]	39.22	10.06	37.00	38.33	4	29.42	35.10	28.42
[48,]	3.24	3.18	13.12	12.71	5	7.70	11.02	9.02
[49,]	7.14	5.29	23.10	19.06	8	12.17	16.03	15.82
[50,]	68.16	12.28	33.58	34.64	15	39.07	42.71	27.67
[51,]	11.88	3.59	13.41	13.82	8	9.96	14.34	9.33

```
[52,]  55.30 11.53  36.73  35.78    6  45.37  51.38 31.61
[53,]  16.20  7.02  26.94  26.30    9  18.23  22.05 17.56
[54,]  82.45 15.52  45.00  44.23   13  59.63  64.41 35.89
[55,]  15.81  6.93  23.91  23.61    7  24.20  28.21 18.74
[56,]   4.65  5.50  20.91  23.39    5  13.53  17.09 15.61
[57,]  41.25  8.41  26.23  25.24   10  28.39  27.65 20.79
[58,]  10.44  5.22  17.10  18.93    3  21.67  26.22 13.66
[59,] 139.65 35.03 119.56 130.83   22 120.17 144.67 88.59
[60,]  16.28  4.81  18.20  18.98    5  15.15  18.04 13.58
[61,]  12.06  2.59   8.74   8.17    5   6.92   7.10  6.35
[62,]   4.20  2.64   9.89  11.25    2   9.35   9.85  7.70
[63,]  19.44  3.83  12.87  13.23    5  13.03  13.40 10.29
[64,]  28.38  8.91  30.95  33.33    8  18.63  24.48 23.13
[65,]  13.50  3.61  15.60  12.39    4  12.28  13.01  9.89
[66,]  23.32  7.10  24.96  28.56   22  16.81  19.72 18.70
[67,]  27.60  9.38  32.29  34.01   20  26.36  28.22 24.46
[68,]  11.70 10.53  37.67  43.60    8  22.85  26.21 28.14
[69,]   4.68  1.85   6.22   5.46    5   8.17   8.70  5.12
[70,]  10.44  4.82  17.13  18.21    9  13.69  14.19 12.99
> d2=randomize(d)
Loading required package: digest
Warning message:
package 'digest' was built under R version 3.1.3
> print(d2)
       [,1]  [,2]   [,3]   [,4] [,5]   [,6]   [,7]  [,8]
 [1,]  11.88  4.97  18.73  18.04    4  12.59  16.85 12.84
 [2,]  10.44  4.82  17.13  18.21    9  13.69  14.19 12.99
 [3,]  43.12 11.31  38.19  35.03    9  36.41  40.62 28.51
 [4,]  11.88  3.59  13.41  13.82    8   9.96  14.34  9.33
 [5,]  52.92 11.67  39.48  39.64    5  39.80  37.73 30.29
 [6,]  16.20  7.02  26.94  26.30    9  18.23  22.05 17.56
 [7,]  39.20 10.80  41.04  41.12    7  26.48  31.36 26.54
 [8,]  11.70 10.53  37.67  43.60    8  22.85  26.21 28.14
 [9,]   4.65  5.50  20.91  23.39    5  13.53  17.09 15.61
[10,]  39.79 22.49  84.77  76.12   11  63.92  79.67 63.11
[11,]  11.62  2.21   6.85   6.37    4   7.81   6.94  5.37
[12,]  31.57 10.08  39.34  40.57    6  37.46  51.02 27.44
[13,]  28.38  8.91  30.95  33.33    8  18.63  24.48 23.13
[14,]   4.68  1.85   6.22   5.46    5   8.17   8.70  5.12
[15,]  61.74 14.50  49.09  42.92    9  46.42  51.59 35.20
[16,]  35.55  6.48  21.69  21.69    6  22.63  25.24 17.15
[17,]  12.48  6.14  23.13  22.46    6  11.64  15.26 15.68
[18,]   7.14  5.29  23.10  19.06    8  12.17  16.03 15.82
[19,]  25.42  9.05  29.69  31.74    4  29.43  42.63 23.34
[20,]  41.40 11.70  44.01  46.35    7  31.62  40.78 31.16
[21,]  22.63  4.43  15.40  15.00    2  17.25  20.80 12.07
[22,]  34.40 11.04  38.16  42.40    8  26.13  29.80 26.29
[23,]  29.26 10.24  41.96  40.65    5  24.69  33.89 26.02
[24,]  44.28 14.14  56.70  52.27   11  39.88  47.72 38.97
[25,]  68.16 12.28  33.58  34.64   15  39.07  42.71 27.67
[26,]  26.66 11.43  41.36  44.63    6  21.82  31.45 27.54
```

```
[27,]  25.44  7.10  26.81  26.23    3  23.11  22.67 17.56
[28,]  61.60 15.56  53.98  50.29    6  34.64  45.46 39.22
[29,]  59.78 13.52  48.80  49.69   15  33.94  39.18 34.61
[30,]  16.28  4.81  18.20  18.98    5  15.15  18.04 13.58
[31,]  30.53  9.30  35.50  35.14    8  24.41  27.16 25.30
[32,]  24.40  8.05  33.42  31.29    7  24.88  25.27 22.43
[33,]  20.79 12.97  54.85  52.07    8  20.29  30.64 33.16
[34,]  34.40 11.61  41.79  41.79    5  30.44  33.53 29.80
[35,]  23.32  7.10  24.96  28.56   22  16.81  19.72 18.70
[36,]  19.74  6.43  24.20  25.66    3  25.72  30.81 16.54
[37,]  31.08  6.26  22.18  21.96    4  11.52  15.14 13.91
[38,]  55.30 11.53  36.73  35.78    6  45.37  51.38 31.61
[39,]  24.96  6.14  24.81  25.15    7  14.94  17.58 16.19
[40,]  38.95 12.82  47.02  48.92    9  32.38  38.55 31.05
[41,]  27.20  9.38  37.80  31.55    4  31.31  38.32 25.03
[42,]  11.20  3.08   9.90  10.06    2   9.91  12.30  7.22
[43,]  10.62  2.55  10.10   9.09    4   6.51   7.02  6.16
[44,]   4.20  2.64   9.89  11.25    2   9.35   9.85  7.70
[45,]  19.35  6.68  22.61  23.31    4  15.96  19.21 15.30
[46,]  41.25  8.41  26.23  25.24   10  28.39  27.65 20.79
[47,]  13.50  3.61  15.60  12.39    4  12.28  13.01  9.89
[48,]   6.30  1.93   7.11   7.68    4   4.59   6.16  4.99
[49,]  14.60  2.88   9.64  11.14    3  10.31  10.86  7.47
[50,]  27.60  9.38  32.29  34.01   20  26.36  28.22 24.46
[51,]  39.22 10.06  37.00  38.33    4  29.42  35.10 28.42
[52,]  15.81  6.93  23.91  23.61    7  24.20  28.21 18.74
[53,]  53.56 13.70  53.04  49.85    7  41.12  43.80 36.54
[54,]  86.13 16.24  48.21  49.69    9  54.53  58.98 38.16
[55,] 139.65 35.03 119.56 130.83   22 120.17 144.67 88.59
[56,]  19.44  3.83  12.87  13.23    5  13.03  13.40 10.29
[57,]   3.24  3.18  13.12  12.71    5   7.70  11.02  9.02
[58,]  36.00 10.15  37.80  39.52    5  37.84  47.85 25.35
[59,]  27.25 14.17  56.46  55.26    9  32.94  36.03 38.19
[60,]  23.92  7.12  25.58  29.01    3  21.00  21.30 18.30
[61,]  46.62 14.65  65.71  57.49   10  43.76  46.64 39.10
[62,]   4.29  5.42  21.45  17.27    5  14.39  18.30 14.33
[63,]  10.44  5.22  17.10  18.93    3  21.67  26.22 13.66
[64,]  12.06  2.59   8.74   8.17    5   6.92   7.10  6.35
[65,]  19.32  5.89  24.01  24.74    6  12.58  15.90 14.42
[66,]  32.64  6.88  28.10  25.45    7  17.06  16.99 17.82
[67,]  16.34  5.84  20.89  22.10    4  19.40  25.18 16.52
[68,]   8.32  3.64  12.92  13.13    2   9.47  11.92  8.85
[69,]  82.45 15.52  45.00  44.23   13  59.63  64.41 35.89
[70,]  28.00  7.61  28.73  27.04    9  27.55  38.19 20.44
> #
> # make a small change in data matrix and randomize again:
> dnew=d
> print(dnew[35,6])
[1] 34.64
> dnew[35,6]=dnew[35,6]+0.0001
> print(dnew[35,6])
```

```
[1] 34.6401
> d3=randomize(dnew)
> print(d3)
        [,1]  [,2]   [,3]   [,4] [,5]     [,6]   [,7]  [,8]
 [1,]  15.81  6.93  23.91  23.61    7  24.2000  28.21 18.74
 [2,]  30.53  9.30  35.50  35.14    8  24.4100  27.16 25.30
 [3,]  39.22 10.06  37.00  38.33    4  29.4200  35.10 28.42
 [4,]  86.13 16.24  48.21  49.69    9  54.5300  58.98 38.16
 [5,]  12.06  2.59   8.74   8.17    5   6.9200   7.10  6.35
 [6,]  22.63  4.43  15.40  15.00    2  17.2500  20.80 12.07
 [7,]   7.14  5.29  23.10  19.06    8  12.1700  16.03 15.82
 [8,]  19.35  6.68  22.61  23.31    4  15.9600  19.21 15.30
 [9,]  24.40  8.05  33.42  31.29    7  24.8800  25.27 22.43
[10,]  43.12 11.31  38.19  35.03    9  36.4100  40.62 28.51
[11,]  41.25  8.41  26.23  25.24   10  28.3900  27.65 20.79
[12,]  44.28 14.14  56.70  52.27   11  39.8800  47.72 38.97
[13,]  29.26 10.24  41.96  40.65    5  24.6900  33.89 26.02
[14,]  52.92 11.67  39.48  39.64    5  39.8000  37.73 30.29
[15,] 139.65 35.03 119.56 130.83   22 120.1700 144.67 88.59
[16,]  46.62 14.65  65.71  57.49   10  43.7600  46.64 39.10
[17,]  39.79 22.49  84.77  76.12   11  63.9200  79.67 63.11
[18,]  24.96  6.14  24.81  25.15    7  14.9400  17.58 16.19
[19,]  35.55  6.48  21.69  21.69    6  22.6300  25.24 17.15
[20,]  26.66 11.43  41.36  44.63    6  21.8200  31.45 27.54
[21,]  82.45 15.52  45.00  44.23   13  59.6300  64.41 35.89
[22,]  11.62  2.21   6.85   6.37    4   7.8100   6.94  5.37
[23,]   4.29  5.42  21.45  17.27    5  14.3900  18.30 14.33
[24,]  28.00  7.61  28.73  27.04    9  27.5500  38.19 20.44
[25,]  41.40 11.70  44.01  46.35    7  31.6200  40.78 31.16
[26,]  32.64  6.88  28.10  25.45    7  17.0600  16.99 17.82
[27,]  10.44  4.82  17.13  18.21    9  13.6900  14.19 12.99
[28,]  53.56 13.70  53.04  49.85    7  41.1200  43.80 36.54
[29,]  11.88  3.59  13.41  13.82    8   9.9600  14.34  9.33
[30,]  23.92  7.12  25.58  29.01    3  21.0000  21.30 18.30
[31,]  16.28  4.81  18.20  18.98    5  15.1500  18.04 13.58
[32,]   4.20  2.64   9.89  11.25    2   9.3500   9.85  7.70
[33,]  61.60 15.56  53.98  50.29    6  34.6401  45.46 39.22
[34,]   4.65  5.50  20.91  23.39    5  13.5300  17.09 15.61
[35,]   8.32  3.64  12.92  13.13    2   9.4700  11.92  8.85
[36,]  34.40 11.04  38.16  42.40    8  26.1300  29.80 26.29
[37,]  20.79 12.97  54.85  52.07    8  20.2900  30.64 33.16
[38,]  39.20 10.80  41.04  41.12    7  26.4800  31.36 26.54
[39,]  27.20  9.38  37.80  31.55    4  31.3100  38.32 25.03
[40,]   4.68  1.85   6.22   5.46    5   8.1700   8.70  5.12
[41,]  55.30 11.53  36.73  35.78    6  45.3700  51.38 31.61
[42,]  19.32  5.89  24.01  24.74    6  12.5800  15.90 14.42
[43,]  34.40 11.61  41.79  41.79    5  30.4400  33.53 29.80
[44,]  25.42  9.05  29.69  31.74    4  29.4300  42.63 23.34
[45,]  19.44  3.83  12.87  13.23    5  13.0300  13.40 10.29
[46,]  10.44  5.22  17.10  18.93    3  21.6700  26.22 13.66
[47,]  16.20  7.02  26.94  26.30    9  18.2300  22.05 17.56
```

```
[48,]  25.44  7.10  26.81  26.23    3  23.1100  22.67 17.56
[49,]  19.74  6.43  24.20  25.66    3  25.7200  30.81 16.54
[50,]  68.16 12.28  33.58  34.64   15  39.0700  42.71 27.67
[51,]   3.24  3.18  13.12  12.71    5   7.7000  11.02  9.02
[52,]  61.74 14.50  49.09  42.92    9  46.4200  51.59 35.20
[53,]  31.57 10.08  39.34  40.57    6  37.4600  51.02 27.44
[54,]  16.34  5.84  20.89  22.10    4  19.4000  25.18 16.52
[55,]  27.25 14.17  56.46  55.26    9  32.9400  36.03 38.19
[56,]  13.50  3.61  15.60  12.39    4  12.2800  13.01  9.89
[57,]  14.60  2.88   9.64  11.14    3  10.3100  10.86  7.47
[58,]  59.78 13.52  48.80  49.69   15  33.9400  39.18 34.61
[59,]  36.00 10.15  37.80  39.52    5  37.8400  47.85 25.35
[60,]  10.62  2.55  10.10   9.09    4   6.5100   7.02  6.16
[61,]   6.30  1.93   7.11   7.68    4   4.5900   6.16  4.99
[62,]  11.88  4.97  18.73  18.04    4  12.5900  16.85 12.84
[63,]  31.08  6.26  22.18  21.96    4  11.5200  15.14 13.91
[64,]  12.48  6.14  23.13  22.46    6  11.6400  15.26 15.68
[65,]  38.95 12.82  47.02  48.92    9  32.3800  38.55 31.05
[66,]  11.70 10.53  37.67  43.60    8  22.8500  26.21 28.14
[67,]  28.38  8.91  30.95  33.33    8  18.6300  24.48 23.13
[68,]  27.60  9.38  32.29  34.01   20  26.3600  28.22 24.46
[69,]  23.32  7.10  24.96  28.56   22  16.8100  19.72 18.70
[70,]  11.20  3.08   9.90  10.06    2   9.9100  12.30  7.22
> #
> # permute the rows and columns of d to see that we get the same
> # randomize matrix:
> set.seed(90001)
> dnew=d[sample.int(70),]
> d4=randomize(d)
> any(!(d2!=d4))
[1] TRUE
>
>
> proc.time()
   user  system elapsed
 16.802   5.148  32.315
```

E Additional Results from Monte Carlo Experiments

Table E.1 gives results from the Monte Carlo experiments described in Section 6 using the DEA estimators with $r = 3$. Tables E.2–E.3 give results from the Monte Carlo experiments described in Section 6 using FDH estimators with $r = 1$ and 2, analogous to Tables 1–2 in the paper. Similarly, Table E.4 gives results obtained using FDH estimators with $r = 3$, analogous to the results in Table E.1.

In addition to the first set of experiments described in Section 6, we perform two additional sets of experiments to examine the performance of the separability tests. In the second set of experiments, we generate X, Y^{eff} and Z as described in Section 6. We then simulate observed output values Y by setting

$$Y = Y^{\text{eff}} \left[(1 + \delta(Z\alpha))^{1/2} |u| + 1 \right]^{-1} \tag{E.1}$$

where $u \sim N^{+}(0, 1)$ and δ and α are defined as in Section 6. Here, the environmental variables Z affect the distribution of inefficiency, but have no effect on the frontier since any effect of Z is wiped out whenever $u = 0$. Consequently, the separability condition in Assumption 2.1 holds. Results from these experiments, for $r \in \{1,\ 2,\ 3\}$ and using DEA estimators appear in Tables E.5–E.7. Corresponding results obtained with FDH estimators appear in Tables E.8–E.10.

In the third set of experiments, we again generate X, Y^{eff} and Z as described in Section 6. We then simulate observed output values Y by setting

$$Y = Y^{\text{eff}} \left[1 + \delta(Z\alpha) \right] \left[(1 + \delta(Z\alpha))^{1/2} |u| + 1 \right]^{-1} \tag{E.2}$$

where where $u \sim N^{+}(0, 1)$ and δ and α are defined as in Section 6. In (E.2), the environmental variables Z affect *both* the frontier as well as the distribution of inefficiency, violating the separability in Assumption 2.1 so that Assumption 2.2 holds instead. Results from these experiments, for $r \in \{1,\ 2,\ 3\}$ and using DEA estimators appear in Tables E.11–E.13. Corresponding results obtained with FDH estimators appear in Tables E.14–E.16.

Table E.1: Rejection Rates for Separability Test using DEA ($r = 3$)

		$T_{1,n}$									$T_{2,n}$					
n	δ	$p=1,\ q=1$			$p=2,\ q=1$			$p=2,\ q=2$			$p=3,\ q=2$			$p=3,\ q=3$		
		.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
100	0.00	0.130	0.061	0.012	0.142	0.067	0.016	0.163	0.095	0.026	0.151	0.088	0.032	0.153	0.080	0.030
	0.10	0.181	0.090	0.026	0.171	0.100	0.028	0.171	0.108	0.038	0.143	0.082	0.025	0.169	0.103	0.034
	0.20	0.206	0.110	0.032	0.168	0.104	0.025	0.175	0.115	0.052	0.166	0.105	0.030	0.152	0.086	0.038
	0.30	0.230	0.147	0.052	0.157	0.089	0.029	0.200	0.125	0.053	0.174	0.099	0.027	0.164	0.113	0.041
	0.40	0.261	0.164	0.066	0.200	0.109	0.029	0.186	0.124	0.060	0.172	0.097	0.026	0.169	0.115	0.048
	0.50	0.245	0.169	0.065	0.183	0.110	0.039	0.211	0.143	0.065	0.162	0.101	0.040	0.184	0.113	0.043
	0.60	0.274	0.180	0.075	0.213	0.138	0.054	0.216	0.151	0.068	0.151	0.102	0.031	0.199	0.132	0.052
	0.70	0.303	0.205	0.085	0.236	0.153	0.056	0.231	0.152	0.075	0.146	0.101	0.045	0.174	0.103	0.044
	0.80	0.320	0.232	0.096	0.215	0.140	0.050	0.234	0.175	0.069	0.153	0.094	0.026	0.177	0.108	0.034
	0.90	0.311	0.214	0.099	0.217	0.134	0.043	0.221	0.155	0.066	0.156	0.090	0.033	0.148	0.087	0.034
	1.00	0.334	0.230	0.114	0.223	0.141	0.057	0.249	0.165	0.072	0.163	0.100	0.030	0.153	0.102	0.043
	1.50	0.348	0.256	0.110	0.222	0.145	0.068	0.228	0.170	0.078	0.167	0.099	0.035	0.168	0.105	0.033
	2.00	0.361	0.255	0.127	0.245	0.159	0.066	0.253	0.182	0.080	0.154	0.091	0.038	0.184	0.116	0.047
200	0.00	0.128	0.058	0.008	0.143	0.068	0.010	0.142	0.080	0.026	0.110	0.056	0.010	0.116	0.056	0.009
	0.10	0.157	0.076	0.015	0.162	0.095	0.021	0.149	0.094	0.029	0.129	0.057	0.009	0.123	0.062	0.008
	0.20	0.215	0.119	0.027	0.186	0.112	0.036	0.183	0.106	0.040	0.134	0.079	0.014	0.135	0.065	0.011
	0.30	0.304	0.203	0.070	0.209	0.137	0.039	0.198	0.128	0.047	0.142	0.068	0.012	0.111	0.055	0.011
	0.40	0.324	0.214	0.089	0.226	0.145	0.048	0.241	0.155	0.062	0.143	0.074	0.014	0.129	0.065	0.017
	0.50	0.366	0.276	0.104	0.264	0.160	0.057	0.246	0.155	0.070	0.159	0.094	0.018	0.129	0.067	0.009
	0.60	0.407	0.300	0.134	0.245	0.163	0.064	0.259	0.183	0.080	0.143	0.075	0.022	0.137	0.075	0.021
	0.70	0.425	0.306	0.162	0.277	0.189	0.065	0.275	0.196	0.081	0.138	0.078	0.018	0.130	0.065	0.015
	0.80	0.436	0.335	0.162	0.303	0.212	0.079	0.266	0.180	0.099	0.129	0.072	0.019	0.164	0.087	0.018
	0.90	0.481	0.367	0.183	0.306	0.214	0.087	0.298	0.207	0.098	0.140	0.080	0.027	0.123	0.067	0.022
	1.00	0.512	0.409	0.213	0.330	0.220	0.103	0.288	0.206	0.097	0.141	0.081	0.020	0.131	0.066	0.017
	1.50	0.534	0.425	0.237	0.296	0.209	0.084	0.309	0.210	0.092	0.133	0.070	0.018	0.143	0.085	0.024
	2.00	0.586	0.466	0.256	0.333	0.237	0.112	0.338	0.244	0.106	0.161	0.091	0.021	0.119	0.076	0.018

Table E.1: Rejection Rates for Separability Test using DEA ($r = 3$, continued)

		$T_{1,n}$									$T_{2,n}$					
n	δ	$p=1,\ q=1$			$p=2,\ q=1$			$p=2,\ q=2$			$p=3,\ q=2$			$p=3,\ q=3$		
		.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
1000	0.00	0.109	0.047	0.003	0.110	0.049	0.003	0.102	0.043	0.003	0.091	0.033	0.003	0.098	0.037	0.004
	0.10	0.215	0.129	0.029	0.183	0.086	0.016	0.200	0.101	0.027	0.101	0.043	0.002	0.114	0.054	0.005
	0.20	0.503	0.350	0.147	0.368	0.251	0.085	0.330	0.219	0.076	0.122	0.054	0.006	0.124	0.049	0.003
	0.30	0.641	0.502	0.263	0.552	0.426	0.196	0.485	0.355	0.177	0.173	0.092	0.021	0.146	0.063	0.012
	0.40	0.722	0.621	0.383	0.664	0.553	0.322	0.586	0.454	0.246	0.232	0.125	0.030	0.148	0.063	0.012
	0.50	0.807	0.704	0.496	0.721	0.622	0.413	0.645	0.552	0.330	0.266	0.161	0.044	0.173	0.098	0.023
	0.60	0.839	0.776	0.574	0.780	0.686	0.459	0.712	0.598	0.394	0.297	0.190	0.055	0.215	0.121	0.022
	0.70	0.883	0.821	0.622	0.816	0.732	0.548	0.735	0.640	0.453	0.307	0.207	0.050	0.213	0.116	0.026
	0.80	0.886	0.817	0.649	0.839	0.762	0.566	0.808	0.717	0.497	0.300	0.180	0.049	0.211	0.123	0.024
	0.90	0.904	0.848	0.721	0.845	0.770	0.581	0.805	0.723	0.524	0.335	0.222	0.066	0.235	0.129	0.036
	1.00	0.933	0.881	0.736	0.863	0.778	0.600	0.812	0.735	0.523	0.333	0.210	0.075	0.228	0.132	0.037
	1.50	0.954	0.929	0.841	0.921	0.879	0.715	0.880	0.828	0.657	0.417	0.282	0.086	0.262	0.155	0.032
	2.00	0.966	0.950	0.878	0.931	0.885	0.757	0.885	0.827	0.694	0.411	0.306	0.110	0.266	0.167	0.055

Table E.2: Rejection Rates for Separability Test using FDH ($r = 1$)

n	δ	$T_{1,n}$						$T_{2,n}$								
		$p=1,\ q=1$			$p=2,\ q=1$			$p=2,\ q=2$			$p=3,\ q=2$			$p=3,\ q=3$		
		.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
100	0.00	0.193	0.132	0.057	0.257	0.214	0.124	0.200	0.132	0.058	0.151	0.101	0.059	0.169	0.122	0.072
	0.10	0.225	0.147	0.068	0.276	0.215	0.122	0.175	0.121	0.057	0.167	0.116	0.062	0.163	0.099	0.052
	0.20	0.300	0.229	0.120	0.275	0.217	0.133	0.181	0.120	0.050	0.162	0.114	0.058	0.162	0.108	0.051
	0.30	0.388	0.293	0.180	0.297	0.246	0.146	0.205	0.131	0.060	0.153	0.107	0.059	0.161	0.108	0.064
	0.40	0.452	0.383	0.243	0.305	0.241	0.161	0.196	0.134	0.055	0.173	0.117	0.055	0.173	0.124	0.067
	0.50	0.556	0.479	0.325	0.343	0.280	0.191	0.211	0.146	0.064	0.168	0.121	0.064	0.181	0.118	0.067
	0.60	0.616	0.528	0.384	0.355	0.290	0.189	0.196	0.136	0.069	0.176	0.110	0.063	0.183	0.127	0.068
	0.70	0.671	0.589	0.420	0.390	0.330	0.223	0.206	0.137	0.073	0.166	0.104	0.055	0.168	0.110	0.057
	0.80	0.711	0.633	0.486	0.386	0.325	0.227	0.200	0.146	0.066	0.184	0.132	0.077	0.159	0.118	0.062
	0.90	0.807	0.723	0.581	0.420	0.352	0.254	0.218	0.141	0.071	0.179	0.123	0.058	0.172	0.113	0.059
	1.00	0.813	0.751	0.603	0.444	0.374	0.274	0.249	0.177	0.078	0.187	0.118	0.069	0.190	0.132	0.070
	1.50	0.901	0.844	0.750	0.514	0.447	0.334	0.280	0.195	0.106	0.150	0.107	0.058	0.178	0.127	0.069
	2.00	0.918	0.883	0.790	0.593	0.528	0.402	0.271	0.181	0.080	0.181	0.130	0.074	0.179	0.122	0.067
200	0.00	0.196	0.132	0.038	0.237	0.175	0.091	0.128	0.085	0.022	0.151	0.106	0.049	0.116	0.085	0.036
	0.10	0.219	0.153	0.055	0.242	0.179	0.102	0.134	0.082	0.031	0.128	0.083	0.029	0.132	0.083	0.038
	0.20	0.378	0.274	0.145	0.260	0.211	0.125	0.162	0.103	0.042	0.128	0.082	0.028	0.118	0.080	0.032
	0.30	0.560	0.472	0.301	0.343	0.286	0.179	0.182	0.104	0.034	0.153	0.096	0.041	0.123	0.078	0.035
	0.40	0.695	0.601	0.424	0.432	0.353	0.235	0.184	0.117	0.048	0.143	0.095	0.036	0.135	0.088	0.047
	0.50	0.802	0.722	0.571	0.521	0.451	0.327	0.238	0.152	0.063	0.142	0.090	0.042	0.155	0.109	0.051
	0.60	0.871	0.826	0.679	0.579	0.507	0.367	0.224	0.155	0.056	0.143	0.089	0.031	0.148	0.105	0.057
	0.70	0.931	0.882	0.766	0.626	0.560	0.438	0.257	0.177	0.084	0.141	0.100	0.039	0.129	0.078	0.032
	0.80	0.950	0.923	0.828	0.679	0.610	0.477	0.297	0.214	0.099	0.150	0.090	0.041	0.155	0.092	0.036
	0.90	0.966	0.947	0.890	0.712	0.653	0.521	0.348	0.244	0.121	0.145	0.091	0.035	0.131	0.088	0.043
	1.00	0.972	0.952	0.896	0.751	0.683	0.565	0.365	0.258	0.133	0.147	0.090	0.037	0.141	0.090	0.041
	1.50	0.996	0.990	0.974	0.855	0.811	0.707	0.417	0.317	0.145	0.181	0.119	0.050	0.120	0.078	0.044
	2.00	0.997	0.996	0.987	0.907	0.872	0.789	0.499	0.368	0.188	0.178	0.111	0.038	0.154	0.098	0.051

Table E.2: Rejection Rates for Separability Test using FDH ($r = 1$, continued)

		$T_{1,n}$						$T_{2,n}$								
n	δ	$p=1,\ q=1$			$p=2,\ q=1$			$p=2,\ q=2$			$p=3,\ q=2$			$p=3,\ q=3$		
		.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
1000	0.00	0.155	0.081	0.020	0.172	0.113	0.042	0.108	0.051	0.013	0.102	0.045	0.007	0.097	0.051	0.017
	0.10	0.413	0.298	0.129	0.327	0.231	0.114	0.118	0.061	0.019	0.105	0.063	0.010	0.119	0.055	0.014
	0.20	0.860	0.751	0.572	0.658	0.571	0.397	0.182	0.103	0.023	0.097	0.047	0.010	0.106	0.056	0.015
	0.30	0.971	0.955	0.896	0.890	0.832	0.709	0.298	0.179	0.053	0.118	0.056	0.011	0.115	0.073	0.016
	0.40	0.998	0.992	0.979	0.972	0.959	0.899	0.412	0.263	0.103	0.174	0.095	0.026	0.122	0.073	0.016
	0.50	1.000	1.000	1.000	0.993	0.985	0.957	0.530	0.403	0.183	0.219	0.125	0.045	0.153	0.095	0.027
	0.60	1.000	1.000	1.000	1.000	1.000	0.993	0.631	0.488	0.276	0.273	0.158	0.063	0.172	0.106	0.041
	0.70	1.000	1.000	1.000	0.999	0.999	0.996	0.687	0.536	0.291	0.292	0.191	0.074	0.168	0.111	0.030
	0.80	1.000	1.000	1.000	0.999	0.998	0.998	0.719	0.590	0.348	0.346	0.225	0.086	0.205	0.119	0.035
	0.90	1.000	1.000	1.000	1.000	1.000	1.000	0.770	0.675	0.454	0.335	0.235	0.095	0.232	0.155	0.056
	1.00	1.000	1.000	1.000	0.999	0.999	0.999	0.814	0.713	0.485	0.387	0.269	0.134	0.257	0.155	0.056
	1.50	1.000	1.000	1.000	1.000	1.000	1.000	0.867	0.803	0.596	0.549	0.412	0.214	0.347	0.242	0.115
	2.00	1.000	1.000	1.000	1.000	1.000	1.000	0.907	0.837	0.658	0.600	0.469	0.223	0.384	0.269	0.113

Table E.3: Rejection Rates for Separability Test using FDH ($r = 2$)

		$T_{1,n}$						$T_{2,n}$								
		— $p = 1,\ q = 1$ —			— $p = 2,\ q = 1$ —			— $p = 2,\ q = 2$ —			— $p = 3,\ q = 2$ —			— $p = 3,\ q = 3$ —		
n	δ	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
100	0.00	0.166	0.102	0.033	0.239	0.181	0.105	0.144	0.091	0.033	0.186	0.138	0.078	0.166	0.104	0.058
	0.10	0.204	0.139	0.060	0.262	0.202	0.113	0.157	0.096	0.039	0.150	0.092	0.049	0.167	0.099	0.041
	0.20	0.262	0.191	0.079	0.244	0.184	0.109	0.168	0.105	0.045	0.172	0.111	0.068	0.147	0.095	0.050
	0.30	0.322	0.237	0.120	0.251	0.205	0.129	0.173	0.116	0.053	0.197	0.128	0.060	0.182	0.124	0.067
	0.40	0.359	0.264	0.151	0.291	0.219	0.132	0.197	0.126	0.057	0.177	0.125	0.066	0.165	0.114	0.059
	0.50	0.407	0.312	0.179	0.271	0.216	0.128	0.177	0.125	0.047	0.179	0.112	0.053	0.161	0.116	0.056
	0.60	0.432	0.345	0.203	0.279	0.231	0.128	0.189	0.126	0.050	0.164	0.108	0.063	0.164	0.107	0.044
	0.70	0.423	0.334	0.204	0.297	0.240	0.139	0.185	0.131	0.056	0.158	0.115	0.063	0.154	0.099	0.052
	0.80	0.511	0.415	0.262	0.315	0.258	0.159	0.171	0.117	0.053	0.143	0.106	0.055	0.180	0.111	0.053
	0.90	0.523	0.428	0.282	0.326	0.264	0.164	0.208	0.132	0.049	0.164	0.113	0.062	0.163	0.116	0.066
	1.00	0.533	0.434	0.248	0.317	0.250	0.162	0.207	0.144	0.058	0.166	0.122	0.055	0.154	0.104	0.060
	1.50	0.594	0.507	0.323	0.334	0.261	0.163	0.203	0.142	0.058	0.170	0.116	0.064	0.153	0.097	0.053
	2.00	0.620	0.535	0.371	0.303	0.241	0.147	0.194	0.126	0.063	0.150	0.112	0.056	0.166	0.112	0.055
200	0.00	0.130	0.065	0.018	0.195	0.138	0.064	0.136	0.081	0.028	0.111	0.063	0.024	0.137	0.104	0.049
	0.10	0.214	0.130	0.050	0.208	0.152	0.072	0.142	0.077	0.028	0.130	0.089	0.037	0.112	0.077	0.038
	0.20	0.310	0.217	0.100	0.271	0.200	0.101	0.144	0.090	0.026	0.148	0.095	0.037	0.149	0.095	0.050
	0.30	0.414	0.311	0.163	0.263	0.202	0.105	0.164	0.100	0.029	0.115	0.081	0.044	0.158	0.098	0.044
	0.40	0.524	0.404	0.239	0.324	0.243	0.130	0.178	0.105	0.022	0.142	0.083	0.030	0.116	0.075	0.036
	0.50	0.621	0.497	0.312	0.363	0.297	0.168	0.177	0.111	0.041	0.150	0.090	0.029	0.149	0.094	0.035
	0.60	0.627	0.539	0.383	0.367	0.291	0.189	0.191	0.122	0.049	0.133	0.088	0.035	0.140	0.084	0.036
	0.70	0.711	0.633	0.440	0.399	0.317	0.208	0.204	0.132	0.050	0.146	0.096	0.032	0.151	0.100	0.042
	0.80	0.742	0.654	0.481	0.425	0.351	0.228	0.213	0.131	0.055	0.133	0.075	0.020	0.139	0.091	0.039
	0.90	0.744	0.662	0.502	0.458	0.369	0.229	0.258	0.153	0.055	0.142	0.087	0.038	0.136	0.088	0.039
	1.00	0.789	0.711	0.538	0.468	0.376	0.255	0.221	0.139	0.047	0.138	0.082	0.035	0.133	0.087	0.043
	1.50	0.859	0.791	0.649	0.512	0.433	0.288	0.250	0.155	0.058	0.133	0.081	0.028	0.145	0.096	0.043
	2.00	0.889	0.841	0.683	0.541	0.446	0.307	0.243	0.154	0.068	0.146	0.085	0.041	0.138	0.093	0.039

Table E.3: Rejection Rates for Separability Test using FDH ($r = 2$, continued)

		$T_{1,n}$						$T_{2,n}$								
n	δ	$p = 1,\ q = 1$			$p = 2,\ q = 1$			$p = 2,\ q = 2$			$p = 3,\ q = 2$			$p = 3,\ q = 3$		
		.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
1000	0.00	0.133	0.062	0.012	0.164	0.086	0.024	0.106	0.039	0.004	0.096	0.048	0.010	0.097	0.054	0.012
	0.10	0.317	0.204	0.073	0.260	0.175	0.068	0.119	0.058	0.008	0.106	0.043	0.006	0.096	0.056	0.010
	0.20	0.696	0.590	0.348	0.567	0.458	0.275	0.175	0.088	0.023	0.104	0.054	0.010	0.098	0.055	0.016
	0.30	0.895	0.840	0.674	0.755	0.651	0.486	0.257	0.155	0.034	0.125	0.059	0.012	0.100	0.045	0.014
	0.40	0.962	0.942	0.862	0.871	0.819	0.657	0.365	0.239	0.078	0.137	0.058	0.012	0.111	0.061	0.024
	0.50	0.991	0.981	0.930	0.937	0.888	0.776	0.411	0.290	0.101	0.168	0.086	0.027	0.131	0.065	0.022
	0.60	0.993	0.987	0.965	0.968	0.955	0.883	0.458	0.326	0.148	0.172	0.090	0.021	0.136	0.069	0.020
	0.70	0.998	0.995	0.978	0.981	0.966	0.916	0.516	0.395	0.168	0.169	0.093	0.023	0.123	0.074	0.019
	0.80	0.999	0.997	0.991	0.983	0.969	0.921	0.528	0.390	0.160	0.202	0.114	0.026	0.138	0.084	0.030
	0.90	0.998	0.997	0.993	0.986	0.976	0.941	0.550	0.413	0.200	0.224	0.133	0.055	0.151	0.094	0.031
	1.00	1.000	0.999	0.995	0.994	0.987	0.966	0.574	0.459	0.224	0.234	0.142	0.036	0.179	0.095	0.030
	1.50	1.000	1.000	1.000	0.997	0.996	0.987	0.645	0.537	0.307	0.250	0.158	0.046	0.168	0.082	0.030
	2.00	1.000	1.000	1.000	0.998	0.998	0.990	0.640	0.533	0.278	0.278	0.183	0.068	0.201	0.117	0.038

Table E.4: Rejection Rates for Separability Test using FDH ($r = 3$)

n	δ	$T_{1,n}$						$T_{2,n}$								
		$p=1,\ q=1$			$p=2,\ q=1$			$p=2,\ q=2$			$p=3,\ q=2$			$p=3,\ q=3$		
		.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
100	0.00	0.150	0.085	0.021	0.219	0.158	0.080	0.171	0.107	0.042	0.144	0.095	0.050	0.151	0.093	0.048
	0.10	0.200	0.132	0.033	0.237	0.171	0.091	0.166	0.106	0.048	0.126	0.083	0.042	0.157	0.093	0.058
	0.20	0.214	0.134	0.056	0.239	0.169	0.095	0.162	0.105	0.040	0.156	0.097	0.051	0.152	0.101	0.049
	0.30	0.266	0.182	0.076	0.225	0.169	0.084	0.170	0.111	0.049	0.154	0.103	0.054	0.163	0.109	0.054
	0.40	0.296	0.199	0.089	0.251	0.180	0.089	0.190	0.127	0.054	0.154	0.108	0.058	0.161	0.112	0.056
	0.50	0.278	0.193	0.097	0.255	0.190	0.101	0.176	0.121	0.048	0.136	0.095	0.046	0.174	0.124	0.070
	0.60	0.297	0.195	0.097	0.271	0.198	0.100	0.194	0.141	0.064	0.153	0.105	0.052	0.168	0.113	0.058
	0.70	0.349	0.252	0.116	0.268	0.215	0.117	0.178	0.111	0.047	0.145	0.094	0.043	0.158	0.104	0.058
	0.80	0.346	0.244	0.124	0.267	0.191	0.103	0.219	0.145	0.065	0.155	0.112	0.049	0.175	0.123	0.062
	0.90	0.351	0.242	0.119	0.257	0.192	0.101	0.186	0.128	0.058	0.147	0.096	0.045	0.162	0.100	0.051
	1.00	0.370	0.285	0.146	0.268	0.213	0.127	0.206	0.141	0.061	0.136	0.098	0.044	0.141	0.100	0.057
	1.50	0.377	0.276	0.138	0.271	0.203	0.131	0.175	0.109	0.040	0.148	0.099	0.048	0.149	0.094	0.047
	2.00	0.393	0.292	0.159	0.296	0.223	0.126	0.186	0.132	0.053	0.162	0.109	0.054	0.164	0.123	0.059
200	0.00	0.152	0.084	0.018	0.194	0.117	0.044	0.128	0.074	0.017	0.137	0.092	0.046	0.102	0.059	0.023
	0.10	0.186	0.103	0.029	0.227	0.159	0.063	0.146	0.085	0.023	0.126	0.082	0.032	0.133	0.087	0.032
	0.20	0.262	0.167	0.060	0.236	0.155	0.067	0.142	0.077	0.013	0.151	0.090	0.035	0.117	0.082	0.038
	0.30	0.346	0.236	0.113	0.239	0.169	0.084	0.134	0.071	0.022	0.121	0.081	0.032	0.099	0.057	0.024
	0.40	0.347	0.252	0.112	0.249	0.165	0.070	0.158	0.100	0.032	0.111	0.078	0.036	0.127	0.075	0.023
	0.50	0.428	0.318	0.162	0.275	0.196	0.093	0.166	0.095	0.030	0.137	0.078	0.039	0.113	0.078	0.029
	0.60	0.469	0.356	0.177	0.307	0.219	0.110	0.170	0.106	0.030	0.143	0.085	0.036	0.131	0.091	0.039
	0.70	0.456	0.350	0.186	0.305	0.235	0.121	0.150	0.093	0.026	0.140	0.087	0.036	0.126	0.080	0.042
	0.80	0.502	0.390	0.229	0.311	0.233	0.112	0.189	0.118	0.036	0.098	0.057	0.013	0.124	0.079	0.027
	0.90	0.532	0.415	0.226	0.320	0.242	0.132	0.192	0.118	0.041	0.145	0.088	0.025	0.125	0.080	0.033
	1.00	0.546	0.438	0.245	0.331	0.259	0.139	0.173	0.106	0.030	0.120	0.075	0.027	0.137	0.090	0.036
	1.50	0.551	0.460	0.294	0.340	0.251	0.134	0.169	0.101	0.037	0.129	0.085	0.039	0.147	0.093	0.033
	2.00	0.617	0.521	0.318	0.369	0.275	0.143	0.183	0.119	0.039	0.133	0.071	0.031	0.137	0.080	0.032

Table E.4: Rejection Rates for Separability Test using FDH ($r = 3$, continued)

		$T_{1,n}$						$T_{2,n}$								
n	δ	$p=1,\ q=1$			$p=2,\ q=1$			$p=2,\ q=2$			$p=3,\ q=2$			$p=3,\ q=3$		
		.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
1000	0.00	0.114	0.053	0.005	0.116	0.071	0.018	0.090	0.038	0.003	0.085	0.034	0.005	0.094	0.047	0.006
	0.10	0.246	0.142	0.042	0.217	0.140	0.036	0.121	0.050	0.008	0.083	0.028	0.002	0.101	0.043	0.006
	0.20	0.528	0.408	0.203	0.392	0.291	0.119	0.167	0.087	0.009	0.086	0.035	0.009	0.105	0.047	0.012
	0.30	0.681	0.586	0.376	0.532	0.425	0.239	0.225	0.111	0.027	0.104	0.045	0.006	0.087	0.044	0.011
	0.40	0.809	0.705	0.500	0.666	0.557	0.351	0.259	0.157	0.050	0.121	0.052	0.008	0.103	0.048	0.016
	0.50	0.874	0.805	0.614	0.710	0.613	0.413	0.282	0.168	0.047	0.165	0.073	0.015	0.109	0.062	0.014
	0.60	0.920	0.869	0.700	0.787	0.693	0.504	0.342	0.208	0.055	0.140	0.074	0.014	0.105	0.039	0.011
	0.70	0.930	0.879	0.730	0.811	0.739	0.543	0.342	0.226	0.061	0.135	0.068	0.011	0.114	0.060	0.016
	0.80	0.938	0.900	0.765	0.835	0.769	0.577	0.372	0.226	0.073	0.161	0.079	0.016	0.120	0.068	0.023
	0.90	0.956	0.919	0.802	0.858	0.789	0.632	0.375	0.229	0.061	0.162	0.085	0.027	0.127	0.069	0.024
	1.00	0.963	0.933	0.844	0.867	0.792	0.652	0.389	0.246	0.075	0.142	0.080	0.023	0.122	0.063	0.012
	1.50	0.975	0.956	0.892	0.909	0.856	0.745	0.409	0.258	0.101	0.164	0.084	0.013	0.129	0.073	0.019
	2.00	0.986	0.973	0.924	0.929	0.885	0.764	0.456	0.318	0.139	0.191	0.095	0.024	0.123	0.062	0.017

Table E.5: Rejection Rates for Separability Test using DEA, Experiment 2 ($r = 1$)

		$T_{1,n}$									$T_{2,n}$					
n	δ	$p=1,\ q=1$			$p=2,\ q=1$			$p=2,\ q=2$			$p=3,\ q=2$			$p=3,\ q=3$		
		.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
100	0.00	0.166	0.108	0.035	0.183	0.117	0.050	0.222	0.162	0.074	0.174	0.113	0.043	0.188	0.119	0.047
	0.10	0.142	0.087	0.026	0.175	0.112	0.040	0.236	0.176	0.082	0.178	0.121	0.040	0.168	0.101	0.037
	0.20	0.178	0.103	0.038	0.178	0.117	0.043	0.220	0.147	0.075	0.188	0.110	0.054	0.175	0.113	0.049
	0.30	0.177	0.110	0.038	0.191	0.125	0.060	0.226	0.166	0.086	0.172	0.112	0.044	0.174	0.108	0.042
	0.40	0.157	0.091	0.027	0.184	0.124	0.047	0.208	0.154	0.078	0.177	0.110	0.051	0.190	0.126	0.058
	0.50	0.189	0.113	0.029	0.181	0.114	0.039	0.199	0.147	0.077	0.151	0.085	0.031	0.147	0.098	0.037
	0.60	0.139	0.087	0.022	0.169	0.126	0.049	0.209	0.147	0.077	0.177	0.120	0.051	0.162	0.098	0.042
	0.70	0.161	0.107	0.034	0.186	0.132	0.059	0.232	0.172	0.076	0.170	0.112	0.039	0.159	0.109	0.041
	0.80	0.143	0.090	0.026	0.193	0.124	0.054	0.234	0.172	0.085	0.175	0.119	0.039	0.158	0.104	0.035
	0.90	0.191	0.112	0.035	0.200	0.129	0.046	0.220	0.163	0.084	0.161	0.104	0.040	0.192	0.135	0.045
	1.00	0.197	0.118	0.031	0.193	0.128	0.059	0.208	0.143	0.080	0.188	0.126	0.054	0.176	0.112	0.045
	1.50	0.224	0.147	0.046	0.188	0.124	0.064	0.230	0.166	0.082	0.171	0.113	0.045	0.197	0.135	0.060
	2.00	0.187	0.121	0.050	0.214	0.144	0.064	0.228	0.162	0.077	0.169	0.117	0.049	0.161	0.106	0.038
200	0.00	0.163	0.093	0.025	0.153	0.092	0.030	0.191	0.135	0.060	0.150	0.092	0.028	0.127	0.072	0.025
	0.10	0.139	0.066	0.023	0.166	0.100	0.035	0.197	0.130	0.045	0.142	0.079	0.026	0.132	0.073	0.028
	0.20	0.145	0.073	0.015	0.153	0.085	0.035	0.179	0.113	0.052	0.135	0.085	0.022	0.125	0.084	0.021
	0.30	0.160	0.108	0.023	0.171	0.108	0.033	0.175	0.104	0.043	0.126	0.076	0.017	0.138	0.076	0.025
	0.40	0.157	0.088	0.023	0.177	0.115	0.037	0.181	0.120	0.035	0.140	0.088	0.028	0.146	0.082	0.027
	0.50	0.144	0.082	0.019	0.188	0.117	0.036	0.175	0.120	0.040	0.158	0.096	0.029	0.136	0.082	0.016
	0.60	0.167	0.103	0.026	0.167	0.102	0.034	0.186	0.128	0.061	0.129	0.074	0.018	0.140	0.075	0.022
	0.70	0.170	0.108	0.030	0.187	0.115	0.037	0.192	0.136	0.060	0.138	0.079	0.025	0.143	0.084	0.021
	0.80	0.140	0.078	0.033	0.166	0.108	0.036	0.202	0.131	0.068	0.142	0.082	0.023	0.142	0.076	0.019
	0.90	0.169	0.100	0.035	0.176	0.119	0.044	0.192	0.128	0.068	0.148	0.089	0.021	0.137	0.079	0.022
	1.00	0.169	0.108	0.027	0.188	0.109	0.040	0.206	0.132	0.059	0.131	0.070	0.012	0.131	0.072	0.023
	1.50	0.176	0.108	0.035	0.221	0.148	0.054	0.200	0.129	0.056	0.133	0.085	0.020	0.129	0.076	0.027
	2.00	0.186	0.113	0.040	0.186	0.114	0.029	0.207	0.129	0.063	0.135	0.086	0.025	0.151	0.096	0.034

Table E.5: Rejection Rates for Separability Test using DEA, Experiment 2 ($r = 1$, continued)

n	δ	$T_{1,n}$									$T_{2,n}$					
		$p=1,\ q=1$			$p=2,\ q=1$			$p=2,\ q=2$			$p=3,\ q=2$			$p=3,\ q=3$		
		.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
1000	0.00	0.119	0.055	0.013	0.154	0.086	0.023	0.141	0.087	0.024	0.117	0.058	0.009	0.114	0.053	0.009
	0.10	0.126	0.069	0.009	0.131	0.071	0.023	0.143	0.076	0.022	0.113	0.061	0.011	0.104	0.052	0.009
	0.20	0.108	0.061	0.011	0.144	0.089	0.022	0.124	0.076	0.021	0.122	0.061	0.012	0.100	0.051	0.005
	0.30	0.118	0.056	0.009	0.156	0.089	0.027	0.125	0.072	0.018	0.111	0.064	0.016	0.127	0.060	0.008
	0.40	0.116	0.051	0.008	0.158	0.101	0.025	0.157	0.090	0.025	0.114	0.061	0.015	0.101	0.054	0.010
	0.50	0.143	0.078	0.016	0.151	0.089	0.027	0.150	0.089	0.019	0.100	0.055	0.013	0.118	0.061	0.012
	0.60	0.136	0.075	0.018	0.165	0.094	0.022	0.165	0.104	0.034	0.117	0.067	0.009	0.111	0.058	0.013
	0.70	0.149	0.079	0.021	0.187	0.117	0.032	0.151	0.095	0.026	0.116	0.057	0.011	0.111	0.055	0.015
	0.80	0.136	0.082	0.022	0.165	0.090	0.029	0.155	0.093	0.032	0.113	0.075	0.012	0.106	0.048	0.009
	0.90	0.134	0.071	0.009	0.200	0.112	0.040	0.162	0.097	0.029	0.111	0.054	0.013	0.110	0.062	0.012
	1.00	0.135	0.069	0.009	0.190	0.120	0.027	0.174	0.112	0.028	0.121	0.072	0.019	0.109	0.049	0.012
	1.50	0.157	0.088	0.021	0.219	0.138	0.055	0.170	0.096	0.036	0.121	0.061	0.013	0.114	0.064	0.013
	2.00	0.174	0.097	0.018	0.256	0.151	0.045	0.174	0.103	0.033	0.124	0.066	0.008	0.108	0.058	0.011

Table E.6: Rejection Rates for Separability Test using DEA, Experiment 2 ($r = 2$)

n	δ	$T_{1,n}$									$T_{2,n}$					
		$p=1,\ q=1$			$p=2,\ q=1$			$p=2,\ q=2$			$p=3,\ q=2$			$p=3,\ q=3$		
		.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
100	0.00	0.161	0.090	0.020	0.152	0.102	0.034	0.162	0.101	0.040	0.156	0.091	0.027	0.185	0.116	0.049
	0.10	0.158	0.088	0.025	0.183	0.111	0.032	0.188	0.118	0.043	0.157	0.097	0.032	0.153	0.098	0.044
	0.20	0.134	0.068	0.015	0.161	0.090	0.031	0.208	0.138	0.052	0.158	0.092	0.034	0.159	0.096	0.030
	0.30	0.161	0.085	0.016	0.195	0.107	0.032	0.175	0.111	0.042	0.170	0.098	0.032	0.169	0.093	0.022
	0.40	0.147	0.086	0.020	0.171	0.097	0.029	0.166	0.108	0.049	0.185	0.109	0.039	0.170	0.103	0.041
	0.50	0.146	0.080	0.017	0.169	0.091	0.031	0.186	0.123	0.039	0.160	0.098	0.029	0.170	0.102	0.040
	0.60	0.130	0.077	0.024	0.168	0.101	0.029	0.174	0.117	0.050	0.159	0.105	0.032	0.187	0.121	0.050
	0.70	0.155	0.078	0.028	0.171	0.100	0.039	0.196	0.134	0.055	0.151	0.093	0.031	0.150	0.087	0.032
	0.80	0.174	0.108	0.030	0.168	0.108	0.040	0.204	0.134	0.058	0.176	0.110	0.043	0.165	0.099	0.033
	0.90	0.157	0.097	0.026	0.183	0.121	0.044	0.176	0.101	0.037	0.165	0.096	0.035	0.157	0.101	0.035
	1.00	0.156	0.093	0.031	0.178	0.111	0.040	0.202	0.134	0.062	0.163	0.101	0.035	0.172	0.112	0.044
	1.50	0.189	0.110	0.032	0.188	0.124	0.049	0.201	0.140	0.056	0.196	0.115	0.040	0.164	0.117	0.036
	2.00	0.173	0.095	0.027	0.181	0.129	0.036	0.205	0.135	0.061	0.164	0.107	0.035	0.182	0.113	0.044
200	0.00	0.120	0.058	0.009	0.143	0.085	0.019	0.150	0.093	0.029	0.146	0.075	0.020	0.113	0.062	0.013
	0.10	0.135	0.072	0.014	0.120	0.076	0.021	0.168	0.092	0.023	0.122	0.067	0.013	0.141	0.083	0.019
	0.20	0.127	0.065	0.013	0.140	0.080	0.026	0.179	0.094	0.033	0.134	0.073	0.012	0.139	0.083	0.017
	0.30	0.119	0.060	0.011	0.137	0.076	0.017	0.151	0.089	0.031	0.127	0.058	0.014	0.133	0.072	0.019
	0.40	0.138	0.081	0.009	0.129	0.071	0.016	0.150	0.094	0.032	0.135	0.079	0.016	0.127	0.082	0.016
	0.50	0.146	0.071	0.009	0.152	0.085	0.024	0.139	0.085	0.026	0.105	0.054	0.012	0.142	0.073	0.017
	0.60	0.125	0.068	0.019	0.146	0.092	0.014	0.154	0.092	0.031	0.145	0.078	0.021	0.150	0.085	0.025
	0.70	0.139	0.067	0.009	0.140	0.085	0.020	0.154	0.082	0.024	0.142	0.078	0.016	0.126	0.068	0.022
	0.80	0.155	0.088	0.019	0.158	0.084	0.028	0.165	0.103	0.033	0.119	0.060	0.016	0.122	0.071	0.015
	0.90	0.128	0.075	0.015	0.139	0.083	0.022	0.172	0.099	0.028	0.143	0.080	0.021	0.111	0.058	0.015
	1.00	0.147	0.073	0.015	0.160	0.093	0.026	0.175	0.091	0.027	0.126	0.078	0.018	0.131	0.084	0.020
	1.50	0.154	0.090	0.023	0.142	0.087	0.018	0.129	0.067	0.021	0.133	0.078	0.022	0.134	0.072	0.019
	2.00	0.155	0.075	0.018	0.189	0.117	0.039	0.163	0.092	0.033	0.138	0.071	0.018	0.123	0.069	0.023

Table E.6: Rejection Rates for Separability Test using DEA, Experiment 2 ($r = 2$, continued)

		$T_{1,n}$									$T_{2,n}$					
		$p=1,\ q=1$			$p=2,\ q=1$			$p=2,\ q=2$			$p=3,\ q=2$			$p=3,\ q=3$		
n	δ	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
1000	0.00	0.098	0.041	0.005	0.140	0.073	0.016	0.118	0.059	0.015	0.113	0.049	0.007	0.089	0.039	0.002
	0.10	0.132	0.076	0.009	0.106	0.052	0.009	0.116	0.055	0.010	0.110	0.048	0.003	0.108	0.055	0.008
	0.20	0.109	0.049	0.010	0.123	0.059	0.011	0.103	0.042	0.011	0.106	0.048	0.005	0.103	0.052	0.004
	0.30	0.110	0.050	0.006	0.099	0.044	0.010	0.134	0.077	0.016	0.099	0.043	0.007	0.096	0.038	0.006
	0.40	0.133	0.059	0.012	0.133	0.073	0.012	0.120	0.074	0.016	0.098	0.038	0.008	0.089	0.041	0.004
	0.50	0.115	0.054	0.008	0.147	0.080	0.015	0.123	0.054	0.013	0.112	0.048	0.007	0.110	0.056	0.005
	0.60	0.114	0.045	0.005	0.133	0.059	0.009	0.123	0.055	0.016	0.131	0.066	0.008	0.118	0.055	0.009
	0.70	0.126	0.060	0.012	0.119	0.060	0.009	0.137	0.074	0.011	0.094	0.046	0.014	0.105	0.039	0.008
	0.80	0.118	0.056	0.005	0.164	0.080	0.013	0.124	0.073	0.021	0.098	0.048	0.006	0.109	0.058	0.007
	0.90	0.136	0.057	0.011	0.160	0.077	0.014	0.137	0.073	0.014	0.124	0.055	0.008	0.101	0.050	0.012
	1.00	0.125	0.059	0.011	0.144	0.077	0.009	0.137	0.076	0.023	0.114	0.050	0.010	0.096	0.040	0.007
	1.50	0.137	0.049	0.006	0.149	0.084	0.016	0.134	0.073	0.020	0.121	0.061	0.008	0.116	0.056	0.013
	2.00	0.163	0.090	0.015	0.170	0.097	0.027	0.150	0.088	0.019	0.093	0.042	0.004	0.102	0.046	0.008

Table E.7: Rejection Rates for Separability Test using DEA, Experiment 2 ($r = 3$)

		$T_{1,n}$									$T_{2,n}$					
n	δ	$p=1,\ q=1$			$p=2,\ q=1$			$p=2,\ q=2$			$p=3,\ q=2$			$p=3,\ q=3$		
		.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
100	0.00	0.130	0.061	0.012	0.143	0.068	0.016	0.163	0.095	0.026	0.150	0.087	0.031	0.152	0.080	0.030
	0.10	0.146	0.087	0.015	0.161	0.090	0.016	0.156	0.097	0.030	0.144	0.086	0.021	0.160	0.083	0.024
	0.20	0.123	0.070	0.015	0.166	0.080	0.014	0.159	0.108	0.039	0.159	0.101	0.032	0.166	0.105	0.020
	0.30	0.149	0.073	0.009	0.131	0.066	0.013	0.173	0.105	0.032	0.164	0.085	0.021	0.158	0.089	0.032
	0.40	0.145	0.073	0.015	0.129	0.062	0.018	0.171	0.109	0.035	0.140	0.076	0.031	0.144	0.091	0.039
	0.50	0.145	0.061	0.012	0.131	0.068	0.018	0.155	0.109	0.038	0.140	0.085	0.032	0.177	0.106	0.032
	0.60	0.139	0.079	0.009	0.160	0.082	0.009	0.168	0.110	0.036	0.160	0.091	0.029	0.152	0.092	0.022
	0.70	0.148	0.068	0.003	0.159	0.093	0.024	0.164	0.102	0.032	0.150	0.077	0.019	0.165	0.103	0.032
	0.80	0.143	0.087	0.014	0.150	0.085	0.023	0.166	0.102	0.029	0.157	0.092	0.023	0.152	0.088	0.023
	0.90	0.141	0.076	0.012	0.114	0.061	0.014	0.188	0.118	0.050	0.128	0.073	0.019	0.148	0.089	0.033
	1.00	0.146	0.074	0.017	0.166	0.091	0.029	0.192	0.108	0.047	0.186	0.109	0.031	0.149	0.085	0.032
	1.50	0.136	0.064	0.020	0.167	0.096	0.032	0.176	0.111	0.045	0.166	0.095	0.036	0.163	0.102	0.040
	2.00	0.158	0.068	0.015	0.162	0.087	0.020	0.186	0.122	0.049	0.141	0.090	0.020	0.166	0.107	0.038
200	0.00	0.128	0.058	0.008	0.143	0.068	0.010	0.142	0.080	0.026	0.111	0.056	0.010	0.116	0.056	0.009
	0.10	0.118	0.047	0.005	0.137	0.068	0.015	0.134	0.072	0.022	0.125	0.049	0.010	0.117	0.057	0.007
	0.20	0.120	0.050	0.003	0.132	0.075	0.015	0.139	0.080	0.017	0.123	0.056	0.007	0.120	0.054	0.005
	0.30	0.121	0.049	0.005	0.126	0.065	0.008	0.123	0.066	0.018	0.134	0.071	0.005	0.119	0.060	0.008
	0.40	0.121	0.046	0.004	0.134	0.073	0.009	0.138	0.087	0.023	0.120	0.046	0.009	0.122	0.063	0.012
	0.50	0.129	0.055	0.003	0.142	0.065	0.016	0.155	0.091	0.024	0.141	0.065	0.015	0.129	0.055	0.012
	0.60	0.120	0.056	0.009	0.128	0.061	0.012	0.160	0.090	0.022	0.116	0.053	0.009	0.113	0.050	0.006
	0.70	0.125	0.055	0.006	0.117	0.053	0.009	0.132	0.074	0.022	0.134	0.067	0.009	0.104	0.054	0.005
	0.80	0.138	0.060	0.009	0.135	0.066	0.019	0.149	0.096	0.029	0.128	0.056	0.013	0.105	0.052	0.006
	0.90	0.116	0.052	0.009	0.134	0.064	0.012	0.148	0.082	0.027	0.127	0.058	0.011	0.115	0.055	0.014
	1.00	0.140	0.060	0.008	0.136	0.066	0.013	0.143	0.080	0.024	0.133	0.068	0.016	0.110	0.049	0.011
	1.50	0.125	0.052	0.002	0.128	0.064	0.010	0.141	0.086	0.029	0.135	0.064	0.012	0.119	0.060	0.011
	2.00	0.143	0.080	0.014	0.151	0.085	0.019	0.180	0.104	0.035	0.132	0.061	0.006	0.121	0.064	0.013

Table E.7: Rejection Rates for Separability Test using DEA, Experiment 2 ($r = 3$, continued)

		$T_{1,n}$									$T_{2,n}$					
n	δ	$p=1,\ q=1$			$p=2,\ q=1$			$p=2,\ q=2$			$p=3,\ q=2$			$p=3,\ q=3$		
		.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
1000	0.00	0.109	0.047	0.003	0.110	0.050	0.003	0.099	0.043	0.003	0.090	0.032	0.003	0.097	0.037	0.004
	0.10	0.105	0.034	0.002	0.099	0.042	0.010	0.119	0.056	0.008	0.082	0.034	0.002	0.108	0.048	0.002
	0.20	0.100	0.047	0.001	0.101	0.044	0.004	0.111	0.056	0.010	0.079	0.032	0.003	0.097	0.039	0.002
	0.30	0.102	0.044	0.006	0.125	0.062	0.007	0.121	0.048	0.009	0.097	0.038	0.003	0.089	0.038	0.003
	0.40	0.110	0.049	0.003	0.111	0.062	0.003	0.122	0.061	0.011	0.108	0.039	0.006	0.090	0.034	0.004
	0.50	0.110	0.040	0.005	0.105	0.048	0.009	0.119	0.052	0.011	0.124	0.050	0.000	0.110	0.049	0.001
	0.60	0.084	0.034	0.001	0.108	0.058	0.007	0.117	0.053	0.011	0.119	0.059	0.007	0.095	0.038	0.006
	0.70	0.116	0.047	0.006	0.134	0.069	0.008	0.120	0.050	0.013	0.118	0.055	0.005	0.098	0.045	0.008
	0.80	0.095	0.040	0.008	0.127	0.063	0.005	0.101	0.046	0.005	0.105	0.042	0.006	0.103	0.043	0.002
	0.90	0.101	0.046	0.005	0.143	0.060	0.006	0.104	0.062	0.012	0.107	0.048	0.003	0.101	0.048	0.007
	1.00	0.101	0.034	0.001	0.121	0.049	0.002	0.122	0.064	0.006	0.110	0.043	0.005	0.095	0.044	0.005
	1.50	0.115	0.055	0.006	0.136	0.068	0.013	0.132	0.060	0.010	0.096	0.034	0.004	0.080	0.023	0.002
	2.00	0.111	0.055	0.007	0.140	0.070	0.011	0.128	0.050	0.007	0.113	0.037	0.004	0.098	0.041	0.003

Table E.8: Rejection Rates for Separability Test using FDH, Experiment 2 ($r = 1$)

n	δ	$T_{1,n}$						$T_{2,n}$								
		$p=1,\ q=1$			$p=2,\ q=1$			$p=2,\ q=2$			$p=3,\ q=2$			$p=3,\ q=3$		
		.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
100	0.00	0.192	0.132	0.057	0.256	0.213	0.123	0.200	0.132	0.058	0.151	0.101	0.059	0.169	0.121	0.071
	0.10	0.194	0.115	0.051	0.254	0.193	0.110	0.179	0.130	0.056	0.167	0.120	0.058	0.151	0.104	0.055
	0.20	0.225	0.154	0.058	0.254	0.197	0.117	0.173	0.117	0.037	0.164	0.114	0.071	0.168	0.106	0.044
	0.30	0.214	0.144	0.062	0.273	0.207	0.128	0.183	0.124	0.045	0.150	0.102	0.055	0.177	0.105	0.055
	0.40	0.196	0.119	0.049	0.261	0.201	0.120	0.138	0.092	0.037	0.166	0.106	0.059	0.182	0.129	0.070
	0.50	0.223	0.147	0.059	0.260	0.204	0.123	0.184	0.129	0.052	0.159	0.110	0.058	0.181	0.138	0.073
	0.60	0.190	0.112	0.049	0.245	0.200	0.122	0.148	0.098	0.042	0.180	0.127	0.057	0.181	0.127	0.065
	0.70	0.198	0.136	0.051	0.263	0.202	0.124	0.174	0.118	0.050	0.159	0.109	0.058	0.163	0.103	0.057
	0.80	0.167	0.111	0.052	0.273	0.209	0.124	0.176	0.114	0.045	0.166	0.109	0.061	0.168	0.115	0.066
	0.90	0.221	0.152	0.063	0.260	0.203	0.127	0.189	0.117	0.053	0.173	0.114	0.065	0.182	0.125	0.066
	1.00	0.222	0.153	0.066	0.254	0.198	0.138	0.193	0.129	0.054	0.170	0.122	0.069	0.175	0.119	0.068
	1.50	0.238	0.168	0.080	0.264	0.214	0.131	0.182	0.116	0.048	0.155	0.107	0.065	0.164	0.124	0.069
	2.00	0.225	0.167	0.069	0.259	0.202	0.119	0.163	0.110	0.051	0.167	0.126	0.078	0.151	0.102	0.052
200	0.00	0.195	0.132	0.038	0.237	0.175	0.091	0.128	0.085	0.022	0.152	0.107	0.049	0.116	0.085	0.036
	0.10	0.156	0.101	0.034	0.236	0.177	0.104	0.126	0.085	0.035	0.127	0.085	0.031	0.137	0.089	0.040
	0.20	0.174	0.109	0.032	0.220	0.163	0.090	0.163	0.098	0.033	0.125	0.069	0.029	0.111	0.069	0.024
	0.30	0.205	0.130	0.044	0.228	0.172	0.101	0.147	0.083	0.023	0.148	0.095	0.046	0.132	0.092	0.046
	0.40	0.175	0.117	0.042	0.238	0.181	0.107	0.153	0.093	0.028	0.129	0.074	0.034	0.132	0.081	0.030
	0.50	0.181	0.109	0.042	0.251	0.176	0.084	0.137	0.079	0.025	0.150	0.092	0.031	0.164	0.111	0.046
	0.60	0.211	0.142	0.063	0.233	0.179	0.087	0.125	0.073	0.029	0.131	0.079	0.036	0.127	0.088	0.041
	0.70	0.190	0.128	0.048	0.256	0.194	0.099	0.135	0.086	0.026	0.132	0.096	0.041	0.117	0.077	0.033
	0.80	0.172	0.109	0.046	0.221	0.166	0.092	0.147	0.083	0.027	0.149	0.104	0.043	0.122	0.078	0.032
	0.90	0.192	0.131	0.054	0.220	0.168	0.090	0.167	0.108	0.029	0.129	0.091	0.041	0.150	0.109	0.053
	1.00	0.205	0.120	0.040	0.252	0.190	0.102	0.145	0.082	0.027	0.148	0.093	0.033	0.122	0.082	0.043
	1.50	0.229	0.142	0.063	0.251	0.188	0.104	0.132	0.075	0.025	0.127	0.080	0.031	0.112	0.074	0.034
	2.00	0.239	0.164	0.067	0.254	0.198	0.099	0.142	0.084	0.034	0.146	0.097	0.035	0.135	0.088	0.041

Table E.8: Rejection Rates for Separability Test using FDH, Experiment 2 ($r = 1$, continued)

		$T_{1,n}$						$T_{2,n}$								
n	δ	$p=1,\ q=1$			$p=2,\ q=1$			$p=2,\ q=2$			$p=3,\ q=2$			$p=3,\ q=3$		
		.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
1000	0.00	0.155	0.081	0.020	0.172	0.113	0.042	0.108	0.051	0.013	0.102	0.045	0.007	0.097	0.051	0.017
	0.10	0.124	0.075	0.015	0.185	0.125	0.052	0.094	0.051	0.015	0.111	0.054	0.015	0.115	0.054	0.015
	0.20	0.133	0.071	0.016	0.185	0.122	0.042	0.103	0.052	0.007	0.085	0.042	0.006	0.109	0.049	0.013
	0.30	0.151	0.076	0.015	0.193	0.127	0.052	0.112	0.051	0.010	0.073	0.038	0.013	0.105	0.058	0.014
	0.40	0.138	0.075	0.019	0.211	0.137	0.048	0.093	0.052	0.007	0.101	0.049	0.012	0.097	0.050	0.014
	0.50	0.198	0.115	0.033	0.196	0.125	0.056	0.105	0.046	0.010	0.103	0.049	0.007	0.132	0.072	0.024
	0.60	0.182	0.108	0.025	0.227	0.159	0.067	0.124	0.068	0.007	0.109	0.050	0.011	0.083	0.050	0.010
	0.70	0.203	0.115	0.033	0.253	0.187	0.079	0.109	0.056	0.006	0.116	0.053	0.010	0.103	0.052	0.013
	0.80	0.192	0.108	0.042	0.212	0.159	0.082	0.112	0.057	0.015	0.111	0.060	0.012	0.098	0.047	0.012
	0.90	0.179	0.104	0.030	0.267	0.172	0.077	0.111	0.055	0.012	0.111	0.050	0.013	0.097	0.059	0.018
	1.00	0.198	0.112	0.028	0.249	0.170	0.071	0.112	0.063	0.013	0.117	0.053	0.015	0.100	0.052	0.017
	1.50	0.213	0.131	0.043	0.299	0.219	0.120	0.122	0.064	0.014	0.112	0.061	0.017	0.108	0.065	0.022
	2.00	0.241	0.156	0.048	0.335	0.248	0.122	0.144	0.074	0.016	0.111	0.056	0.017	0.107	0.066	0.015

Table E.9: Rejection Rates for Separability Test using FDH, Experiment 2 ($r = 2$)

		$T_{1,n}$						$T_{2,n}$								
n	δ	$p = 1,\ q = 1$			$p = 2,\ q = 1$			$p = 2,\ q = 2$			$p = 3,\ q = 2$			$p = 3,\ q = 3$		
		.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
100	0.00	0.165	0.101	0.032	0.239	0.181	0.105	0.143	0.091	0.034	0.186	0.138	0.077	0.166	0.104	0.058
	0.10	0.190	0.122	0.037	0.256	0.191	0.098	0.169	0.107	0.030	0.161	0.102	0.053	0.164	0.110	0.054
	0.20	0.181	0.109	0.031	0.241	0.181	0.103	0.164	0.105	0.036	0.169	0.117	0.062	0.159	0.105	0.057
	0.30	0.193	0.121	0.040	0.241	0.185	0.103	0.160	0.100	0.045	0.177	0.112	0.056	0.162	0.112	0.057
	0.40	0.179	0.106	0.040	0.244	0.178	0.093	0.154	0.086	0.039	0.201	0.132	0.068	0.157	0.108	0.054
	0.50	0.182	0.114	0.042	0.235	0.167	0.083	0.158	0.086	0.027	0.158	0.102	0.054	0.171	0.121	0.078
	0.60	0.177	0.109	0.024	0.251	0.192	0.098	0.166	0.112	0.046	0.179	0.119	0.054	0.173	0.114	0.065
	0.70	0.177	0.112	0.046	0.234	0.165	0.094	0.177	0.109	0.052	0.156	0.111	0.061	0.145	0.108	0.052
	0.80	0.185	0.118	0.043	0.254	0.188	0.103	0.167	0.108	0.036	0.155	0.104	0.050	0.162	0.105	0.059
	0.90	0.187	0.123	0.055	0.235	0.182	0.101	0.181	0.114	0.035	0.184	0.132	0.069	0.176	0.123	0.065
	1.00	0.203	0.142	0.057	0.247	0.193	0.099	0.196	0.133	0.052	0.176	0.122	0.063	0.166	0.121	0.066
	1.50	0.222	0.157	0.054	0.266	0.195	0.111	0.193	0.121	0.049	0.190	0.129	0.063	0.171	0.115	0.067
	2.00	0.206	0.138	0.056	0.248	0.172	0.095	0.172	0.115	0.057	0.175	0.115	0.064	0.172	0.118	0.057
200	0.00	0.130	0.065	0.018	0.195	0.138	0.064	0.137	0.081	0.028	0.111	0.063	0.024	0.137	0.104	0.049
	0.10	0.183	0.106	0.029	0.190	0.124	0.058	0.137	0.080	0.024	0.139	0.091	0.040	0.114	0.075	0.038
	0.20	0.166	0.100	0.029	0.213	0.135	0.056	0.128	0.073	0.029	0.132	0.081	0.033	0.130	0.090	0.046
	0.30	0.156	0.081	0.020	0.179	0.124	0.057	0.144	0.078	0.020	0.118	0.078	0.030	0.131	0.081	0.043
	0.40	0.151	0.101	0.029	0.208	0.151	0.062	0.128	0.077	0.028	0.124	0.079	0.032	0.119	0.077	0.039
	0.50	0.157	0.098	0.021	0.222	0.162	0.080	0.127	0.069	0.021	0.140	0.085	0.035	0.144	0.085	0.039
	0.60	0.161	0.106	0.026	0.185	0.144	0.049	0.133	0.076	0.017	0.135	0.081	0.031	0.155	0.095	0.052
	0.70	0.174	0.105	0.030	0.216	0.152	0.057	0.140	0.078	0.022	0.140	0.083	0.023	0.131	0.086	0.037
	0.80	0.180	0.116	0.033	0.220	0.147	0.067	0.126	0.070	0.021	0.110	0.070	0.020	0.121	0.081	0.030
	0.90	0.160	0.096	0.023	0.211	0.138	0.048	0.148	0.080	0.023	0.121	0.079	0.035	0.123	0.075	0.045
	1.00	0.192	0.112	0.034	0.217	0.133	0.048	0.134	0.080	0.019	0.129	0.085	0.029	0.136	0.085	0.032
	1.50	0.210	0.131	0.042	0.228	0.149	0.070	0.138	0.078	0.019	0.132	0.072	0.025	0.155	0.102	0.034
	2.00	0.187	0.110	0.037	0.235	0.172	0.080	0.154	0.085	0.022	0.153	0.096	0.033	0.131	0.093	0.051

Table E.9: Rejection Rates for Separability Test using FDH, Experiment 2 ($r = 2$, continued)

		$T_{1,n}$						$T_{2,n}$								
		$p=1,\ q=1$			$p=2,\ q=1$			$p=2,\ q=2$			$p=3,\ q=2$			$p=3,\ q=3$		
n	δ	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
1000	0.00	0.134	0.062	0.014	0.164	0.086	0.024	0.106	0.039	0.004	0.095	0.048	0.010	0.097	0.054	0.012
	0.10	0.119	0.072	0.010	0.155	0.093	0.024	0.096	0.042	0.005	0.095	0.047	0.003	0.101	0.055	0.015
	0.20	0.123	0.065	0.012	0.151	0.088	0.017	0.093	0.036	0.006	0.099	0.041	0.006	0.111	0.055	0.016
	0.30	0.120	0.063	0.008	0.151	0.090	0.018	0.093	0.049	0.006	0.089	0.037	0.005	0.088	0.047	0.010
	0.40	0.156	0.073	0.022	0.163	0.107	0.031	0.127	0.058	0.005	0.088	0.034	0.004	0.094	0.050	0.015
	0.50	0.138	0.066	0.013	0.175	0.103	0.027	0.109	0.052	0.013	0.100	0.046	0.011	0.101	0.053	0.014
	0.60	0.140	0.067	0.009	0.174	0.093	0.029	0.105	0.046	0.004	0.086	0.037	0.009	0.102	0.053	0.016
	0.70	0.148	0.082	0.020	0.176	0.109	0.035	0.109	0.044	0.010	0.085	0.041	0.010	0.100	0.059	0.015
	0.80	0.143	0.082	0.016	0.202	0.126	0.036	0.105	0.047	0.008	0.089	0.038	0.010	0.098	0.049	0.018
	0.90	0.160	0.089	0.019	0.194	0.120	0.037	0.105	0.035	0.010	0.109	0.052	0.011	0.096	0.049	0.006
	1.00	0.155	0.080	0.017	0.202	0.135	0.046	0.118	0.059	0.011	0.084	0.035	0.005	0.091	0.047	0.014
	1.50	0.168	0.090	0.019	0.205	0.124	0.043	0.133	0.058	0.008	0.098	0.046	0.008	0.098	0.051	0.014
	2.00	0.191	0.114	0.026	0.200	0.132	0.047	0.120	0.073	0.008	0.080	0.040	0.011	0.103	0.051	0.012

Table E.10: Rejection Rates for Separability Test using FDH, Experiment 2 ($r = 3$)

		$T_{1,n}$						$T_{2,n}$								
n	δ	$p=1,\ q=1$			$p=2,\ q=1$			$p=2,\ q=2$			$p=3,\ q=2$			$p=3,\ q=3$		
		.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
100	0.00	0.150	0.085	0.021	0.218	0.157	0.079	0.171	0.107	0.042	0.144	0.095	0.050	0.151	0.094	0.048
	0.10	0.169	0.101	0.023	0.228	0.158	0.084	0.162	0.108	0.048	0.137	0.093	0.048	0.156	0.102	0.055
	0.20	0.155	0.074	0.020	0.227	0.158	0.073	0.166	0.097	0.037	0.149	0.104	0.051	0.163	0.100	0.055
	0.30	0.186	0.108	0.035	0.201	0.138	0.065	0.166	0.106	0.037	0.161	0.108	0.053	0.152	0.102	0.048
	0.40	0.166	0.091	0.025	0.209	0.150	0.079	0.170	0.119	0.040	0.144	0.096	0.037	0.157	0.114	0.056
	0.50	0.163	0.088	0.026	0.215	0.157	0.079	0.165	0.106	0.046	0.132	0.089	0.045	0.177	0.131	0.072
	0.60	0.167	0.087	0.024	0.217	0.160	0.077	0.185	0.128	0.054	0.156	0.102	0.051	0.149	0.098	0.051
	0.70	0.171	0.107	0.030	0.231	0.171	0.082	0.159	0.094	0.037	0.152	0.103	0.051	0.144	0.086	0.041
	0.80	0.176	0.116	0.038	0.238	0.171	0.083	0.199	0.140	0.056	0.167	0.103	0.045	0.170	0.113	0.061
	0.90	0.197	0.109	0.034	0.218	0.158	0.074	0.178	0.120	0.042	0.126	0.084	0.047	0.159	0.102	0.049
	1.00	0.188	0.114	0.035	0.233	0.158	0.076	0.194	0.126	0.058	0.158	0.102	0.050	0.173	0.104	0.055
	1.50	0.172	0.100	0.031	0.254	0.198	0.107	0.171	0.117	0.050	0.177	0.118	0.047	0.150	0.104	0.057
	2.00	0.174	0.112	0.033	0.203	0.161	0.074	0.165	0.111	0.045	0.162	0.109	0.063	0.190	0.131	0.067
200	0.00	0.152	0.084	0.018	0.199	0.123	0.047	0.127	0.074	0.017	0.137	0.092	0.046	0.102	0.059	0.023
	0.10	0.139	0.063	0.014	0.213	0.127	0.047	0.137	0.070	0.015	0.131	0.074	0.029	0.121	0.072	0.038
	0.20	0.150	0.087	0.016	0.190	0.127	0.050	0.112	0.051	0.013	0.142	0.078	0.033	0.118	0.070	0.033
	0.30	0.142	0.084	0.019	0.172	0.101	0.038	0.127	0.059	0.009	0.134	0.096	0.035	0.119	0.072	0.026
	0.40	0.139	0.075	0.016	0.198	0.114	0.051	0.117	0.060	0.020	0.117	0.074	0.034	0.128	0.081	0.031
	0.50	0.159	0.096	0.014	0.179	0.113	0.045	0.126	0.065	0.019	0.124	0.087	0.033	0.123	0.082	0.041
	0.60	0.139	0.085	0.022	0.178	0.108	0.036	0.131	0.061	0.009	0.149	0.092	0.039	0.135	0.082	0.035
	0.70	0.139	0.082	0.016	0.183	0.110	0.045	0.137	0.064	0.016	0.130	0.078	0.026	0.123	0.088	0.034
	0.80	0.189	0.096	0.024	0.189	0.116	0.044	0.128	0.074	0.021	0.124	0.071	0.023	0.113	0.071	0.024
	0.90	0.153	0.076	0.019	0.178	0.121	0.052	0.143	0.073	0.017	0.127	0.070	0.035	0.130	0.088	0.035
	1.00	0.157	0.089	0.018	0.204	0.132	0.050	0.131	0.071	0.014	0.157	0.094	0.026	0.127	0.076	0.031
	1.50	0.149	0.086	0.017	0.197	0.126	0.068	0.148	0.072	0.019	0.135	0.085	0.031	0.154	0.090	0.032
	2.00	0.199	0.115	0.038	0.197	0.133	0.058	0.151	0.087	0.024	0.135	0.089	0.034	0.146	0.097	0.040

Table E.10: Rejection Rates for Separability Test using FDH, Experiment 2 ($r = 3$, continued)

		$T_{1,n}$						$T_{2,n}$									
n	δ	$p=1,\ q=1$			$p=2,\ q=1$			$p=2,\ q=2$			$p=3,\ q=2$			$p=3,\ q=3$			
		.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	
1000	0.00	0.114	0.052	0.005	0.116	0.071	0.018	0.089	0.038	0.003	0.084	0.034	0.005	0.093	0.046	0.006	
	0.10	0.103	0.061	0.009	0.125	0.060	0.012	0.103	0.040	0.003	0.078	0.026	0.004	0.093	0.043	0.005	
	0.20	0.124	0.059	0.006	0.131	0.075	0.012	0.094	0.029	0.002	0.073	0.033	0.002	0.077	0.034	0.009	
	0.30	0.122	0.051	0.005	0.133	0.071	0.013	0.102	0.032	0.002	0.083	0.033	0.006	0.091	0.047	0.008	
	0.40	0.114	0.055	0.006	0.138	0.076	0.018	0.099	0.054	0.003	0.089	0.041	0.006	0.100	0.045	0.011	
	0.50	0.134	0.063	0.016	0.156	0.088	0.025	0.091	0.045	0.003	0.102	0.040	0.005	0.081	0.039	0.006	
	0.60	0.108	0.049	0.009	0.148	0.086	0.015	0.108	0.041	0.007	0.094	0.037	0.004	0.085	0.047	0.011	
	0.70	0.115	0.052	0.011	0.166	0.099	0.023	0.097	0.041	0.005	0.097	0.045	0.009	0.079	0.037	0.014	
	0.80	0.119	0.053	0.006	0.159	0.095	0.021	0.110	0.051	0.006	0.093	0.044	0.007	0.083	0.045	0.010	
	0.90	0.108	0.050	0.011	0.156	0.085	0.011	0.080	0.044	0.004	0.075	0.036	0.006	0.092	0.044	0.006	
	1.00	0.128	0.065	0.005	0.169	0.090	0.020	0.101	0.042	0.005	0.081	0.032	0.007	0.095	0.050	0.011	
	1.50	0.136	0.067	0.012	0.183	0.115	0.029	0.101	0.042	0.005	0.081	0.039	0.007	0.102	0.049	0.007	
	2.00	0.139	0.060	0.009	0.181	0.094	0.019	0.112	0.050	0.003	0.109	0.051	0.005	0.088	0.043	0.009	

Table E.11: Rejection Rates for Separability Test using DEA, Experiment 3 ($r = 1$)

		$T_{1,n}$									$T_{2,n}$					
		$p=1,\ q=1$			$p=2,\ q=1$			$p=2,\ q=2$			$p=3,\ q=2$			$p=3,\ q=3$		
n	δ	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
100	0.00	0.166	0.108	0.035	0.182	0.117	0.050	0.221	0.161	0.074	0.174	0.113	0.043	0.188	0.119	0.047
	0.10	0.188	0.110	0.028	0.188	0.111	0.046	0.239	0.185	0.088	0.193	0.116	0.051	0.177	0.104	0.044
	0.20	0.263	0.174	0.085	0.231	0.144	0.060	0.245	0.185	0.111	0.182	0.123	0.059	0.191	0.124	0.056
	0.30	0.320	0.240	0.111	0.267	0.195	0.106	0.272	0.215	0.129	0.184	0.123	0.050	0.188	0.125	0.052
	0.40	0.384	0.301	0.171	0.262	0.188	0.095	0.309	0.236	0.144	0.203	0.131	0.064	0.217	0.155	0.069
	0.50	0.433	0.363	0.221	0.302	0.219	0.125	0.302	0.246	0.144	0.179	0.120	0.059	0.170	0.120	0.055
	0.60	0.501	0.421	0.266	0.310	0.254	0.150	0.305	0.248	0.159	0.205	0.155	0.071	0.170	0.113	0.055
	0.70	0.545	0.451	0.302	0.352	0.265	0.169	0.377	0.313	0.201	0.202	0.135	0.066	0.202	0.133	0.064
	0.80	0.573	0.490	0.338	0.365	0.296	0.198	0.365	0.284	0.183	0.194	0.138	0.058	0.204	0.134	0.065
	0.90	0.656	0.582	0.418	0.417	0.337	0.212	0.372	0.308	0.206	0.209	0.131	0.057	0.218	0.157	0.083
	1.00	0.689	0.602	0.445	0.406	0.334	0.207	0.381	0.324	0.214	0.243	0.175	0.091	0.226	0.161	0.086
	1.50	0.768	0.699	0.542	0.488	0.419	0.274	0.451	0.384	0.284	0.232	0.167	0.088	0.222	0.172	0.084
	2.00	0.782	0.722	0.583	0.569	0.493	0.356	0.486	0.427	0.317	0.247	0.176	0.085	0.216	0.153	0.080
200	0.00	0.163	0.093	0.025	0.153	0.092	0.030	0.191	0.135	0.060	0.148	0.090	0.027	0.130	0.073	0.025
	0.10	0.163	0.102	0.036	0.169	0.118	0.057	0.207	0.142	0.056	0.139	0.082	0.031	0.143	0.076	0.024
	0.20	0.320	0.226	0.089	0.247	0.156	0.069	0.239	0.175	0.084	0.158	0.098	0.029	0.119	0.073	0.028
	0.30	0.459	0.345	0.177	0.334	0.245	0.126	0.287	0.234	0.127	0.151	0.091	0.030	0.152	0.090	0.035
	0.40	0.589	0.488	0.313	0.411	0.323	0.193	0.335	0.267	0.160	0.180	0.120	0.044	0.169	0.108	0.035
	0.50	0.685	0.600	0.412	0.507	0.418	0.262	0.423	0.367	0.232	0.187	0.120	0.053	0.185	0.107	0.030
	0.60	0.759	0.689	0.523	0.534	0.447	0.289	0.473	0.400	0.260	0.205	0.133	0.060	0.166	0.112	0.041
	0.70	0.818	0.736	0.585	0.623	0.544	0.375	0.500	0.427	0.302	0.197	0.134	0.055	0.174	0.107	0.030
	0.80	0.855	0.803	0.634	0.639	0.546	0.409	0.548	0.469	0.349	0.200	0.128	0.050	0.203	0.127	0.042
	0.90	0.888	0.830	0.710	0.693	0.624	0.451	0.636	0.570	0.440	0.227	0.151	0.074	0.195	0.128	0.040
	1.00	0.899	0.859	0.721	0.715	0.639	0.491	0.659	0.593	0.456	0.236	0.175	0.078	0.193	0.122	0.051
	1.50	0.950	0.923	0.830	0.838	0.781	0.651	0.715	0.651	0.539	0.261	0.189	0.088	0.249	0.178	0.067
	2.00	0.964	0.944	0.882	0.862	0.820	0.706	0.799	0.756	0.642	0.314	0.224	0.110	0.267	0.190	0.080

Table E.11: Rejection Rates for Separability Test using DEA, Experiment 3 ($r = 1$, continued)

		$T_{1,n}$									$T_{2,n}$					
n	δ	$p = 1,\ q = 1$			$p = 2,\ q = 1$			$p = 2,\ q = 2$			$p = 3,\ q = 2$			$p = 3,\ q = 3$		
		.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
1000	0.00	0.119	0.059	0.015	0.154	0.086	0.023	0.140	0.087	0.024	0.116	0.059	0.010	0.114	0.053	0.009
	0.10	0.357	0.245	0.099	0.275	0.186	0.064	0.260	0.167	0.055	0.126	0.056	0.013	0.110	0.054	0.012
	0.20	0.773	0.658	0.429	0.659	0.567	0.340	0.597	0.494	0.303	0.211	0.131	0.039	0.122	0.068	0.014
	0.30	0.940	0.898	0.754	0.900	0.836	0.687	0.836	0.758	0.584	0.313	0.210	0.077	0.205	0.121	0.029
	0.40	0.984	0.968	0.910	0.975	0.951	0.875	0.950	0.919	0.822	0.429	0.311	0.144	0.282	0.171	0.049
	0.50	0.999	0.997	0.986	0.989	0.982	0.940	0.977	0.963	0.913	0.553	0.413	0.222	0.377	0.239	0.086
	0.60	0.998	0.995	0.987	0.997	0.994	0.977	0.995	0.990	0.967	0.619	0.499	0.295	0.422	0.312	0.133
	0.70	1.000	1.000	0.996	1.000	0.999	0.995	0.999	0.993	0.980	0.693	0.585	0.375	0.463	0.342	0.164
	0.80	1.000	1.000	0.996	1.000	1.000	0.996	0.998	0.998	0.990	0.748	0.639	0.421	0.525	0.393	0.197
	0.90	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.998	0.791	0.667	0.455	0.572	0.453	0.231
	1.00	1.000	1.000	1.000	1.000	0.999	0.998	1.000	1.000	1.000	0.849	0.762	0.533	0.609	0.499	0.271
	1.50	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.923	0.871	0.717	0.756	0.643	0.423
	2.00	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.957	0.916	0.786	0.788	0.686	0.462

Table E.12: Rejection Rates for Separability Test using DEA, Experiment 3 ($r = 2$)

		$T_{1,n}$									$T_{2,n}$					
		$p=1,\ q=1$			$p=2,\ q=1$			$p=2,\ q=2$			$p=3,\ q=2$			$p=3,\ q=3$		
n	δ	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
100	0.00	0.161	0.090	0.020	0.151	0.102	0.034	0.163	0.101	0.040	0.158	0.092	0.027	0.187	0.118	0.051
	0.10	0.182	0.119	0.045	0.199	0.139	0.051	0.189	0.120	0.042	0.176	0.100	0.034	0.153	0.107	0.040
	0.20	0.199	0.134	0.051	0.193	0.127	0.057	0.223	0.158	0.066	0.166	0.106	0.045	0.185	0.120	0.040
	0.30	0.250	0.177	0.094	0.235	0.149	0.065	0.230	0.168	0.069	0.188	0.120	0.046	0.193	0.128	0.050
	0.40	0.276	0.194	0.092	0.235	0.155	0.073	0.238	0.180	0.081	0.198	0.128	0.053	0.192	0.128	0.050
	0.50	0.312	0.222	0.090	0.239	0.157	0.070	0.250	0.182	0.090	0.180	0.130	0.053	0.196	0.123	0.058
	0.60	0.327	0.228	0.113	0.249	0.174	0.084	0.245	0.185	0.104	0.202	0.132	0.049	0.207	0.140	0.065
	0.70	0.330	0.241	0.113	0.255	0.197	0.103	0.278	0.205	0.109	0.181	0.122	0.044	0.184	0.119	0.052
	0.80	0.366	0.286	0.150	0.292	0.215	0.107	0.255	0.198	0.101	0.205	0.140	0.062	0.194	0.133	0.061
	0.90	0.387	0.307	0.169	0.278	0.207	0.105	0.274	0.234	0.133	0.183	0.131	0.061	0.203	0.139	0.064
	1.00	0.396	0.294	0.157	0.300	0.217	0.119	0.303	0.228	0.124	0.229	0.164	0.078	0.201	0.145	0.068
	1.50	0.405	0.303	0.177	0.306	0.224	0.120	0.297	0.236	0.160	0.215	0.154	0.075	0.202	0.139	0.081
	2.00	0.447	0.361	0.221	0.278	0.216	0.114	0.302	0.245	0.144	0.214	0.150	0.078	0.195	0.133	0.057
200	0.00	0.120	0.058	0.009	0.142	0.085	0.019	0.151	0.093	0.029	0.146	0.075	0.020	0.113	0.062	0.013
	0.10	0.184	0.099	0.020	0.155	0.085	0.021	0.206	0.123	0.036	0.126	0.069	0.020	0.150	0.091	0.025
	0.20	0.255	0.171	0.054	0.184	0.115	0.046	0.211	0.145	0.062	0.150	0.087	0.023	0.153	0.087	0.024
	0.30	0.313	0.219	0.101	0.241	0.168	0.065	0.223	0.161	0.071	0.133	0.075	0.025	0.146	0.088	0.029
	0.40	0.359	0.262	0.125	0.251	0.166	0.071	0.253	0.182	0.087	0.161	0.096	0.025	0.148	0.092	0.027
	0.50	0.419	0.328	0.171	0.308	0.222	0.110	0.279	0.210	0.111	0.166	0.098	0.032	0.161	0.096	0.024
	0.60	0.440	0.345	0.198	0.315	0.222	0.119	0.303	0.235	0.125	0.186	0.128	0.033	0.171	0.108	0.030
	0.70	0.506	0.400	0.219	0.305	0.228	0.092	0.305	0.224	0.105	0.183	0.117	0.043	0.161	0.097	0.043
	0.80	0.493	0.399	0.238	0.367	0.284	0.145	0.326	0.245	0.146	0.207	0.136	0.045	0.151	0.096	0.040
	0.90	0.493	0.407	0.240	0.379	0.283	0.146	0.355	0.274	0.162	0.179	0.121	0.053	0.148	0.097	0.035
	1.00	0.553	0.430	0.258	0.372	0.280	0.158	0.371	0.294	0.160	0.189	0.115	0.043	0.156	0.097	0.038
	1.50	0.604	0.504	0.325	0.389	0.295	0.182	0.355	0.270	0.164	0.201	0.139	0.062	0.170	0.106	0.044
	2.00	0.609	0.495	0.319	0.419	0.346	0.219	0.382	0.302	0.177	0.207	0.136	0.059	0.165	0.113	0.052

Table E.12: Rejection Rates for Separability Test using DEA, Experiment 3 ($r = 2$, continued)

		$T_{1,n}$									$T_{2,n}$					
		$p=1,\ q=1$			$p=2,\ q=1$			$p=2,\ q=2$			$p=3,\ q=2$			$p=3,\ q=3$		
n	δ	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
1000	0.00	0.098	0.041	0.005	0.136	0.070	0.014	0.118	0.059	0.015	0.112	0.049	0.007	0.090	0.039	0.002
	0.10	0.281	0.176	0.049	0.203	0.127	0.035	0.189	0.119	0.033	0.130	0.059	0.009	0.109	0.062	0.006
	0.20	0.514	0.378	0.189	0.434	0.321	0.138	0.344	0.234	0.102	0.153	0.079	0.012	0.129	0.058	0.009
	0.30	0.724	0.619	0.404	0.597	0.470	0.281	0.563	0.456	0.252	0.185	0.102	0.026	0.132	0.071	0.017
	0.40	0.847	0.765	0.580	0.721	0.619	0.432	0.684	0.585	0.381	0.237	0.144	0.047	0.173	0.099	0.025
	0.50	0.901	0.833	0.648	0.828	0.736	0.529	0.749	0.659	0.460	0.302	0.211	0.061	0.221	0.131	0.037
	0.60	0.914	0.871	0.732	0.884	0.834	0.660	0.798	0.717	0.547	0.341	0.223	0.075	0.250	0.167	0.056
	0.70	0.945	0.911	0.795	0.898	0.845	0.683	0.850	0.776	0.627	0.338	0.233	0.088	0.260	0.163	0.048
	0.80	0.955	0.923	0.832	0.900	0.847	0.714	0.888	0.821	0.654	0.382	0.270	0.117	0.253	0.164	0.055
	0.90	0.967	0.939	0.839	0.917	0.878	0.745	0.884	0.839	0.688	0.390	0.281	0.142	0.283	0.182	0.076
	1.00	0.968	0.950	0.863	0.941	0.892	0.782	0.881	0.829	0.704	0.408	0.293	0.129	0.282	0.181	0.062
	1.50	0.982	0.966	0.914	0.948	0.915	0.830	0.920	0.889	0.783	0.452	0.332	0.150	0.345	0.234	0.090
	2.00	0.986	0.979	0.939	0.942	0.921	0.847	0.928	0.904	0.809	0.481	0.383	0.186	0.355	0.247	0.104

Table E.13: Rejection Rates for Separability Test using DEA, Experiment 3 ($r = 3$)

		$T_{1,n}$									$T_{2,n}$					
n	δ	$p=1,\ q=1$			$p=2,\ q=1$			$p=2,\ q=2$			$p=3,\ q=2$			$p=3,\ q=3$		
		.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
100	0.00	0.130	0.061	0.012	0.143	0.068	0.016	0.163	0.095	0.026	0.151	0.088	0.032	0.153	0.080	0.030
	0.10	0.177	0.098	0.024	0.177	0.096	0.023	0.168	0.112	0.039	0.141	0.089	0.024	0.172	0.101	0.032
	0.20	0.182	0.102	0.032	0.172	0.098	0.021	0.168	0.112	0.055	0.178	0.105	0.027	0.163	0.094	0.033
	0.30	0.225	0.129	0.046	0.154	0.090	0.029	0.209	0.129	0.065	0.179	0.109	0.022	0.183	0.113	0.040
	0.40	0.229	0.145	0.056	0.173	0.115	0.032	0.204	0.130	0.071	0.170	0.095	0.035	0.167	0.117	0.054
	0.50	0.218	0.132	0.052	0.182	0.114	0.039	0.209	0.150	0.074	0.159	0.108	0.042	0.193	0.123	0.055
	0.60	0.245	0.150	0.055	0.200	0.133	0.052	0.238	0.172	0.078	0.175	0.111	0.046	0.197	0.133	0.052
	0.70	0.250	0.157	0.069	0.223	0.149	0.056	0.223	0.158	0.086	0.176	0.113	0.046	0.172	0.110	0.049
	0.80	0.258	0.191	0.079	0.215	0.137	0.062	0.260	0.177	0.083	0.163	0.114	0.035	0.194	0.116	0.049
	0.90	0.266	0.172	0.071	0.199	0.128	0.052	0.235	0.165	0.081	0.186	0.117	0.044	0.184	0.112	0.044
	1.00	0.289	0.193	0.094	0.226	0.140	0.057	0.252	0.187	0.095	0.178	0.113	0.036	0.166	0.116	0.052
	1.50	0.282	0.194	0.087	0.236	0.162	0.080	0.251	0.191	0.102	0.201	0.130	0.054	0.183	0.132	0.049
	2.00	0.293	0.205	0.096	0.241	0.178	0.077	0.263	0.209	0.119	0.176	0.119	0.055	0.193	0.136	0.062
200	0.00	0.128	0.058	0.008	0.143	0.068	0.010	0.142	0.080	0.026	0.109	0.056	0.010	0.116	0.056	0.009
	0.10	0.152	0.072	0.013	0.158	0.090	0.022	0.151	0.093	0.032	0.134	0.053	0.010	0.135	0.061	0.006
	0.20	0.193	0.113	0.024	0.167	0.106	0.029	0.184	0.108	0.033	0.149	0.082	0.016	0.134	0.069	0.009
	0.30	0.247	0.147	0.045	0.192	0.120	0.042	0.193	0.124	0.053	0.152	0.084	0.016	0.115	0.067	0.015
	0.40	0.239	0.155	0.055	0.204	0.136	0.041	0.229	0.149	0.063	0.155	0.082	0.015	0.132	0.067	0.018
	0.50	0.280	0.184	0.065	0.231	0.148	0.050	0.232	0.156	0.080	0.166	0.100	0.026	0.153	0.076	0.015
	0.60	0.280	0.202	0.075	0.224	0.138	0.052	0.232	0.166	0.080	0.158	0.081	0.026	0.145	0.083	0.027
	0.70	0.275	0.198	0.084	0.236	0.144	0.045	0.259	0.180	0.088	0.162	0.096	0.026	0.144	0.083	0.018
	0.80	0.314	0.225	0.097	0.237	0.160	0.055	0.274	0.188	0.094	0.148	0.076	0.024	0.164	0.097	0.024
	0.90	0.318	0.220	0.075	0.255	0.173	0.067	0.267	0.197	0.097	0.158	0.089	0.029	0.160	0.087	0.027
	1.00	0.333	0.245	0.114	0.259	0.174	0.075	0.248	0.176	0.090	0.157	0.095	0.022	0.137	0.080	0.029
	1.50	0.335	0.232	0.095	0.228	0.148	0.074	0.270	0.191	0.086	0.153	0.097	0.026	0.179	0.101	0.031
	2.00	0.321	0.231	0.105	0.264	0.179	0.077	0.307	0.227	0.128	0.182	0.107	0.035	0.160	0.103	0.027

Table E.13: Rejection Rates for Separability Test using DEA, Experiment 3 ($r = 3$, continued)

		$T_{1,n}$									$T_{2,n}$					
n	δ	$p=1,\ q=1$			$p=2,\ q=1$			$p=2,\ q=2$			$p=3,\ q=2$			$p=3,\ q=3$		
		.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
1000	0.00	0.109	0.047	0.003	0.110	0.050	0.003	0.102	0.043	0.003	0.091	0.033	0.003	0.098	0.037	0.004
	0.10	0.179	0.105	0.021	0.147	0.071	0.015	0.175	0.094	0.023	0.102	0.044	0.003	0.109	0.051	0.003
	0.20	0.352	0.229	0.079	0.270	0.180	0.042	0.256	0.163	0.049	0.121	0.048	0.005	0.119	0.052	0.005
	0.30	0.465	0.320	0.139	0.388	0.278	0.105	0.335	0.233	0.098	0.140	0.074	0.014	0.123	0.058	0.008
	0.40	0.526	0.394	0.194	0.440	0.313	0.142	0.394	0.282	0.132	0.177	0.091	0.025	0.123	0.059	0.013
	0.50	0.579	0.435	0.222	0.506	0.375	0.182	0.440	0.317	0.147	0.191	0.109	0.022	0.163	0.080	0.012
	0.60	0.592	0.475	0.255	0.524	0.403	0.192	0.479	0.363	0.180	0.216	0.120	0.037	0.172	0.096	0.015
	0.70	0.647	0.534	0.311	0.548	0.418	0.218	0.479	0.358	0.188	0.218	0.131	0.041	0.162	0.086	0.023
	0.80	0.642	0.537	0.328	0.566	0.451	0.247	0.509	0.388	0.192	0.206	0.113	0.030	0.177	0.088	0.013
	0.90	0.667	0.547	0.309	0.566	0.446	0.246	0.548	0.429	0.252	0.207	0.113	0.029	0.172	0.099	0.030
	1.00	0.675	0.561	0.344	0.562	0.458	0.249	0.509	0.391	0.205	0.220	0.137	0.038	0.171	0.093	0.018
	1.50	0.712	0.610	0.403	0.623	0.512	0.296	0.564	0.449	0.266	0.250	0.155	0.038	0.180	0.099	0.024
	2.00	0.729	0.617	0.387	0.607	0.495	0.299	0.565	0.471	0.275	0.241	0.151	0.043	0.190	0.103	0.028

Table E.14: Rejection Rates for Separability Test using FDH, Experiment 3 ($r = 1$)

		$T_{1,n}$						$T_{2,n}$								
n	δ	$p=1,\ q=1$			$p=2,\ q=1$			$p=2,\ q=2$			$p=3,\ q=2$			$p=3,\ q=3$		
		.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
100	0.00	0.193	0.132	0.057	0.256	0.213	0.123	0.200	0.133	0.058	0.150	0.100	0.059	0.169	0.121	0.071
	0.10	0.210	0.139	0.063	0.274	0.215	0.116	0.175	0.120	0.056	0.172	0.118	0.062	0.159	0.098	0.054
	0.20	0.281	0.203	0.100	0.263	0.215	0.135	0.188	0.115	0.047	0.169	0.123	0.055	0.173	0.116	0.055
	0.30	0.327	0.259	0.142	0.295	0.245	0.148	0.200	0.139	0.058	0.165	0.104	0.055	0.173	0.111	0.064
	0.40	0.379	0.309	0.179	0.287	0.229	0.148	0.184	0.131	0.054	0.168	0.116	0.059	0.178	0.127	0.081
	0.50	0.433	0.351	0.227	0.311	0.248	0.161	0.220	0.157	0.067	0.167	0.123	0.068	0.192	0.133	0.072
	0.60	0.482	0.412	0.277	0.317	0.255	0.174	0.197	0.143	0.071	0.170	0.127	0.064	0.191	0.130	0.062
	0.70	0.503	0.416	0.283	0.347	0.287	0.201	0.221	0.154	0.086	0.167	0.113	0.054	0.191	0.118	0.056
	0.80	0.539	0.446	0.301	0.338	0.286	0.199	0.195	0.144	0.066	0.194	0.136	0.073	0.180	0.123	0.062
	0.90	0.613	0.538	0.385	0.357	0.297	0.200	0.216	0.149	0.066	0.188	0.118	0.059	0.168	0.122	0.063
	1.00	0.639	0.575	0.448	0.366	0.308	0.224	0.230	0.162	0.072	0.211	0.148	0.072	0.203	0.142	0.071
	1.50	0.700	0.619	0.479	0.406	0.352	0.264	0.237	0.174	0.085	0.171	0.124	0.054	0.213	0.160	0.080
	2.00	0.711	0.656	0.529	0.447	0.390	0.270	0.239	0.169	0.076	0.185	0.145	0.091	0.180	0.137	0.077
200	0.00	0.196	0.132	0.038	0.237	0.175	0.091	0.128	0.085	0.022	0.152	0.107	0.049	0.116	0.085	0.036
	0.10	0.214	0.135	0.051	0.241	0.178	0.102	0.131	0.083	0.032	0.132	0.084	0.026	0.134	0.082	0.039
	0.20	0.310	0.211	0.107	0.254	0.197	0.112	0.162	0.102	0.037	0.126	0.083	0.029	0.116	0.081	0.036
	0.30	0.450	0.372	0.201	0.312	0.243	0.157	0.179	0.097	0.031	0.157	0.104	0.041	0.131	0.090	0.039
	0.40	0.525	0.442	0.297	0.373	0.303	0.186	0.173	0.117	0.047	0.143	0.097	0.035	0.135	0.086	0.044
	0.50	0.647	0.561	0.404	0.417	0.355	0.249	0.208	0.120	0.054	0.149	0.105	0.041	0.175	0.118	0.059
	0.60	0.734	0.660	0.504	0.437	0.368	0.250	0.193	0.132	0.044	0.146	0.094	0.038	0.152	0.098	0.059
	0.70	0.789	0.720	0.558	0.491	0.413	0.302	0.224	0.154	0.073	0.153	0.099	0.042	0.135	0.084	0.031
	0.80	0.809	0.754	0.616	0.514	0.456	0.333	0.233	0.164	0.071	0.155	0.104	0.048	0.159	0.090	0.044
	0.90	0.858	0.815	0.696	0.540	0.471	0.337	0.251	0.187	0.082	0.157	0.104	0.043	0.152	0.108	0.052
	1.00	0.875	0.823	0.691	0.578	0.500	0.375	0.297	0.204	0.093	0.151	0.097	0.038	0.162	0.106	0.043
	1.50	0.934	0.906	0.810	0.663	0.600	0.481	0.304	0.212	0.107	0.174	0.111	0.052	0.136	0.087	0.043
	2.00	0.950	0.925	0.862	0.724	0.664	0.550	0.352	0.253	0.132	0.168	0.118	0.047	0.168	0.114	0.048

Table E.14: Rejection Rates for Separability Test using FDH, Experiment 3 ($r = 1$, continued)

		$T_{1,n}$						$T_{2,n}$								
n	δ	$p=1,\ q=1$			$p=2,\ q=1$			$p=2,\ q=2$			$p=3,\ q=2$			$p=3,\ q=3$		
		.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
1000	0.00	0.155	0.081	0.020	0.170	0.113	0.042	0.107	0.050	0.012	0.101	0.045	0.007	0.097	0.051	0.017
	0.10	0.336	0.233	0.088	0.277	0.192	0.097	0.111	0.058	0.018	0.105	0.059	0.012	0.118	0.058	0.017
	0.20	0.748	0.658	0.475	0.526	0.427	0.260	0.166	0.095	0.017	0.097	0.050	0.007	0.109	0.054	0.016
	0.30	0.938	0.904	0.771	0.764	0.688	0.518	0.222	0.129	0.040	0.113	0.047	0.012	0.115	0.066	0.017
	0.40	0.981	0.968	0.926	0.909	0.858	0.731	0.292	0.190	0.059	0.149	0.079	0.025	0.112	0.070	0.015
	0.50	1.000	0.998	0.985	0.936	0.899	0.826	0.387	0.288	0.112	0.152	0.089	0.028	0.142	0.087	0.028
	0.60	0.998	0.996	0.987	0.975	0.958	0.915	0.466	0.346	0.160	0.209	0.113	0.043	0.140	0.089	0.028
	0.70	1.000	0.999	0.998	0.990	0.975	0.941	0.528	0.392	0.176	0.245	0.146	0.038	0.149	0.084	0.019
	0.80	1.000	1.000	1.000	0.995	0.990	0.976	0.536	0.401	0.209	0.214	0.135	0.041	0.163	0.084	0.032
	0.90	1.000	1.000	0.999	0.997	0.994	0.982	0.624	0.505	0.273	0.247	0.158	0.055	0.182	0.111	0.032
	1.00	1.000	1.000	0.999	0.997	0.997	0.992	0.640	0.520	0.296	0.294	0.193	0.077	0.185	0.120	0.043
	1.50	1.000	1.000	1.000	1.000	1.000	0.998	0.718	0.606	0.371	0.386	0.269	0.120	0.242	0.155	0.053
	2.00	1.000	1.000	1.000	0.999	0.999	0.999	0.793	0.691	0.486	0.403	0.269	0.125	0.264	0.171	0.067

Table E.15: Rejection Rates for Separability Test using FDH, Experiment 3 ($r = 2$)

		$T_{1,n}$						$T_{2,n}$								
		$p=1,\ q=1$			$p=2,\ q=1$			$p=2,\ q=2$			$p=3,\ q=2$			$p=3,\ q=3$		
n	δ	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
100	0.00	0.165	0.101	0.032	0.239	0.181	0.105	0.142	0.091	0.034	0.186	0.139	0.078	0.166	0.104	0.058
	0.10	0.210	0.141	0.059	0.261	0.208	0.108	0.155	0.095	0.034	0.154	0.097	0.052	0.168	0.105	0.046
	0.20	0.234	0.160	0.070	0.247	0.190	0.115	0.184	0.120	0.053	0.175	0.104	0.070	0.150	0.100	0.052
	0.30	0.289	0.207	0.119	0.254	0.198	0.140	0.172	0.112	0.050	0.208	0.137	0.059	0.179	0.129	0.066
	0.40	0.299	0.228	0.122	0.264	0.205	0.121	0.194	0.131	0.054	0.194	0.136	0.069	0.182	0.128	0.066
	0.50	0.316	0.247	0.131	0.270	0.200	0.121	0.181	0.123	0.049	0.188	0.122	0.051	0.165	0.111	0.057
	0.60	0.339	0.262	0.134	0.281	0.227	0.136	0.199	0.135	0.062	0.181	0.123	0.061	0.166	0.116	0.056
	0.70	0.346	0.262	0.154	0.272	0.219	0.132	0.185	0.116	0.049	0.173	0.127	0.062	0.170	0.107	0.052
	0.80	0.385	0.295	0.159	0.301	0.230	0.148	0.186	0.124	0.049	0.162	0.119	0.060	0.193	0.126	0.056
	0.90	0.413	0.323	0.194	0.303	0.250	0.158	0.214	0.138	0.060	0.181	0.125	0.070	0.183	0.122	0.069
	1.00	0.385	0.303	0.173	0.295	0.230	0.137	0.213	0.141	0.061	0.176	0.124	0.057	0.185	0.125	0.068
	1.50	0.420	0.341	0.195	0.321	0.247	0.164	0.234	0.166	0.081	0.191	0.138	0.071	0.184	0.132	0.061
	2.00	0.464	0.388	0.249	0.287	0.224	0.149	0.219	0.145	0.077	0.184	0.124	0.067	0.188	0.128	0.068
200	0.00	0.130	0.065	0.018	0.195	0.138	0.064	0.137	0.081	0.028	0.112	0.064	0.025	0.137	0.104	0.049
	0.10	0.204	0.120	0.046	0.201	0.146	0.070	0.142	0.077	0.030	0.130	0.097	0.037	0.120	0.080	0.042
	0.20	0.260	0.167	0.080	0.249	0.180	0.095	0.153	0.088	0.028	0.142	0.095	0.036	0.148	0.098	0.051
	0.30	0.321	0.244	0.125	0.240	0.173	0.091	0.166	0.097	0.032	0.124	0.085	0.044	0.145	0.103	0.042
	0.40	0.394	0.295	0.151	0.280	0.221	0.112	0.175	0.105	0.030	0.141	0.093	0.038	0.130	0.082	0.039
	0.50	0.436	0.346	0.180	0.316	0.247	0.149	0.191	0.114	0.033	0.153	0.099	0.037	0.152	0.097	0.048
	0.60	0.441	0.357	0.210	0.272	0.228	0.144	0.187	0.117	0.051	0.144	0.097	0.035	0.169	0.105	0.046
	0.70	0.498	0.404	0.240	0.322	0.251	0.158	0.197	0.126	0.045	0.152	0.100	0.034	0.163	0.114	0.043
	0.80	0.504	0.420	0.262	0.335	0.266	0.169	0.177	0.112	0.049	0.138	0.083	0.033	0.157	0.106	0.048
	0.90	0.508	0.408	0.262	0.340	0.261	0.168	0.221	0.142	0.049	0.142	0.089	0.039	0.148	0.097	0.046
	1.00	0.551	0.455	0.283	0.366	0.276	0.179	0.194	0.127	0.050	0.148	0.098	0.043	0.151	0.103	0.047
	1.50	0.612	0.513	0.340	0.381	0.305	0.201	0.215	0.147	0.061	0.147	0.088	0.036	0.171	0.103	0.044
	2.00	0.600	0.512	0.343	0.387	0.311	0.204	0.228	0.151	0.062	0.165	0.112	0.045	0.180	0.124	0.054

Table E.15: Rejection Rates for Separability Test using FDH, Experiment 3 ($r = 2$, continued)

		$T_{1,n}$						$T_{2,n}$								
n	δ	$p=1,\ q=1$			$p=2,\ q=1$			$p=2,\ q=2$			$p=3,\ q=2$			$p=3,\ q=3$		
		.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
1000	0.00	0.133	0.062	0.012	0.163	0.085	0.023	0.105	0.039	0.004	0.098	0.048	0.010	0.097	0.054	0.012
	0.10	0.244	0.162	0.055	0.220	0.146	0.058	0.115	0.059	0.007	0.107	0.045	0.005	0.099	0.057	0.011
	0.20	0.552	0.424	0.207	0.421	0.334	0.166	0.141	0.069	0.013	0.094	0.049	0.006	0.097	0.058	0.017
	0.30	0.743	0.666	0.446	0.551	0.456	0.274	0.200	0.100	0.027	0.107	0.056	0.008	0.098	0.047	0.011
	0.40	0.859	0.804	0.625	0.683	0.561	0.371	0.264	0.151	0.035	0.118	0.051	0.012	0.107	0.062	0.020
	0.50	0.909	0.852	0.691	0.750	0.668	0.498	0.290	0.171	0.054	0.127	0.065	0.019	0.117	0.066	0.023
	0.60	0.929	0.898	0.782	0.810	0.735	0.572	0.313	0.212	0.072	0.143	0.066	0.022	0.137	0.069	0.018
	0.70	0.958	0.931	0.838	0.839	0.787	0.638	0.349	0.219	0.059	0.137	0.075	0.020	0.116	0.068	0.018
	0.80	0.964	0.938	0.841	0.850	0.795	0.650	0.342	0.237	0.076	0.156	0.079	0.018	0.132	0.072	0.026
	0.90	0.971	0.950	0.883	0.863	0.806	0.683	0.350	0.236	0.097	0.171	0.092	0.024	0.122	0.067	0.026
	1.00	0.978	0.959	0.885	0.893	0.848	0.734	0.373	0.264	0.112	0.152	0.077	0.024	0.148	0.095	0.017
	1.50	0.987	0.973	0.927	0.909	0.869	0.759	0.418	0.293	0.131	0.167	0.097	0.026	0.149	0.083	0.023
	2.00	0.991	0.983	0.951	0.922	0.882	0.766	0.426	0.290	0.128	0.196	0.105	0.032	0.164	0.087	0.022

Table E.16: Rejection Rates for Separability Test using FDH, Experiment 3 ($r = 3$)

		$T_{1,n}$						$T_{2,n}$								
		$p=1,\ q=1$			$p=2,\ q=1$			$p=2,\ q=2$			$p=3,\ q=2$			$p=3,\ q=3$		
n	δ	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
100	0.00	0.150	0.085	0.021	0.217	0.156	0.078	0.170	0.106	0.042	0.144	0.095	0.050	0.151	0.094	0.048
	0.10	0.196	0.128	0.028	0.238	0.175	0.096	0.167	0.105	0.051	0.126	0.091	0.043	0.161	0.100	0.058
	0.20	0.191	0.129	0.051	0.237	0.183	0.088	0.177	0.107	0.041	0.162	0.101	0.052	0.151	0.102	0.058
	0.30	0.246	0.167	0.071	0.229	0.164	0.087	0.168	0.107	0.053	0.166	0.117	0.052	0.179	0.114	0.053
	0.40	0.251	0.172	0.067	0.227	0.170	0.092	0.199	0.135	0.063	0.163	0.112	0.053	0.185	0.116	0.055
	0.50	0.234	0.166	0.082	0.248	0.198	0.107	0.182	0.122	0.054	0.154	0.095	0.047	0.196	0.150	0.080
	0.60	0.266	0.164	0.072	0.260	0.198	0.112	0.221	0.156	0.073	0.166	0.119	0.048	0.162	0.114	0.048
	0.70	0.299	0.217	0.097	0.273	0.225	0.114	0.179	0.118	0.058	0.152	0.108	0.055	0.170	0.113	0.056
	0.80	0.288	0.212	0.106	0.262	0.191	0.098	0.222	0.146	0.070	0.169	0.117	0.056	0.183	0.136	0.071
	0.90	0.293	0.206	0.098	0.254	0.191	0.108	0.213	0.152	0.072	0.148	0.095	0.044	0.170	0.113	0.052
	1.00	0.320	0.224	0.115	0.274	0.220	0.126	0.218	0.163	0.070	0.157	0.103	0.051	0.174	0.122	0.059
	1.50	0.315	0.221	0.103	0.283	0.224	0.134	0.190	0.134	0.063	0.175	0.120	0.051	0.173	0.115	0.057
	2.00	0.307	0.227	0.128	0.273	0.216	0.131	0.207	0.142	0.071	0.168	0.118	0.063	0.203	0.149	0.082
200	0.00	0.152	0.084	0.018	0.198	0.122	0.047	0.127	0.074	0.017	0.137	0.092	0.046	0.102	0.059	0.022
	0.10	0.173	0.100	0.026	0.225	0.155	0.058	0.149	0.083	0.021	0.131	0.080	0.029	0.131	0.087	0.030
	0.20	0.228	0.141	0.055	0.225	0.161	0.058	0.149	0.074	0.014	0.151	0.098	0.033	0.121	0.083	0.033
	0.30	0.285	0.186	0.078	0.223	0.141	0.071	0.141	0.074	0.021	0.137	0.080	0.035	0.119	0.068	0.025
	0.40	0.269	0.176	0.071	0.230	0.158	0.071	0.153	0.091	0.030	0.118	0.084	0.037	0.128	0.082	0.031
	0.50	0.326	0.232	0.100	0.255	0.177	0.080	0.173	0.108	0.040	0.150	0.085	0.041	0.126	0.089	0.035
	0.60	0.334	0.236	0.102	0.246	0.188	0.086	0.167	0.101	0.028	0.154	0.103	0.035	0.144	0.090	0.036
	0.70	0.335	0.234	0.103	0.268	0.197	0.093	0.182	0.103	0.037	0.153	0.089	0.042	0.148	0.099	0.038
	0.80	0.357	0.269	0.131	0.273	0.188	0.088	0.178	0.105	0.031	0.119	0.060	0.022	0.152	0.087	0.036
	0.90	0.344	0.243	0.114	0.298	0.216	0.119	0.191	0.123	0.055	0.149	0.088	0.034	0.151	0.097	0.042
	1.00	0.366	0.267	0.135	0.307	0.240	0.121	0.176	0.111	0.037	0.142	0.090	0.036	0.161	0.099	0.032
	1.50	0.354	0.266	0.150	0.273	0.194	0.114	0.178	0.108	0.038	0.146	0.100	0.045	0.169	0.109	0.048
	2.00	0.371	0.285	0.158	0.283	0.214	0.107	0.210	0.134	0.049	0.138	0.089	0.037	0.165	0.100	0.042

Table E.16: Rejection Rates for Separability Test using FDH, Experiment 3 ($r = 3$, continued)

n	δ	$T_{1,n}$						$T_{2,n}$								
		$p=1,\ q=1$			$p=2,\ q=1$			$p=2,\ q=2$			$p=3,\ q=2$			$p=3,\ q=3$		
		.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
1000	0.00	0.114	0.053	0.005	0.116	0.071	0.018	0.090	0.038	0.003	0.084	0.034	0.005	0.093	0.046	0.006
	0.10	0.199	0.119	0.029	0.178	0.105	0.026	0.117	0.050	0.006	0.088	0.031	0.002	0.104	0.042	0.006
	0.20	0.376	0.263	0.110	0.300	0.200	0.068	0.152	0.065	0.007	0.095	0.033	0.007	0.097	0.044	0.011
	0.30	0.496	0.374	0.182	0.373	0.277	0.137	0.157	0.078	0.022	0.089	0.036	0.003	0.092	0.046	0.011
	0.40	0.575	0.464	0.254	0.430	0.324	0.174	0.200	0.120	0.028	0.131	0.049	0.010	0.114	0.051	0.017
	0.50	0.654	0.546	0.299	0.494	0.390	0.208	0.187	0.100	0.025	0.141	0.069	0.009	0.108	0.053	0.009
	0.60	0.673	0.558	0.338	0.533	0.419	0.227	0.224	0.138	0.032	0.122	0.056	0.009	0.110	0.049	0.009
	0.70	0.693	0.586	0.396	0.518	0.431	0.263	0.230	0.128	0.035	0.121	0.055	0.010	0.099	0.056	0.011
	0.80	0.689	0.581	0.401	0.544	0.452	0.258	0.231	0.126	0.031	0.132	0.065	0.012	0.116	0.074	0.022
	0.90	0.719	0.636	0.435	0.559	0.458	0.267	0.224	0.124	0.027	0.129	0.064	0.014	0.124	0.069	0.015
	1.00	0.730	0.615	0.418	0.565	0.454	0.275	0.236	0.131	0.027	0.126	0.060	0.019	0.123	0.064	0.013
	1.50	0.766	0.679	0.454	0.605	0.508	0.321	0.245	0.162	0.051	0.134	0.068	0.011	0.127	0.073	0.018
	2.00	0.766	0.669	0.472	0.614	0.508	0.311	0.262	0.169	0.061	0.145	0.073	0.019	0.118	0.066	0.015

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