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Core Existence in Vertically Differentiated Markets

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CORE EXISTENCE IN VERTICALLY DIFFERENTIATED MARKETS*

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ABSTRACT. We prove that a sufficient condition for the core existence in a n-firm vertically differentiated market is that the qualities of firms' products are equally-spaced along the quality spectrum. This result contributes to see that a fully collusive agreement among firms in such markets is more easily reachable when product qualities are not distributed too asymmetrically along the quality ladder.

Keywords: *Vertically Differentiated Markets, Price Collusion, Core, Grand Coalition, Coalition Stability, Games with Externalities, Partition Function Games.*

JEL Classification: D42, D43, L1, L12, L13, L41.

1. Introduction

The main aim of this note is to prove that a sufficient - albeit not necessary - condition for the core existence in a partition function game associated to a n-firm version of the classical vertically differentiated market (e.g., Mussa and Rosen 1978, Gabszewicz and Thisse 1979) is that the qualities of products sold by the firms are equally-spaced along the quality spectrum. In addition, we show that, when this regularity condition is relaxed, the core can be easily empty.

There exist very few contributions dealing with the existence of core in oligopoly games with heterogeneous firms.[1] Our result contributes to see that a fully collusive agreement among firms in such markets is more easily reachable when the product qualities are not distributed too asymmetrically along the quality ladder.

Given that the vertical differentiated market is a setting with strategic interdependence, the most appropriate coalitional game derived from it is a game in partition function (Thrall and Lucas 1963). This in line with the recent interest in coalitional games with externalities (see, e.g., Maskin 2003, Ray 2007, Hafalir 2007, de Clippel and Serrano 2008, Bloch and van den Nouweland 2014, Ray and Vohra 2015). It is well known that, when externalities are at work across coalitions, the use of a coalitional worth requires some assumptions on the expected behaviour of players outside every deviating coalition. In such cases, core allocations may fail to exist even in convex games, for instance when players in the complementary coalition are expected to remain together, as in the *delta core* (Hart and Kurz 1983), also denoted *projection core* in the recent axiomatization by Bloch and van den Nouweland 2014. Moreover, since in the case of vertically differentiated markets the coalitional worth possesses positive coalition externalities,[2] the delta or projection-core is the smallest core and, therefore, its existence implies the existence of all other possible versions of core in games with simultaneous moves. In this paper, we use this notion of core to provide the strongest existence result for the class of games considered here.

2. Vertically Differentiated Market

Let n firms $i = 1, 2, ..., n$ offer n quality variants $q_1, q_2, ..., q_n$, respectively, with $q_i \in (0, \infty)$ and $q_n > q_{n-1} > ... > q_1$ to a population of consumers. As in Mussa and Rosen (1978) consumers are indexed by θ and uniformly distributed in the interval $[0, \beta]$, with $\beta < \infty$. As usual, the parameter θ captures consumers' willingness to pay for quality. Each consumer can either buy one one unit of a variant or not buying at all. Formally, consumer's utility is given by

$$(2.1) \qquad U(\theta) = \begin{cases} \theta q_i - p_i \text{ when buying variant } i \\ 0 \text{ when not buying.} \end{cases}$$

From the above formulation, the marginal consumer buying variant $i = 1$ is

$$\theta_1 = \frac{p_1}{q_1},$$

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[1]Zhao (2013) examines the existence of α-, γ- and δ-core in a three-firm linear Cournot oligopoly with different marginal costs. In a differentiated quantity oligopoly with three (or four firms) Watanabe and Matsubayashi (2013) show that for any degree of product differentiation the $\gamma-$core is nonempty while the δ-core only exists in presence of high product differentiation. For a more detailed account of the works dealing with coalitional agreements in oligopoly games, see Marini 2009 and Currarini and Marini 2015.

[2]This means that every firm is advantaged when rivals merge in coalitions.

and the market is *uncovered,* with some consumers excluded from buying even the bottom-quality variant. In general, the consumer indifferent between buying variant $i+1$ and i for $i = 2, 3, ..., n$ is

$$\theta_i = \frac{p_i - p_{i-1}}{q_i - q_{i-1}}.$$

When considering price competition, the payoffs of all firms can be easily characterized by the payoff of three types of firms in the quality spectrum: (i) top quality (ii) intermediate quality and (iii) bottom quality firm. Since in the model product qualities are exogenously given, we disregard costs to simplify calculations.[3] The top quality firm (denoted $i = n$) sets a price $p_n \in [0, \overline{p}]$ maximizing its profit

$$(2.2) \qquad \Pi_n = \left(\beta - \frac{p_n - p_{n-1}}{q_n - q_{n-1}} \right) p_n,$$

whereas every intermediate firm $i = 2, 3, ..., n - 1$ selects a price $p_i \in [0, \overline{p}]$ to maximize

$$(2.3) \qquad \Pi_i = \left(\frac{p_{i+1} - p_i}{q_{i+1} - q_i} - \frac{p_i - p_{i-1}}{q_i - q_{i-1}} \right) p_i.$$

Finally, the bottom-quality firm ($i = 1$), sets a price $p_1 \in [0, \overline{p}]$ to maximize

$$(2.4) \qquad \Pi_1 = \left(\frac{p_2 - p_1}{q_2 - q_1} - \frac{p_1}{q_1} \right) p_1.$$

Note that, from (2.2)-(2.4), firms' profit functions are continuous and concave in their own prices. Moreover, firms' choice sets are compact and convex and best-replies are *contractions,*[4] so the existence of a unique (noncooperative) Nash equilibrium n-price vector $\mathbf{p}^*$ associated to the n variants $(q_1, q_2, ..., q_n)$ is guaranteed for any (finite) number of firms competing in the market.[5] Moreover, the optimal reply of every firm is given by

$$(2.6) \qquad p_n(p_{n-1}) = \gamma p_{n-1} + \frac{\beta}{2} (q_n - q_{n-1})$$

for the top-quality firm ($i = n$)

$$(2.7) \qquad p_i(p_{i-1}, p_{+1}) = \frac{\gamma p_{i-1}(q_{i+1} - q_i) + \lambda p_{i+1}(q_i - q_{i-1})}{(q_{i+1} - q_{i-1})},$$

for all intermediate firms $i = 2, 3, ..., (n - 1)$ and

$$(2.8) \qquad p_1(p_2) = \gamma \frac{q_1}{q_2} p_2$$

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[3]It can be shown that the presence of quality-dependent fixed costs does not change the nature of the results obtained here.

[4]A sufficient condition for the contraction property to hold is (see, for instance, Vives 2000, p.47):

$$\frac{\partial^2 \Pi_i}{\partial (p_i)^2} + \sum_{j \neq i} \left| \frac{\partial^2 \Pi_i}{\partial p_i \partial p_j} \right| < 0,$$

which, using (2.3) for all intermediate firms $i = 2, ..., n - 1$, becomes

$$(2.5) \qquad -\frac{2 (q_{i+1} - q_{i-1})}{(q_{i+1} - q_i)(q_i - q_{i-1})} + \frac{q_{i+1} - q_{i-1}}{(q_{i+1} - q_i)(q_i - q_{i-1})} = \frac{q_{i-1} - q_{i+1}}{(q_{i+1} - q_i)(q_i - q_{i-1})} < 0$$

which is respected for $q_n > q_{n-1} > ... > q_1$. The same applies for top and bottom quality firms.

[5]See, for instance Friedman (1991), p.84.

for the bottom-quality firm $i = 1$, where $\gamma = \lambda = 1/2$ at the noncooperative equilibrium, $\gamma = \lambda = 1$ both under full collusion and when a firm lies *inside* a coalition of firms, and $\gamma = 1/2$ and $\lambda = 1$ (or $\gamma = 1$ and $\lambda = 1/2$) when the firm competes with its left (right) neighbour and colludes with its right (left) neighbour. This implies that every firm benefits from rivals' cartelisation and the coalitional worth (joint profit) of firms exhibits positive *coalitional* externalities: from (2.6)-(2.8) it ensues that all firms' optimal replies are positively sloped and their slope increases with (partial or full) collusion. Thus, rivals' cartelisation increases all firms' prices and, hence, their payoffs.

2.1. **Grand Coalition Payoff.** When all firms form a cartel they maximize the sum of firms' payoffs. As shown in Gabszewicz *et al.* (2016), under full price collusion all firms set prices p_i^c such that their market shares are nil for all firms but the top-quality one $(i = n)$. This is is easy to show. Using (2.6)-(2.8) with $\gamma = \lambda = 1$ for all firms, the following price is obtained

$$(2.9) \qquad\qquad p_i^c = \frac{1}{2}\beta \sum_{j \le i} \delta_j,$$

where $\delta_j = (q_j - q_{j-1})$ is the quality gap of every firm j selling goods of lower or equal quality than firm i, and $\delta_1 = (q_1 - q_0) = q_1$. Inserting (2.9) in every firm's market share D_i, we obtain:

$$D_1(p_1^c, p_2^c) = \left(\frac{p_2^c - p_1^c}{\delta_2} - \frac{p_1^c}{\delta_1}\right) = \left(\frac{\frac{1}{2}\beta\left(\delta_1 + \delta_2\right) - \frac{1}{2}\beta\delta_1}{\delta_2} - \frac{\frac{1}{2}\beta\delta_1}{\delta_1}\right) = 0$$

for the bottom quality firm,

$$\begin{aligned}
D_i(p_{i-1}^c, p_i^c, p_{i+1}^c) &= \left(\frac{p_{i+1}^c - p_i^c}{\delta_{i+1}} - \frac{p_i^c - p_{i-1}^c}{\delta_{i-1}}\right) = \\
&= \left(\frac{\frac{1}{2}\beta\sum_{j \le i+1}\delta_j - \frac{1}{2}\beta\sum_{j \le i}\delta_j}{\delta_{i+1}} - \frac{\frac{1}{2}\beta\sum_{j \le i}\delta_j - \frac{1}{2}\beta\sum_{j \le i-1}\delta_j}{\delta_{i-1}}\right) = \\
&= \left(\frac{\frac{1}{2}\beta\delta_{i+1}}{\delta_{i+1}} - \frac{\frac{1}{2}\beta\delta_i}{\delta_i}\right) = 0.
\end{aligned}$$

for any intermediate quality firm, and

$$\begin{aligned}
D_n(p_{n-1}^c, p_n^c) &= \left(\beta - \frac{p_n^c - p_{n-1}^c}{q_n - q_{n-1}}\right) = \left(\beta - \frac{\frac{1}{2}\beta\sum_{j \le n}\delta_j - \frac{1}{2}\beta\sum_{j \le n-1}\delta_j}{\delta_n}\right) = \\
&= \left(\beta - \frac{\frac{1}{2}\beta\delta_n}{\delta_n}\right) = \frac{1}{2}\beta,
\end{aligned}$$

for the top quality firm. Thus, when colluding together all firms cover only half of the market and the grand coalition payoff is:

$$(2.10) \qquad\qquad v(N) = \sum_{i \in N} \Pi_i^{\{N\}} = \sum_{i=1}^n p_i D_i = \frac{1}{4}\beta^2 q_n.$$

2.2. **Coalitional Payoffs.** The n firms can also collude organizing themselves in partition $P = (S_1, S_2, ..., S_m)$ different from the grand coalition. Every firm can actively collude in prices only with its left (lower quality), with its right (higher quality) or with both its closest competitors by forming *bottom, intermediate* or *top* quality cartels.[6]

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[6]Without forming cartels among *consecutive* firms, i.e producing adjacent variants firms' collusion does not affect price behaviour.

Definition 1. *(i) A bottom cartel $S_B \subset N$ is a coalition formed by consecutive intermediate firms $i = 2, ..., n - 1$ also including the bottom quality firm $i = 1$. (ii) An intermediate cartel $S_I \subset N$ is a coalition only formed by consecutive intermediate firms $i = 2, ..., n - 1$. (iii) A top cartel $S_T \subset N$ is a coalition formed by consecutive intermediate firms $i = 2, ..., n - 1$, also including the top quality firm $i = n$.*

In the next proposition, we characterize the variants produced by the firms belonging to: (i) an intermediate cartel; (ii) a bottom cartel; (iii) a top cartel. The detailed proofs of these propositions are contained in Gabszewicz *et al.* (2016).

Proposition 1. *(i) A bottom cartel only produces in equilibrium the top quality variant among those formerly produced by its firms. (ii) Any intermediate cartel only produces in equilibrium the top and the bottom quality variants among those formerly produced by its firms. (iii) Any top cartel only produces in equilibrium the top and the bottom quality variants among those formerly produced by its firms.*

Proof. See Gabszewicz *et al.* (2016). □

Proposition 1 enables to characterize the number of variants marketed by the firms in any feasible partition $P = (S_1, S_2, ..., S_m)$ for $m \leq n$ and will be used extensively to prove the main paper result.

3. Core stability

This section analyses the stability of full price collusion, i.e. the situation in which all firms in the industry collude in prices. In particular, the next proposition shows that, when all firm quality variants are *equally spaced*, it is always possible to find a division of the monopoly profit which makes the whole industry cartel stable against individual or coalitional deviations by firms.

We can formally associate to the described vertically differentiated market a *partition function game* $G = (N, v(S; P))$, where N is the set of firms and $v(S; P) : 2^N \times \mathcal{P} \to \mathcal{R}$ is the worth associated to every coalition of firms $S \subset N$ embedded in a partition $P \in \mathcal{P}$, where $\mathcal{P}$ is the set of all feasible partitions of the N firms. We can now define the core of a partition function game.

Definition 2. *A vector of payoffs $x = (x_1, x_2, ..., x_n)$ with $\sum_{i \in N} x_i = v(N)$ is in the core of the partition function game G if, for every $S \subset N$ and every partition P in which S can be embedded, $\sum_{i \in S} x_i \geq v(S; P)$.*

We are now ready to prove our main result:

Proposition 2. *Let market variants $q_1, q_2, ..., q_n$ be equally spaced with $(q_i - q_{i-1}) = \delta \in (0, \infty)$ for every $i = 1, 2, ..., n$, and $q_0 = 0$. Thus, the core of the partition function game G associated to the n-firm vertically differentiated market is nonempty.*

Proof. In our model of vertical differentiation, when a coalition of firms $S \subset N$ forms, its maximal coalitional payoff is obtained when the remaining firms in $N - S$ stick together in the complementary coalition $\{N - S\}$. Therefore, if the core is nonempty when the coalitional worth $v(S; P)$ is computed for $P = \{S, N - S\}$, it will *a fortiori* be nonempty under any other partition $P \in \mathcal{P}$ in which S can be embedded. For this reason, in what follows, we only need to prove that there exists an allocation $\mathbf{x} = (x_1, x_2, ..., x_n)$ of the grand coalition

payoff $v(N)$ such that, for all $S \subset N$, $\sum_{i \in S} x_i \geq v(S; \{S, N - S\})$. In particular, we prove this result by constructing a specific allocation respecting this requirement. Since the payoff obtained by every firm i in partition $P = \{i, N - i\}$ is crucial to build such allocation, let us start from it. We consider first the payoff of the top quality firm (denoted $i = n$), in partition $P = \{n, N - n\}$. In this case, by Proposition 1, only two variants remain on sale, q_n from firm n and q_{n-1} from the remaining firms merged in the *bottom cartel* $S_B = \{N - n\}$. As a result, in the new equilibrium under equally spaced variants[7]

$$(3.1) \qquad v(n; \{n, N - n\}) = \Pi_n^{\{n, N-n\}} = \frac{4\beta^2 q_n^2 (q_n - q_{n-1})}{(4q_n - q_{n-1})^2} = \frac{4\beta^2 \delta n^2}{(3n + 1)^2}.$$

As a second step, let us consider the payoff of the bottom-quality firm in partition $P = \{1, N - 1\}$. By Proposition 1, in this case only three variants remain on sale, q_1, q_2 and q_n, where q_2 and q_n are offered by the firms merged in the *top cartel* $S_T = \{N - 1\}$. In this new equilibrium, the payoff obtained by firm $i = 1$ is

$$(3.2) \qquad v(1; \{1, N - 1\}) = \Pi_1^{\{1, N-1\}} = \frac{\beta^2 q_1 q_2 (q_2 - q_1)}{(4q_2 - q_1)^2} = \frac{2\beta^2 \delta}{49}.$$

Finally, let us consider the payoff obtained by every *intermediate* firm $i = 2, ..., n - 1$ in partition $P = \{S_B, i, S_T\}$, where S_B and S_T are the bottom and top cartel neighbouring firm i. In this case, at most four variants remain on sale, namely q_{i-1} from S_B, q_i from i and q_{i+1} and q_n from S_T, yielding:

$$(3.3) \quad v(i; \{i, N - i\}) = \Pi_i^{\{i, N-i\}} = \frac{q_i^2 \beta^2 (q_i - q_{i-1})(q_{i+1} - q_i)(q_{i+1} - q_{i-1})}{(2q_{i-1}q_i + q_{i-1}q_{i+1} - 4q_i q_{i+1} + q_i^2)^2} = \frac{\delta \beta^2 (i)^2}{(6i + 1)^2}.$$

Now, using (2.10) and (3.1)-(3.3) it is easy to see that, under equally-spaced variants, inequality

$$v(N) \geq v(1; \{1, N - 1\}) + \sum_{i=2}^{n-1} v(i, \{i, N - i\}) + v(n; \{n, N - n\}),$$

writes as

$$(3.4) \qquad \frac{n}{4} \geq \frac{2}{49} + \sum_{i=2}^{n-1} \frac{(i)^2}{(6i + 1)^2} + \frac{4n^2}{(3n + 1)^2},$$

and the latter expression holds with strict inequality for any number of firms $n \geq 2$.

Let us now construct a specific allocation $\widehat{x} = (\widehat{x}_1, \widehat{x}_2, ..., \widehat{x}_n)$ assigning to every firm $i = 1, 2, ..., n$ a share s_i of the grand coalition payoff $v(N)$ equal to

$$s_i = \frac{v(i; \{i, N - i\})}{\sum_{i \in N} v(i; \{i, N - i\})},$$

such that $\sum_{i \in N} s_i = 1$, that is

$$(3.5) \qquad \widehat{x} = (s_1 v(N), s_2 v(N), ..., s_n v(N)).$$

Thus, since (3.4) holds with strict inequality, it ensues that for every firm $i = 1, 2, ..., n$

$$\widehat{x}_i = s_i v(N) > s_i \sum_{i=1}^{n} v(i; \{i, N - i\}) = v(i; \{i, N - i\}),$$

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[7]That is, for $(q_1 - q_0) = (q_2 - q_1) = .. = (q_n - q_{n-1}) = \delta$, and, hence, $q_1 = \delta$, $q_2 = 2\delta, ..., q_n = n\delta$.

implying that the selected allocation $\widehat{x} \in \mathcal{R}^n$ is robust against any individual firm's deviations.

As a second step, we need to look at the payoff obtained by any feasible coalition of firms. Let us assume again that every forming coalition expects the remaining firms to stick together in the complementary coalition (*delta* or *projection* expectations). As a result, when a coalition of firms in a bottom cartel $S_B \subset N$ forms under partition $P = \{S_B, N - S_B\}$, by Proposition 1 only variants q_h, q_{h+1} and q_n remain on sale (q_h from S_B and q_{h+1} and q_n from $\{N - S_B\}$), where q_h denotes the highest quality variant in S_B. The worth of any bottom cartel S_B is, therefore,

(3.6)

$$v\left(S_B; \{S_B, N - S_B\}\right) = \sum_{i \in S_B} \Pi_i = \Pi_{h=\max\{i\}_{i \in S_B}} = \frac{\beta^2 q_h q_{h+1}\left(q_{h+1} - q_h\right)}{\left(4q_{h+1} - q_h\right)^2} = \frac{\delta\beta^2 h\left(h + 1\right)}{\left(3h + 4\right)^2}.$$

From (3.5) and (3.6), for every $S_B \subset N$ inequality

$$\sum_{i \in S_B} \widehat{x}_i \geq v\left(S_B; \{S_B, N - S_B\}\right)$$

writes, under equally-spaced variants, as

(3.7)
$$\frac{\frac{2}{49} + \displaystyle\sum_{i=2}^{h} \frac{(i)^2}{(6i+1)^2}}{\frac{2}{49} + \displaystyle\sum_{i=2}^{n-1} \frac{(i)^2}{(6i+1)^2} + \frac{4n^2}{(3n+1)^2}} \frac{n}{4} \geq \frac{h\left(h + 1\right)}{\left(3h + 4\right)^2},$$

and (3.7) holds with strict inequality for every number of firms n and for every $h = 2, ...n - 1$. Expression (3.7) ensures that no *bottom cartel* S_B can improve upon $\sum_{i \in S_B} \widehat{x}_i$, the joint payoff assigned by allocation $\widehat{x} \in \mathcal{R}^n$ to members of S_B.

When, in turn, a *top cartel* $S_T \subset N$ forms under partition $P = \{S_T, N - S_T\}$, only three variants remain on sale, q_{l-1} from $N - S_T$ and q_l and q_n from S_T, for l denoting the lowest quality firm in S_T. Hence, under equally-spaced variants,

$$\begin{aligned}
v(S_T; \{S_T, N - S_T\}) &= \sum_{i \in S_T} \Pi_i^{\{S_T, N-S_T\}} = \Pi_{l=\min\{i\}_{i \in S_T}} + \Pi_n = \\
&= \frac{\beta^2 q_l q_{l-1}\left(q_l - q_{l-1}\right)}{\left(4q_l - q_{l-1}\right)^2} + \frac{1}{4}\frac{\beta^2\left(4q_l q_n - q_{l-1}q_n - 3q_{l-1}q_l\right)}{\left(4q_l - q_{l-1}\right)} = \\
&= \frac{\delta\beta^2 l(l + 1)}{\left(3l + 4\right)^2} + \frac{1}{4}\frac{\delta\beta^2\left(3l + n + 3l \cdot n - 3l^2\right)}{\left(3l + 1\right)}.
\end{aligned}$$

Note that this expression is decreasing in l, since the highest l the smaller is the size of the top cartel S_T. Now, for every $S_T \subset N$,

$$\sum_{i \in S_T} \widehat{x}_i \geq v\left(S_T; \{S_T, N - S_T\}\right)$$

under equally-spaced variants corresponds to

$$\frac{\sum\limits_{i=l}^{n-1}\frac{(i)^2}{(6i+1)^2}+\frac{4n^2}{(3n+1)^2}}{\frac{2}{49}+\sum\limits_{i=2}^{n-1}\frac{(i)^2}{(6i+1)^2}+\frac{4n^2}{(3n+1)^2}}\frac{n}{4}\geq\frac{l(l+1)}{(3l+4)^2}+\frac{1}{4}\frac{(3l+n+3l\cdot n-3l^2)}{(3l+1)},$$

which holds with strict inequality for every number of firms n and every $l=2,...n-1$. Finally, when an intermediate cartel $S_I\subset N$ forms under partition $P=\{S_B,S_I,S_T\}$, by Proposition 1 at most five variants remain on sale: q_{l-1} from S_B, q_l and q_h from S_I, and q_{h+1} and q_n from S_T, where, in turn, l and h stands for the lowest and highest quality firms in cartel S_I. The payoff obtainable by an intermediate cartel is, therefore,

$$v(S_I;\{S_B,S_I,S_T\})=\Pi_{S_I}^{\{S_B,S_I,S_T\}}=\frac{4\beta^2 q_{l-1}q_l\left(q_l-q_{l-1}\right)\left(q_{h+1}-q_h\right)^2}{\left(q_{l-1}q_h-9q_{l-1}q_l-4q_{l-1}q_{h+1}-4q_lq_h+16q_lq_{h+1}\right)^2}+$$

$$+\frac{\beta^2\left(q_{h+1}-q_h\right)\left(4q_lq_{h+1}-3q_{l-1}q_l-q_{l-1}q_{h+1}\right)\left(4q_lq_h-q_{l-1}q_h-3q_{l-1}q_l\right)}{\left(q_{l-1}q_h-9q_{l-1}q_l-4q_{l-1}q_{h+1}-4q_lq_h+16q_lq_{h+1}\right)^2},$$

that, under equally spaced variants, can be written as

$$\Pi_{S_I}^{\{S_B,S_I,S_T\}}=\frac{1}{16}\frac{\delta\beta^2\left(3l^2-6l-3hl-h-1\right)\left(3l^2-3l-3hl-h\right)}{\left(h+5l+2hl-2l^2+1\right)^2}+\frac{\delta\beta^2\left(l-1\right)l}{4\left(3h+21l+9hl-9l^2+4\right)^2}.$$

Thus, for every $S_I\subset N$

$$\sum_{i\in S_I}\widehat{x}_i\geq v\left(S_I;\{S_I,N-S_I\}\right)$$

is

$$\frac{\sum\limits_{i=l}^{h}\frac{(i)^2}{(6i+1)^2}\Big)}{\frac{2}{49}+\sum\limits_{i=2}^{n-1}\frac{(i)^2}{(6i+1)^2}+\frac{4n^2}{(3n+1)^2}}\frac{n}{4}\geq\frac{1}{16}\frac{\left(3l^2-6l-3hl-h-1\right)\left(3l^2-3l-3hl-h\right)}{\left(h+5l+2hl-2l^2+1\right)^2}+\frac{(l-1)l}{4\left(3h+21l+9hl-9l^2+4\right)^2},$$

which, again, holds for any number of firms n and any $l=2,...n-2$ and $h=3,...,n-1$, with $l<h$. As a result, the selected allocation $\widehat{x}$ distributes the grand coalition payoff in a way that no coalition of firms $S\subset N$ can, by leaving the grand coalition N, obtain a better payoff. The core is, therefore, nonempty. $\square$

3.1. Endogenous Qualities.

It can be shown that, when $N=\{1,2,3\}$, the core is nonempty also when firms are allowed to select endogenously both qualities and prices. Following Gabszewicz *et al.* (2015), the grand coalition sets endogenously a product quality $q^{\{N\}}=0.25$ and, hence, $v(N)=0.03125\beta$, which is sufficient to prevent individual deviations, given that:

$$v(N)=0.03125\beta>v(1;\{1,23\})+v(2;\{2,13\})+v(3;\{12,3\})=0.00152\beta+0.00152\beta+0.02443\beta.$$

Moreover, there exist allocations $x=(x_1,x_2,x_3)$ distributing $v(N)$ in such a way that no coalition $S\subset N$, by selecting its optimal quality and price, has an incentive to deviate. Using

our sharing rule $s = (s_1, s_2, s_3) = (0.0533, 0.0533, 0.8893)$, we obtain that

$$\sum_{i \in \{1,2\}} \widehat{x}_i = 0.0033\beta > v(12; \{12, 3\}) = 0.00152\beta,$$

$$\sum_{i \in \{1,3\}} \widehat{x}_i = 0.0945\beta > v(13; \{13, 2\}) = 0.02443\beta,$$

$$\sum_{i \in \{2,3\}} \widehat{x}_i = 0.0945\beta > v(23; \{1, 23\}) = 0.02443\beta,$$

and the core is, therefore, nonempty. However, it can be shown that, with only three firms, the nonemptiness of core always holds for any distribution of the product qualities. For core emptiness to arise, the existence of at least four firms are required, as the next example will show.

3.2. **An Empty Core Example.** Let us consider the case of four firms selling four different variants q_1, q_2, q_3 and q_4. In this case, if the top cartel $S_T = \{234\}$ decides to leave the grand coalition $\{N\}$ and partition $P = \{1, 234\}$ forms, it gains:

$$v((234); \{1, 234\}) = \Pi_{234}^{(\{1\}, \{2,3,4\})} = \frac{\beta^2 q_2 q_3 (q_3 - q_2)}{(4q_3 - q_2)^2} + \frac{1}{4}\frac{\beta^2 (4q_2 q_4 - q_1 q_4 - 3q_1 q_2)}{(4q_2 - q_1)},$$

while firm 1 obtains

$$v(1; \{1, 234\}) = \Pi_1^{\{1,234\}} = \frac{\beta^2 q_1 q_2 (q_2 - q_1)}{(4q_2 - q_1)^2}.$$

Note that, for $\beta = 1$, $q_1 = 1$, $q_2 = 5$ and $q_4 = 10$ and $q_3 > 7.26$, the quality gap between q_2 and q_3 (both produced inside the cartel) becomes sufficiently high for

$$\Pi_1^{\{1,234\}} + \Pi_{234}^{\{1,234\}} > \Pi_N^{\{N\}} = v(N) = \frac{1}{4}\beta^2 q_4 = 2.5$$

and the core is, as a result, empty. If, instead, products are equally spaced, with $q_1 = 2.5$, $q_2 = 5$, $q_3 = 7.5$ and $q_4 = 10$,

$$\Pi_1^{\{1,234\}} + \Pi_{234}^{\{1,234\}} = 2.21 < \Pi_N^{\{N\}}$$

and, in addition, also all other feasible deviations by single or coalitions of firms cannot in any way improve upon the grand coalition payoff. Core existence is, in such a way, re-established.

4. CONCLUDING REMARKS

In this paper we have shown that in a vertically differentiated market when the variants marketed by the firms are equally spaced, a price fully collusive agreement is core-stable. When this regularity condition is relaxed, the core can be easily empty.

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Proposition 1. *(i) A bottom cartel only produces in equilibrium the top quality variant among those formerly produced by its firms. (ii) Any intermediate cartel only produces in equilibrium the top and the bottom quality variants among those formerly produced by its firms. (iii) Any top cartel only produces in equilibrium the top and the bottom quality variants among those formerly produced by its firms.*

Proof. (Gabszewicz et al. 2016)**.** We first prove (ii) and then the proof easily extends to (iii) and, with slight modifications, to (i).Take a generic *intermediate* cartel $S_I \subset N$ made of k firms, with $k \leq |N - 2|$. Before the cartel is formed, these firms are selling variants denoted $(q_i, q_{i+1}, q_{i+2}, ..., q_{i+k})$ and competing with, in turn, a left-hand fringe of independent firms selling lower quality variants $q_1, q_2, ..., q_{i-1}$, and with a righ-hand fringe selling higher quality variants $q_{i+k+1}, q_{i+k+2}, ..., q_n$. The optimal-replies of firms in the cartel can be written as, respectively,

$$p_i(p_{i-1}, p_{i+1}) = \frac{\frac{1}{2}p_{i-1}(q_{i+1} - q_i) + p_{i+1}(q_i - q_{i-1})}{(q_{i+1} - q_{i-1})}$$

$$p_{i+1}(p_i, p_{i+2}) = \frac{p_i(q_{i+2} - q_{i+1}) + p_{i+2}(q_{i+1} - q_i)}{(q_{i+2} - q_i)}$$

$$p_{i+2}(p_{i+1}, p_{i+3}) = \frac{p_{i+1}(q_{i+3} - q_{i+2}) + p_{i+3}(q_{i+2} - q_{i+1})}{(q_{i+3} - q_{i+1})}$$
$$..............................$$
$$..............................$$
$$..............................$$
$$p_{i+k}(p_{i+k-1}, p_{i+k+1}) = \frac{p_{i+k-1}(q_{i+k+1} - q_{i+k}) + \frac{1}{2}p_{i+k+1}(q_{i+k} - q_{i+k-1})}{q_{i+k+1} - q_{i+k-1}},$$

where only the two extreme firms i and $i + k$ in the cartel are directly competing with firms outside. Without loss of generality, take a generic firm inside the cartel selling an intermediate variant (i.e neither the bottom nor the top quality in the cartel), say firm $i+1$. Using both the optimal reply of firm $i + 1$ and those of the firms connected to it (i.e. firms i and $i + 2$) and re-arranging, we obtain the optimal replies of these three firms as functions of p_{i-1} and p_{i+3} only.

$$\tilde{p}_i = p_i(p_{i-1}, p_{i+3}) = \frac{1}{2}\frac{p_{i-1}(q_{i+3} - q_i) + 2p_{i+3}(q_i - q_{i-1})}{q_{i+3} - q_{i-1}},$$

$$\tilde{p}_{i+1} = p_{i+1}(p_{i-1}, p_{i+3}) = \frac{1}{2}\frac{p_{i-1}(q_{i+3} - q_{i+1}) + 2p_{i+3}(q_{i+1} - q_{i-1})}{q_{i+3} - q_{i-1}},$$

$$\tilde{p}_{i+2} = p_{i+2}(p_{i-1}, p_{i+3}) = \frac{1}{2}\frac{p_{i-1}(q_{i+3} - q_{i+2}) + 2p_{i+3}(q_{i+2} - q_{i-1})}{q_{i+3} - q_{i-1}}.$$

Using the above, we can easily compute the optimal market share of firm $(i + 1)$ as

$$D_{i+1}(\tilde{p}_i, \tilde{p}_{i+1}, \tilde{p}_{i+2}) = \frac{\tilde{p}_{i+2} - \tilde{p}_{i+1}}{q_{i+2} - q_{i+1}} - \frac{\tilde{p}_{i+1} - \tilde{p}_i}{q_{i+1} - q_i} = 0$$

which proves that under partial collusion every intermediate firm of an *intermediate* cartel obtains zero market share. Repeating now the same procedure for the firm producing the

lowest quality in the cartel (here firm i), we obtain instead that

$$D_i(\tilde{p}_i, \tilde{p}_{i+1}, \tilde{p}_{i-1}) = \frac{\tilde{p}_{i+1} - \tilde{p}_i}{q_{i+1} - q_i} - \frac{\tilde{p}_i - \tilde{p}_{i-1}}{q_i - q_{i-1}} = \frac{1}{2}\frac{\tilde{p}_{i-1}}{(q_i - q_{i-1})} > 0$$

for $\tilde{p}_{i-1} > 0$. Finally, computing the optimal replies of the highest quality firm in the cartel, i.e. firm $(i + k)$, and of the firms directly connected to it, we obtain

$$\tilde{p}_{i+k-1}(p_{i+k-2}, p_{i+k}) = \frac{p_{i+k-2}(q_{i+k-1} - q_{i+k-2}) + p_{i+k}(q_{i+k-1} - q_{i+k-2})}{q_{i+k} - q_{i+k-2}}$$

$$\tilde{p}_{i+k}(p_{i+k-1}, p_{i+k+1}) = \frac{p_{i+k-1}(q_{i+k+1} - q_{i+k}) + \frac{1}{2}p_{i+k+1}(q_{i+k} - q_{i+k-1})}{q_{i+k+1} - q_{i+k-1}}$$

$$\tilde{p}_{i+k+1}(p_{i+k}, p_{i+k+2}) = \frac{1}{2}\frac{p_{i+k}(q_{i+k+2} - q_{i+k+1}) + p_{i+k+2}(q_{i+k+1} - q_{i+k})}{q_{i+k+2} - q_{i+k}}.$$

Using the above,

$$D_{i+k}(\tilde{p}_{i+k-1}, \tilde{p}_{i+k}, \tilde{p}_{i+k+1}) = \frac{\tilde{p}_{i+k+1} - \tilde{p}_{i+k}}{q_{i+k+1} - q_{i+k}} - \frac{\tilde{p}_{i+k} - \tilde{p}_{i+k-1}}{q_{i+k} - q_{i+k-1}} =$$

$$= \frac{1}{2}\frac{\tilde{p}_{i+k+1}}{(q_{i+k} - q_{i+k-1})} > 0.$$

showing that only the variants produced by the two firms at the extremes of this (generic) intermediate cartel are sold at prices implying *positive* market shares.

(iii) Exactly the same procedure can be replicated to prove that, in a *top cartel* $S_T \subset N$ only the highest and lowest quality variants initially sold by the cartel remain on sale. (i) Finally, let us consider a *bottom cartel* $S_B \subset N$, i.e. a cartel formed by firms $1, 2, ..., k$ initially selling k variants $q_1, q_2, ...q_k$ and competing with $(n - k)$ independent firms selling higher quality variants $q_{k+1}, q_{k+2}, ..., q_n$. Again, we can apply the same argument used above to show that every firm in the *interior* of the cartel (i.e neither selling its lowest quality nor its highest quality variant in the cartel) obtains zero market share. Also, for the top quality firm in the cartel (here firm k), we obtain that $D_k(\tilde{p}_k, \tilde{p}_{k-1}, \tilde{p}_{k+1}) > 0$. Finally, when considering the firm selling the lowest quality variant in the *bottom* cartel, its market share is:

$$D_1(p_2, p_1) = \frac{p_2 - p_1}{q_2 - q_1} - \frac{p_1}{q_1} = 0,$$

that, by simply substituting firm 1 optimal reply

$$p_1(p_2) = \frac{q_1}{q_2}p_2$$

becomes

$$D_1(p_2, \tilde{p}_1) = \frac{p_2 - \frac{q_1}{q_2}p_2}{q_2 - q_1} - \frac{\frac{q_1}{q_2}p_2}{q_1} = 0,$$

showing that, differently from all other cartels, the *bottom cartel* only produces its top-quality variant q_k. $\square$

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