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It is a matter of hierarchy: a Nash equilibrium problem perspective on bilevel programming

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Abstract

Inspired by the optimal value approach, we propose a new reformulation of the optimistic Bilevel programming Problem (BP) as a suitable Generalized Nash Equilibrium Problem (GNEP). We provide a complete analysis of the relationship between the original hierarchical BP and the corresponding "more democratic" GNEP. Moreover, we investigate solvability and convexity issues of our reformulation. Finally, relying on the vast literature on solution methods for GNEPs, we devise a new effective algorithmic framework for the solution of significant classes of BPs.

Keywords: Bilevel programming, Generalized Nash Equilibrium Problems (GNEP), parametric optimization, numerical approaches.

1 Introduction

Bilevel programming is a fruitful modeling framework that is widely used in many fields, ranging from economy and engineering to natural sciences (see [4], the fundamental [5], [6], the recent [10], the references therein, and the seminal paper [20]).

Bilevel programs have a hierarchical structure involving two decision levels. At the *upper* level an optimization is carried out with respect to two blocks of variables, namely x and y; but, in turn, y is implicitly constrained by the reaction of a subaltern (*lower* level) part to the choice of the first variable block x. Thus, bilevel problems can be viewed, in some sense, as a special two-agents optimization. The two agents play here an asymmetric role, in that the variable block x is controlled only by the upper level agent, while the second block y is controlled by both the upper and the lower level agents. It is precisely this asymmetrically shared control on the variable blocks that makes bilevel programs inherently hard to solve. It is worth noting that, whenever there is not such a thorny relationship between the agents, things become much simpler. Indeed, on the one hand, if all the variables are controlled by both the agents, we have a (conceptually) simpler pure hierarchical problem; while, on the other hand, with x being controlled by the upper level agent, if y is controlled only by the lower level agent, we get a game, in which the two agents act as players.

In this paper, we address the optimistic Bilevel programming Problem (BP) (see [5, 10]). This problem, in its full generality, is structurally nonconvex and nonsmooth (see [8]); furthermore, it is hard to define suitable constraint qualification conditions for it, see, e.g., [9, 22]. In fact, the study of provably convergent and practically implementable algorithms for the solution of BPs is still in its infancy (see, as a matter of example, [3, 6, 18, 21]), as also witnessed by the scarcity of results in the literature. Yet, some approaches have been proposed mainly in order to investigate optimality conditions and constraint qualifications: to date, the most studied and promising are optimal value and KKT one level reformulations (see [10], the references therein and [17, 23]). As far as the KKT reformulation is concerned, it should be remarked that the BP has often be considered as a special case of Mathematical Program with Complementarity Constraints (MPCC). Actually, this is not the case, as shown in [7]: indeed, a local solution of the MPCC, which is what one can expect to compute, may happen not to be a local optimal solution of the corresponding BP.

The main contributions of our analysis are: (i) we propose a new reformulation of a BP as a suitable Generalized Nash Equilibrium Problem (GNEP) (see [11, 12, 13, 14, 15]); (ii) we investigate properties, e.g., solvability issues and convexity, of the resulting GNEP; (iii) we capitalize on the proposed GNEP reformulation to devise an algorithmic framework for the solution of significant classes of BPs.

We remark that our GNEP reformulation is slightly related to the optimal value approach, in that, when passing from the bilevel problem to the "more democratic" GNEP, we exploit the value function idea to mimic the originally hierarchical relationship between the agents. We study connections between the original problem and our GNEP reformulation for classes of BPs and we point out strengths, as well as inevitable drawbacks of our approach. Moreover, we give new sufficient optimality conditions for BPs based on our reformulation.

Numerical testing of our algorithmic framework is underway but preliminary results show the effectiveness of the proposed approach.

2 Generalized Nash equilibrium problem reformulation

We consider the following optimistic bilevel programming problem (BP):

$$\begin{array}{ll} \underset{x,y}{\text{minimize}} & F(x,y) \\ \text{s.t.} & x \in X \\ & y \in S(x), \end{array} \tag{1}$$

where $F : \mathbb{R}^{n_0} \times \mathbb{R}^{n_1} \to \mathbb{R}$ is a continuous function and $X \subseteq \mathbb{R}^{n_0}$ is a closed set; the set-valued mapping $S : \mathbb{R}^{n_0} \rightrightarrows \mathbb{R}^{n_1}$ describes the solution set of the following lower level parametric optimization problem:

$$\begin{array}{ll} \underset{w}{\text{minimize}} & f(x,w) \\ \text{s.t.} & w \in U \\ & g(x,w) \leq 0, \end{array} \tag{2}$$

where $f : \mathbb{R}^{n_0} \times \mathbb{R}^{n_1} \to \mathbb{R}$ and $g : \mathbb{R}^{n_0} \times \mathbb{R}^{n_1} \to \mathbb{R}^m$ are continuous functions, and $U \subseteq \mathbb{R}^{n_1}$ is a closed set. Besides, let $W \triangleq \{(x,y) \in X \times S(x)\}$ and $U \cap K(x)$, with $K(x) \triangleq \{v \in \mathbb{R}^{n_1} : g(x,v) \leq 0\}$, denote the feasible sets of problems (1) and (2), respectively.

A point (x^*, y^*) is a global solution of problem (1) if

$$\begin{aligned} &(x^*, y^*) \in W, \\ &F(x^*, y^*) \leq F(x,y), \qquad \forall\, (x,y) \in W. \end{aligned}$$

The previous relations, which state feasibility and optimality of (x^*, y^*), respectively, can be equivalently rewritten, more explicitly, in the following manner:

$$(x^*, y^*) \in X \times U,\ f(x^*, y^*) \leq f(x^*, y)\ \forall y \in U \cap K(x^*),\ g(x^*, y^*) \leq 0 \tag{3}$$

$$F(x^*, y^*) \leq F(x,y)\ \forall\, (x,y) \in W, \tag{4}$$

where $W = \big\{(u,v) \in X \times U :\ f(u,v) \leq f(u,w)\ \forall w \in U \cap K(u),\ g(u,v) \leq 0\big\}$.

Now, let us consider the following Generalized Nash Equilibrium Problem (GNEP) with two players:

$$\begin{array}{ll} \underset{x,y}{\text{minimize}} & F(x,y) \\ \text{s.t.} & (x,y) \in X \times U \\ & f(x,y) \leq f(x,w) \\ & g(x,y) \leq 0 \end{array} \qquad\qquad \begin{array}{ll} \underset{w}{\text{minimize}} & f(x,w) \\ \text{s.t.} & w \in U \\ & g(x,w) \leq 0. \end{array} \tag{5}$$

We say that the player controlling x and y is the leader, while the other player is the follower.

GNEP (5) is strongly related to the original BP (1), as the following considerations clearly show (see Theorems 2.1 and 2.5, Corollary 2.3 and Examples 2.2 and 2.4). As a side remark, we point out that, in order to recast BP (1) as GNEP (5), we draw inspiration from the optimal value approach (see [10, 17, 23]). Indeed, the structure of leader's feasible set in (5) (in particular, constraint $f(x,y) \leq f(x,w)$) is intended to mimic, in some sense, and to deal with the value function implicit constraint $f(x,y) \leq \min_y\{f(x,y) :\ y \in K(x) \cap U\}$. We note that, as one can expect, it is precisely the "difficult" constraint $f(x,y) \leq f(x,w)$ that makes, in general, problem (5) not easily solvable: because of the presence of such constraint, GNEP (5) lacks convexity and suitable constraint qualifications are not readily at hand. However, one can still define classes of BPs for which reformulation (5) can lead to efficient algorithmic schemes by referring to the vast literature on solution methods for GNEPs [11]-[15].

We denote by

$$T \triangleq \{(x,y) \in X \times U \,:\, g(x,y) \leq 0\} \;\text{ and }\; U$$

the "private" constraints sets, and by

$$H(w) \triangleq \{(x,y) \in \mathbb{R}^{n_0} \times \mathbb{R}^{n_1} \,:\, f(x,y) \leq f(x,w)\} \;\text{ and }\; K(x)$$

the "coupling" constraints sets of the leader and the follower, respectively. Moreover, let $V(w) \triangleq T \cap H(w)$ be the feasible set of the leader.

A solution, or an equilibrium, of GNEP (5) is a triple (x^*, y^*, w^*) such that

$$(x^*, y^*) \in X \times U, \;\; f(x^*, y^*) \leq f(x^*, w^*), \;\; g(x^*, y^*) \leq 0, \tag{6}$$

$$F(x^*, y^*) \leq F(x, y), \; \forall\, (x, y) \in V(w^*), \tag{7}$$

$$w^* \in U, \;\; g(x^*, w^*) \leq 0, \tag{8}$$

$$f(x^*, w^*) \leq f(x^*, w), \;\; \forall w \in U \cap K(x^*), \tag{9}$$

where $V(w^*) = \big\{(u,v) \in X \times U : \; f(u,v) \leq f(u,w^*), \; g(u,v) \leq 0\big\}$. Conditions (6)-(7) and (8)-(9) state feasibility and optimality of (x^*, y^*, w^*) for leader's problem and for follower's problem, respectively.

The following theorem establish relations between solutions of GNEP (5) and of BP (1).

Theorem 2.1 *Let (x^*, y^*, w^*) be an equilibrium of GNEP (5). If $g(x, w^*) \leq 0$ for all x such that there exists y with $(x, y) \in W$ and $F(x, y) \leq F(x^*, y^*)$, then (x^*, y^*) is a solution of BP (1).*

Proof. Under the assumptions of the theorem, (x^*, y^*, w^*) satisfy relations (6)-(9); we now show that (3) and (4) hold at (x^*, y^*).

We observe that (6), (8) and (9) together imply that (x^*, y^*) satisfies (3).

Furthermore, let us denote by $\mathcal{L}^*$ the level set of F at (x^*, y^*), and by $(\mathcal{L}^*)^c$ its complement:

$$\mathcal{L}^* \triangleq \{(x,y) \in \mathbb{R}^{n_0} \times \mathbb{R}^{n_1} : F(x,y) \leq F(x^*, y^*)\},$$

$$(\mathcal{L}^*)^c \triangleq \{(x,y) \in \mathbb{R}^{n_0} \times \mathbb{R}^{n_1} : F(x,y) > F(x^*, y^*)\}.$$

Let $(\bar{x}, \bar{y})$ be any couple in $W \cap \mathcal{L}^*$: by assumptions, we have $g(\bar{x}, w^*) \leq 0$. Therefore, $w^* \in U \cap K(\bar{x})$ and, since $(\bar{x}, \bar{y}) \in W$, in turn $(\bar{x}, \bar{y}) \in V(w^*)$ and

$$W \cap \mathcal{L}^* \subseteq V(w^*). \tag{10}$$

Thanks to (7) and (10), and noting that for every $(x, y) \in W \cap (\mathcal{L}^*)^c$ we have $F(x, y) > F(x^*, y^*)$, (4) holds at (x^*, y^*). Hence, (x^*, y^*) is a solution of BP (1). □

The following example gives a picture of the relationship between GNEP (5) and BP (1), as stated in Theorem 2.1.

Example 2.2 Let us consider the following bilevel problem:

$$\begin{array}{ll} \underset{x,y}{\text{minimize}} & x^2 + y^2 \\ \text{s.t.} & x \geq 1 \\ & y \in S(x), \end{array} \tag{11}$$

where $S(x)$ denotes the solution set of the lower level problem

$$\begin{array}{ll} \underset{w}{\text{minimize}} & w \\ & x + w \geq 1, \end{array}$$

and the corresponding GNEP, that is,

$$\begin{array}{ll} \underset{x,y}{\text{minimize}} & x^2 + y^2 \\ \text{s.t.} & x \geq 1 \\ & y \leq w \\ & x + y \geq 1 \end{array} \qquad \begin{array}{ll} \underset{w}{\text{minimize}} & w \\ \text{s.t.} & x + w \geq 1. \end{array} \tag{12}$$

Point $(1, 0)$ is the unique solution of problem (11), while all the infinitely many points $(1 - \lambda, \lambda, \lambda)$, with $\lambda \leq 0$, are equilibria of GNEP (12). In particular, we remark that $(1, 0, 0)$ is the only solution of GNEP (12) that satisfies the assumptions of Theorem 2.1 (see Figure 1 and Figure 2).

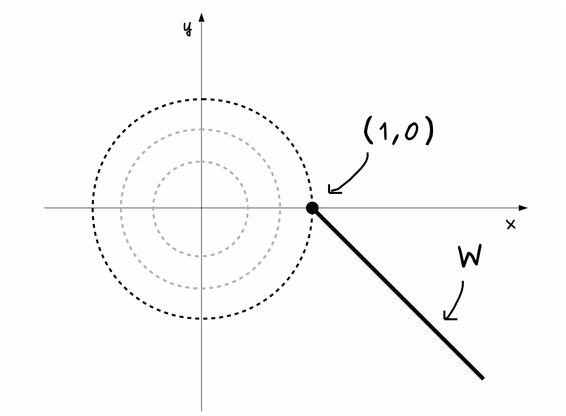

Figure 1: The feasible set W and the unique solution of BP (11)

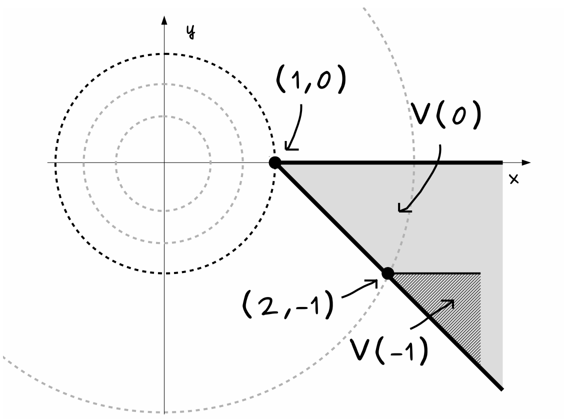

Figure 2: A sketch of leader's problem in GNEP (12): the feasible set $V(w)$ and the corresponding solution are depicted for different values of w, namely $w = 0$ and $w = -1$.

We further observe that if the lower level feasible set does not depend on the upper level variables x, then the requirements of Theorem 2.1 can be weakened, thus obtaining the following result whose proof is omitted.

Corollary 2.3 *Suppose that the lower level feasible set $U \cap K(x)$ does not depend on the upper level variables x. If (x^*, y^*, w^*) is an equilibrium of GNEP* (5)*, then (x^*, y^*) is a solution of BP* (1).

Example 2.4 shows that the implications in Theorem 2.1 and Corollary 2.3 can not be reversed: indeed, in general, given a solution (x^*, y^*) of BP (1), (x^*, y^*, y^*) may not be an equilibrium for GNEP (5), even when the lower level feasible set does not depend on the upper level variables x.

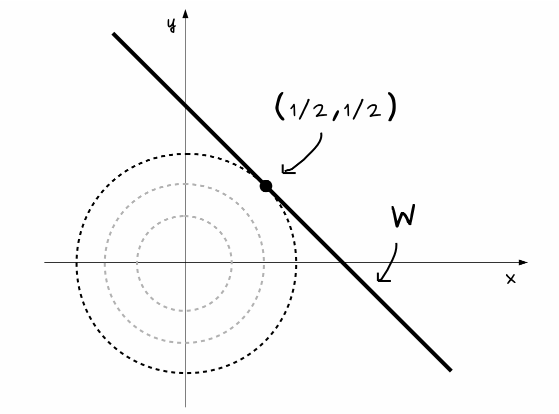

Figure 3: The feasible set W and the unique solution of BP (13)

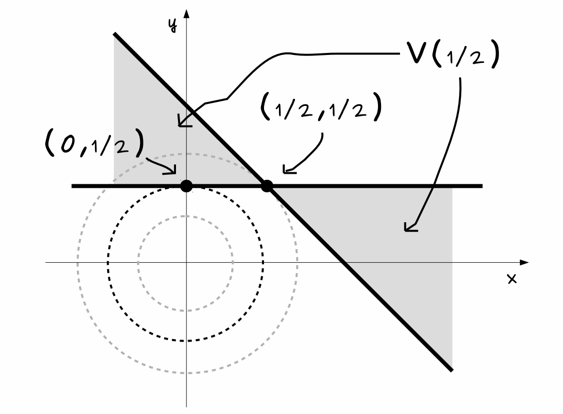

Figure 4: A sketch of leader's problem in GNEP (14): the feasible set $V(w)$ and the corresponding solution are depicted for $w = 1/2$.

Example 2.4 Let us consider the following bilevel problem:

$$\begin{array}{ll} \underset{x,y}{\text{minimize}} & x^2 + y^2 \\ \text{s.t.} & y \in S(x), \end{array} \tag{13}$$

where $S(x)$ denotes the solution set of the lower level problem

$$\underset{w}{\text{minimize}} \quad (x + w - 1)^2$$

and the corresponding GNEP

$$\begin{array}{ll} \underset{x,y}{\text{minimize}} & x^2 + y^2 \\ \text{s.t.} & (x + y - 1)^2 \leq (x + w - 1)^2 \end{array} \qquad \underset{w}{\text{minimize}} \quad (x + w - 1)^2. \tag{14}$$

The unique solution of problem (13) is $(x^*, y^*) = \left(\frac{1}{2}, \frac{1}{2}\right)$. However, the triple $(x^*, y^*, w^*) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ is not an equilibrium of GNEP (14), since point $(\tilde{x}, y^*, w^*) = \left(0, \frac{1}{2}, \frac{1}{2}\right)$ is feasible for the first player and $\tilde{x}^2 + (y^*)^2 < (x^*)^2 + (y^*)^2$ (see Figure 3 and Figure 4).

On the other hand, we notice that, whenever, at the lower level, the whole dependence on x is dropped, thus S does not depend on x, being the solution set of the optimization problem

$$\begin{array}{ll} \underset{w}{\text{minimize}} & f(w) \\ \text{s.t.} & w \in U, \end{array}$$

then an even deeper connection between BP (1) and GNEP (5) could be established. Indeed, in this case, the implications in Theorem 2.1 can be reversed.

Theorem 2.5 *Suppose that set S in* (1) *does not depend on the upper level variables x. The following implications hold:*

(i) if (x^*, y^*, w^*) is an equilibrium of GNEP (5)*, then (x^*, y^*) is a solution of BP* (1)*;*

(ii) if (x^*, y^*) is a solution of BP (1)*, then, for all $w^* \in U$ such that $f(w^*) = f(y^*)$, (x^*, y^*, w^*) is a solution of GNEP* (5)*.*

Proof. (i) The claim follows from Corollary 2.3.

(ii) Preliminarily, for the sake of clarity, we specialize relations (3) and (4) and (6)-(9) in case S does not depend on the upper level variables. Hence, (x^*, y^*) is a solution of (1) if

$$\begin{aligned} &(x^*, y^*) \in X \times U, \quad f(y^*) \leq f(y) \ \ \forall y \in U \\ &F(x^*, y^*) \leq F(x, y), \quad \forall\, (x, y) \in W, \end{aligned} \tag{15}$$

where $W = \big\{(x, y) \in X \times U : \ f(y) \leq f(w) \ \ \forall w \in U\big\}$.

We aim at proving that (x^*, y^*, w^*) is a solution of GNEP (5), that is,

$$\begin{aligned} &(x^*, y^*) \in X \times U, \\ &F(x^*, y^*) \leq F(x, y), \ \forall\, (x, y) \in V(w^*), \\ &w^* \in U, \\ &f(w^*) \leq f(w), \quad \forall w \in U, \end{aligned}$$

where $V(w^*) = \big\{(x, y) \in X \times U : \ f(y) \leq f(w^*)\big\}$.

Since y^* is optimal for the lower-level problem and $f(w^*) = f(y^*)$, then also w^* is optimal for the lower-level problem. Thus, relying on (15), we obtain $W \equiv V(w^*)$ and the thesis follows readily. □

Theorem 2.1 clearly indicates the path: we aim at computing a solution of the original problem (1) by solving GNEP (5). It goes without saying that this approach is viable only if an equilibrium of GNEP (5) exists; clearly, in this case, also a solution of BP (1) is guaranteed to exist. As a matter of example, relying on classical equilibrium existence results (see [1, 12, 16]), whenever the lower level functions f and g are affine, one can refer to the following proposition.

Proposition 2.6 *Let F be quasi-convex on T, f and g be affine functions, and X and U be polyhedral compact sets. Suppose that $T \times U$ is nonempty, and, for every $(x, y, w) \in T \times U$, both $T \cap H(w)$ and $U \cap K(x)$ are nonempty. Then, an equilibrium of GNEP* (5) *exists.*

Proof. Since, by assumptions, both the graph of $T \cap H$ and the graph of $U \cap K$ are polyhedral, then both $T \cap H$ and $U \cap K$ are Lipschitz continuous, see e.g. [19, Example 9.35]. For the sake of clarity, we remark that sets $T \cap H(w) \subseteq T$ and $U \cap K(x) \subseteq U$ are nonempty and convex for all $w \in U$ and all $(x, y) \in T$. Finally, by noting that F and f are quasi-convex, the assertion is a consequence of the classical theorem on the existence of an equilibrium, see e.g. [12, Theorem 4.1]. □

An important class of BPs is that in which the objective function of the lower level problem does not depend on the upper level variables x, while its feasible set depends on them. For these problems, convexity and solvability for the corresponding GNEP reformulation can be established by imposing simple assumptions. The following proposition concerns convexity, while Theorem 3.2 in the next section copes with the solvability issue.

Proposition 2.7 *Suppose that function f does not depend on variables x, that is $f(x,\cdot) = f(\cdot)$. If X and U are convex sets, and F, f and g are convex functions on T, U and T, respectively, then both leader's and follower's problems in GNEP* (5) *are convex.*

3 Algorithmic Framework

Building on the previous theoretical considerations and in view of the relationship between the original general BP (1) and its GNEP reformulation (5), we aim at solving (1) by finding a suitable equilibrium of (5). This is a twofold issue: on the one hand, one must be able to compute an equilibrium of GNEP (5); on the other hand, in order to obtain a solution of the original problem, this equilibrium must satisfy the assumptions of Theorem 2.1. As regards the first issue, Algorithm 1 defines an algorithmic framework which is intended to compute an equilibrium of (5). Algorithm 2 deals with the second issue.

Algorithm 1 : L/F Alternating Optimization Algorithm for BP

Data: $x^0 \in X$, $w^0 \in U \cap K(x^0)$, set $k = 0$.

(`S.1`) *(Leader optimization)* Calculate (x^{k+1}, y^{k+1}), solution of

$$\begin{aligned} \underset{x,y}{\text{minimize}} \quad & F(x,y) \\ \text{s.t.} \quad & (x,y) \in X \times U \\ & f(x,y) \le f(x^k, w^k) \\ & g(x,y) \le 0. \end{aligned} \tag{16}$$

(`S.2`) *(Follower optimization)* Calculate w^{k+1}, solution of

$$\begin{aligned} \underset{w}{\text{minimize}} \quad & f(x^{k+1}, w) \\ \text{s.t.} \quad & w \in U \\ & g(x^{k+1}, w) \le 0. \end{aligned} \tag{17}$$

(`S.3`) $k \leftarrow k+1$. If $f(x^k, y^k) = f(x^k, w^k)$, then `STOP`. Otherwise, go to step (`S.1`).

We remark that, in some sense, this scheme may be referred to the Gauss-Seidel approach for GNEPs (see [12]), in that it consists in the alternating optimization of players' problems. However, we point out that the presence of x^k in the constraints of problem (16) makes Algorithm 1 different from the standard Gauss-Seidel method.

The following Theorem 3.1 summarizes the main properties of Algorithm 1.

Theorem 3.1 *Suppose that X and U are compact and there exist $x^0 \in X$ such that $U \cap K(x^0) \neq \emptyset$. Let $\{(x^k, y^k, w^k)\}$ be the sequence generated by Algorithm 1. The following properties hold:*

(i) *Algorithm 1 is well defined;*

(ii) $f(x^{k+1}, w^{k+1}) \leq f(x^k, w^k)$ *for every* $k \geq 0$*;*

(iii) $F(x^{k+1}, y^{k+1}) \geq F(x^k, y^k)$ *for every* $k \geq 1$*;*

(iv) *Every point* (x^k, w^k) *belongs to* W*, thus is feasible for BP* (1)*, for every* $k \geq 1$*;*

(v) *sequence* $\{(x^k, y^k, w^k)\}$ *is bounded and such that either Algorithm 1 terminates after a finite number of iterations, or each of its limit points satisfies the stopping criterion at step (S.3) and sequence* $f^k \triangleq \{f(x^k, w^k)\}$ *converges.*

Proof. (i) By the assumptions of the theorem, at least point (x^k, w^k) belongs to the feasible set of problem (16) for every k; moreover, certainly y^{k+1} is feasible for problem (17). Thus, thanks to the Weierstrass theorem, steps (S.1) and (S.2) are well defined.

(ii) We observe that, for every $k \geq 0$,

$$f(x^{k+1}, w^{k+1}) \leq f(x^{k+1}, y^{k+1}) \leq f(x^k, w^k), \tag{18}$$

where the second inequality holds because (x^{k+1}, y^{k+1}) is feasible for problem (16), while, since y^{k+1} is feasible and w^{k+1} optimal for problem (17), respectively, we get $f(x^{k+1}, w^{k+1}) \leq f(x^{k+1}, y^{k+1})$.

(iii) By relations (18), the sequence of the right hand side terms of the constraint $f(x, y) \leq f(x^k, w^k)$ in (16), that is $\{f(x^k, w^k)\}$, is monotonic nonincreasing. This fact, in turn, entails $F(x^{k+1}, y^{k+1}) \geq F(x^k, y^k)$ for every $k \geq 1$.

(iv) Since w^{k+1} is optimal for (17), we have $(x^{k+1}, w^{k+1}) \in W$.

(v) Preliminarily, we note that, in view of the compactness of X and U, the sequence generated by the algorithm is bounded. Moreover, if, for every k, the stopping criterion at step (S.3) is not fulfilled, then, however, by (ii) and the compactness assumption, sequence $\{f(x^k, w^k)\}$ converges, implying $\lim_{k\to\infty} f(x^k, w^k) - f(x^{k+1}, w^{k+1}) = 0$; furthermore, by (18), $\lim_{k\to\infty} f(x^{k+1}, y^{k+1}) - f(x^{k+1}, w^{k+1}) = 0$ and the thesis follows from the continuity of f and the boundedness of $\{(x^k, y^k, w^k)\}$. □

Actually, in general, Algorithm 1 is not guaranteed to converge to an equilibrium of GNEP (5). Nonetheless, one can establish suitable conditions and, thus, classes of problems for which the previous method converges to a solution of (5). As a matter of example, whenever function f does not depend on x, Algorithm 1 reduces to the standard Gauss-Seidel method and the following result holds.

Theorem 3.2 *Under the same assumptions of Theorem 3.1, suppose that function* f *does not depend on variables* x*, that is* $f(x, \cdot) = f(\cdot)$*. Then, if Algorithm 1 terminates after a finite number of iterations, it provides an equilibrium of GNEP* (5)*. Furthermore, let both* $T \cap H(w)$ *and* $U \cap K(x)$ *be nonempty for every* $(x, y, w) \in T \times U$*. The following properties hold:*

(i) *if* F *is quasi-convex on* T*,* f *and* g *are affine, and* X *and* U *are polyhedral, then each limit point of the sequence generated by Algorithm 1 is an equilibrium of GNEP* (5)*;*

(ii) *suppose that* f *is quasi-convex on the convex set* U*; if* $g(x, \cdot)$ *is convex on* U *for every* $x \in X$ *and the Slater constraint qualification holds in* $U \cap K(x)$ *for every* $x \in X$*, then each limit point* $(\bar{x}, \bar{y}, \bar{w})$ *of the sequence generated by the algorithm is such that* $(\bar{x}, \bar{w})$ *is feasible for BP* (1)*, that is* $(\bar{x}, \bar{w}) \in W$*.*

Proof. Whenever f does not depend on variables x, steps (S.1) and (S.2) in Algorithm 1 become

(S.1) *(Leader optimization)* Calculate (x^{k+1}, y^{k+1}), solution of

$$\begin{array}{ll} \underset{x,y}{\text{minimize}} & F(x,y) \\ \text{s.t.} & (x,y) \in X \times U \\ & f(y) \le f(w^k) \\ & g(x,y) \le 0. \end{array} \tag{19}$$

(S.2) *(Follower optimization)* Calculate w^{k+1}, solution of

$$\begin{array}{ll} \underset{w}{\text{minimize}} & f(w) \\ \text{s.t.} & w \in U \\ & g(x^{k+1}, w) \le 0. \end{array} \tag{20}$$

As regards the finite termination of the algorithm, in view of Theorem 3.1, it is not hard to prove the claim.

In order to prove properties (i) and (ii), preliminarily, we remark that (19) and (20) are parametric optimization problems in which the feasible sets vary with respect to w^k and x^{k+1}, respectively.

(i) Let the set-valued mapping $R : \mathbb{R} \rightrightarrows \mathbb{R}^{n_0} \times \mathbb{R}^{n_1}$ describe the solution set of problem (19): hence, for all k, we can write $(x^{k+1}, y^{k+1}) \in R(\rho^k)$, with $\rho^k = f(w^k)$. We recall that $S : \mathbb{R}^{n_0} \rightrightarrows \mathbb{R}^{n_1}$ describes the solution set of problem (20). Since, by assumptions, the graph of $T \cap \bar{H}(\rho)$, where $\bar{H}(\rho) \triangleq \{(x,y) \in \mathbb{R}^{n_0} \times \mathbb{R}^{n_1} : f(y) \le \rho\}$, and the graph of $U \cap K(x)$ are polyhedral, then both $T \cap \bar{H}$ and $U \cap K$ are Lipschitz continuous, see e.g. [19, Example 9.35]. Moreover, the sets $R(\rho)$ and $S(x)$ are nonempty and bounded for every $\rho \in f(U)$ and $x \in X$; by [2, Theorem 4.3.3] and [2, Lemma 2.2.1] (see also [19, Theorem 5.19]), R and S are outer semicontinuous relative to $f(U)$ and X, respectively.

Let $(\bar{x}, \bar{y}, \bar{w})$ be a limit point of the sequence generated by Algorithm 1. We can write $(x^{k+1}, y^{k+1}) \in R(\rho^k)$, and $w^{k+1} \in S(x^{k+1})$ for every k. Furthermore, we remark that, by (v) in Theorem 3.1, sequence $\{f(w^k)\}$ converges and, thus, without loss of generality, $\lim_{k\to\infty} \rho^k = \bar{\rho}$. Therefore, by virtue of the outer semicontinuity of R and S relative to $f(U)$ and X, we have $(\bar{x}, \bar{y}) \in R(\bar{\rho})$, and $\bar{w} \in S(\bar{x})$.

Since f is continuous and ρ^k convergent, we get $\bar{\rho} = f(\bar{w})$ and, hence, $(\bar{x}, \bar{y})$ is a solution of Problem (19) given $\bar{w}$. Moreover, $\bar{w}$ is a solution of problem (20) given $\bar{x}$. But this, in turn, is equivalent to say that $(\bar{x}, \bar{y}, \bar{w})$ is an equilibrium of GNEP (5).

(ii) By [2, Theorem 3.1.6] the mapping $U \cap K$ is inner semicontinuous, while by [19, Example 5.8], it is also outer semicontinous. Furthermore, $S(x)$ is nonempty and bounded and $U \cap K(x)$ is convex for every $x \in X$. Thus, thanks again to [2, Theorem 4.3.3], S is outer semicontinuos relative to X: letting $(\bar{x}, \bar{y}, \bar{w})$ be a limit point of the sequence generated by Algorithm 1, since $w^{k+1} \in S(x^{k+1})$ for every k, we have $\bar{w} \in S(\bar{x})$ with $\bar{x} \in X$ and, in turn, $(\bar{x}, \bar{w}) \in W$. □

In the light of Theorem 2.1, we can obtain a solution of the original BP (1) by solving GNEP (5), provided that the computed equilibrium satisfies further conditions. The generic algorithmic framework 2 produces a sequence of feasible points for BP (1) aimed at satisfying the assumptions of Theorem 2.1. We preliminarily define the following set, given fixed x^k and y^k, for each $i \in \{1, \dots, m\}$:

$$M^i(x^k, y^k) \triangleq \{(x, y) \in W : g_i(x, y^k) > 0, F(x, y) \leq F(x^k, y^k)\}. \tag{21}$$

We observe that, in general, finding a point belonging to $M^i(x^k, y^k)$ is not an easy task. Indeed, this set is the intersection between W, which is the feasible set of (1), and a set defined by some side constraints. However, in our forthcoming numerical analysis, we will provide some procedures aiming at efficiently implementing the following scheme.

Algorithm 2 : Improving Algorithm for BP

Data: $(x^0, y^0) \in W$, set $k = 0$.

(`S.1`) If $M^i(x^k, y^k) = \emptyset \ \ \forall i \in \{1, \dots, m\}$, then `STOP`.

(`S.2`) For any $M^j(x^k, y^k) \neq \emptyset$, find $(x^{k+1}, y^{k+1}) \in M^j(x^k, y^k)$; $k \leftarrow k + 1$, go to step (`S.1`).

Theorem 3.3 states that the sequence generated by Algorithm 2 is feasible with respect to BP (1) and such that the corresponding values of F are nonincreasing. Moreover, letting $(\bar{x}, \bar{y})$ be a point provided by Algorithm 2, if $(\bar{x}, \bar{y}, \bar{y})$ happens to be a solution of GNEP (5), then $(\bar{x}, \bar{y})$ is also a solution of the original BP (1).

Theorem 3.3 *Let $\{(x^k, y^k)\}$ be the sequence generated by Algorithm 2. The following properties hold:*

(i) $F(x^k, y^k) \leq F(x^{k-1}, y^{k-1})$ and $(x^k, y^k) \in W$ for every $k \geq 1$;

(ii) let $(\bar{x}, \bar{y})$ be a point satisfying the stopping criterion in (`S.1`) and suppose further that $(\bar{x}, \bar{y}, \bar{y})$ is an equilibrium of GNEP (5); then $(\bar{x}, \bar{y})$ is a solution of BP (1).

Proof. (i) The property is a direct consequence of the fact that $(x^k, y^k) \in M^i(x^{k-1}, y^{k-1})$, for one $i \in \{1, \dots, m\}$.

(ii) The assertion follows from Theorem 2.1. In fact, $(\bar{x}, \bar{y}, \bar{y})$ is an equilibrium of GNEP (5) and $g(x, \bar{y}) \leq 0$ for all x such that there exists y with $(x, y) \in W$ and $F(x, y) \leq F(\bar{x}, \bar{y})$. □

We note that, since the sequence generated by the previous scheme is not guaranteed to lead to an equilibrium of GNEP (5), one can rely on Algorithm 2 mainly as an *a posteriori* optimality check of the output of Algorithm 1.

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