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Control of Mechanical Systems with Second-Order Nonholonomic Constraints: Underactuated Manipulators

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Abstract

An analysis of underactuated manipulators from both the dynamic and control points of view is presented. While the unactuated joints dynamic equation is recognized to be a nonholonomic constraint in the general case, necessary and sufficient conditions are given to identify special cases in which such a constraint is integrable. In contrast to most examples in the literature, the unactuated joints dynamics are an instance of second-order nonholonomic constraint. It is shown that smooth feedback stabilization to a single equilibrium point is not possible, and a feedback scheme providing stabilization to a manifold of equilibrium positions is proposed.

1. Introduction

The control of nonholonomic mechanical systems, i.e. systems with non-integrable differential constraints on generalized coordinates, has attracted growing attention in recent years. Even if the existence of non-integrable constraints in a system is strictly a mechanical property, the control problem for nonholonomic systems exhibits distinctive features. In fact, many nonholonomic systems naturally fit into the category of *underactuated* mechanisms, defined as systems in which the dimension of the configuration space exceeds that of the control input space. Examples of nonholonomic underactuated systems are wheeled mobile robots [1–3], legged robots in flight phase [4], space robots without jets or momentum wheels [5].

The difficulty of the control problem for underactuated mechanisms is obviously due the reduced dimension of the input space. A main question is whether the controllability of the system is affected by this feature, i.e. if it is still possible to reach any configuration by properly steering the input. Even when controllability is guaranteed, the control problem for underactuated systems is generally harder. A general theorem [6] can be often applied to these systems in order to infer a negative result, namely that smooth feedback stabilization to an equilibrium point is not possible. In this case non-smooth stabilization must be pursued, or different control objectives must be addressed.

In this paper, we tackle the problem of controlling an underactuated robot manipulator, i.e. a manipulator in which some joints are not equipped with actuators. The basic feature which may allow to control the whole mechanism through a limited number of actuators is obviously the coupling between links of the kinematic chain; such coupling is however highly nonlinear. The motivation for studying such a structure is multiple. First, underactuated manipulators may prove useful in particular instances in which it is important to reduce weight, cost or energy consumption, while still mantaining an adequate degree of dexterity; for example, one might think of space applications or some low-cost process automation.

Moreover, underactuated robot manipulators are a particularly challenging example of underactuated systems, because the associated control problem has a number of additional difficulties. In fact, it is possible to show that existing conditions do not provide any conclusion about system controllability. Besides, not only the system is not smoothly feedback-stabilizable, but further the presence of a drift term in the state-space model makes very difficult to devise intuitive open-loop strategies, as possible in simpler cases.

The final motivation stands in the fact that underactuated robots are an example of a mechanical system with *second-order* nonholonomic constraints. In particular, the dynamic equation of the unactuated joints is a second-order constraint on generalized coordinates which is in general non-integrable. This is in contrast with the vast majority of literature on nonholonomic systems, where only first-order (kinematic) constraints have been presented.

Underactuated robots have previously appeared in [7], [8]. However, in both cases there was a particular feature of the system which simplified the control problem. In [7], the unactuated joints were equipped with a braking mechanism, so that it was possible to eliminate the coupling between the links when desired; by appropriately combining brake on/brake off phases, control of the whole manipulator was achieved. In [8], the inclusion of gravity made the system linearization around equilibrium points controllable (and hence smoothly stabilizable); on the other hand, it also reduced the set of equilibrium points. For these reasons, the presence of a gravitational torque will be considered in establishing conditions for the integrability of the unactuated joints dynamics, but will be removed when addressing the control problem.

In the next section, the problem is stated and conditions for the integrability of the unactuated joints dynamic equation are given; in Section 3 some examples are presented to clarify these results. The general control problem for underactuated manipulators is discussed in Section 5. In Section 6 a control scheme achieving stabilization to a manifold of equilibrium points is presented, and Section 7 offers some conclusive remarks.

2. Integrability conditions

The dynamic model of a robot manipulator with n joints (which is a reasonably general model of mechanical systems) is expressed as

$$\mathbf{H}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{c}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{e}(\mathbf{q}) = \mathbf{u}, \tag{2.1}$$

where $\mathbf{q}$ is the vector of joint variables (generalized coordinates). $\mathbf{H}(\mathbf{q})$ is the $n \times n$ inertia matrix, $\mathbf{c}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}}$ is the vector of Coriolis and centripetal torques, $\mathbf{e}(\mathbf{q})$ are the gravitational terms, and $\mathbf{u}$ is the vector of input torques. If only m joints are equipped with actuators, vector $\mathbf{q}$ can be partitioned without loss of generality as $(\mathbf{q}_a \; \mathbf{q}_u)$, where $\mathbf{q}_a \in I\!R^m$ represents the actuated joints, while $\mathbf{q}_u \in I\!R^{(n-m)}$ represents the unactuated ones. The dynamic model (2.1) is then written as

$$\begin{bmatrix} \mathbf{H}_a \\ \mathbf{H}_u \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}_a \\ \ddot{\mathbf{q}}_u \end{bmatrix} + \begin{bmatrix} \mathbf{c}_a \\ \mathbf{c}_u \end{bmatrix} + \begin{bmatrix} \mathbf{e}_a \\ \mathbf{e}_u \end{bmatrix} = \begin{bmatrix} \mathbf{u}_a \\ \mathbf{0} \end{bmatrix}, \tag{2.2}$$

and in particular the dynamic equation relative to the unactuated joints is

$$\mathbf{H}_u(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{c}_u(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{e}_u(\mathbf{q}) = \mathbf{0}. \tag{2.3}$$

No input term explicitly appears in the $(n-m)$ equations (2.3), which may thus be interpreted as an $(n-m)$-dimensional *constraint* involving generalized coordinates as well as their first and second-order time derivatives. It is important to determine if this constraint is integrable, for in this case it can be used to reduce the system dimension by eliminating some generalized coordinates.

As a preliminary step, we check if equation (2.3) may be integrated to the form $\mathbf{g}(\mathbf{q}, \dot{\mathbf{q}}, t) = \mathbf{0}$. In the following, this property will be referred to as *partial integrability*. If partial integrability holds, possible further integrability to a constraint of the form $\mathbf{f}(\mathbf{q}, t) = \mathbf{0}$ must be investigated. If such a constraint exists, constraint (2.3) will be said to possess the *complete integrability* property or, according to the standard mechanics terminology, to be *holonomic*. In fact, although almost all examples of nonholonomic constraints presented in the literature are kinematic, also differential constraints of higher order can be considered nonholonomic in principle [9].

A common approach in the most recent literature concening nonholonomic systems is to infer the nonholonomic nature of the constraints from the accessibility of the system [1,2,10]. However, this procedure cannot be applied to underactuated manipulators. In fact, in this case the existing sufficient conditions for accessibility (i.e. that the Lie Algebra of system vector fields generates an n-dimensional distribution) are not satisfied, while the necessary ones are [11]. It is still possible to use Frobenius Theorem [12] to discuss the integrability (both partial and complete) of constraint (2.3). A difficulty derives from the fact that the latter is a second-order constraint, so that some transformation is needed to put it in a first-order form, to which Frobenius Theorem can be directly applied [11]. We will establish integrability conditions by following here a different approach, taking advantage of the special pattern of constraint (2.3), which in turns depends on the robot dynamic model structure.

2.1 Conditions for partial integrability

Consider the generic $(n-m)$-dimensional constraint of the first order (i.e. involving $\mathbf{q}$ and $\dot{\mathbf{q}}$), expressed as

$$\mathbf{g}(\mathbf{q}, \dot{\mathbf{q}}, t) = \mathbf{0}, \quad \mathbf{g} \in I\!R^{n-m}. \tag{2.4}$$

Note that the explicit presence of the time variable t is considered in (2.4), because this enlarges the class of integrable unactuated joint dynamics, as will be shown in the following. According to the previously introduced terminology, constraint (2.3) is partially integrable if can be integrated to (2.4). Differentiating (2.4) w.r.t. t yields

$$\frac{\partial \mathbf{g}}{\partial \mathbf{q}}\dot{\mathbf{q}} + \frac{\partial \mathbf{g}}{\partial \dot{\mathbf{q}}}\ddot{\mathbf{q}} + \frac{\partial \mathbf{g}}{\partial t} = \mathbf{0}. \tag{2.5}$$

For (2.3) to be partially integrable, it must be structurally equivalent to (2.5). We determine the integrability conditions comparing the two equations.

First, it appears that $\partial \mathbf{g}/\partial t$ must be constant, since t does not appear explicitly in equation (2.3). Thus, we may restrict our attention to constraints of the form

$$\mathbf{g}(\mathbf{q}, \dot{\mathbf{q}}, t) = \mathbf{g}_1(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{k}_1 t = \mathbf{0}, \tag{2.6}$$

where $\mathbf{k}_1 \in I\!R^{n-m}$ is an arbitrary constant vector. Differentiating (2.6) we have

$$\frac{\partial \mathbf{g}_1}{\partial \mathbf{q}}\dot{\mathbf{q}} + \frac{\partial \mathbf{g}_1}{\partial \dot{\mathbf{q}}}\ddot{\mathbf{q}} + \mathbf{k}_1 = \mathbf{0}. \tag{2.7}$$

We obtain necessary conditions by considering special cases. Setting $\dot{\mathbf{q}} = \mathbf{0}$ in (2.3) and (2.7)

$$(2.3) \;\Rightarrow\; \mathbf{H}_u\ddot{\mathbf{q}} + \mathbf{e}_u = \mathbf{0}, \quad (2.7) \;\Rightarrow\; \frac{\partial \mathbf{g}_1}{\partial \dot{\mathbf{q}}}\ddot{\mathbf{q}} + \mathbf{k}_1 = \mathbf{0}.$$

For these two equations to provide the same feasible accelerations, the following must hold [11]

$$\mathbf{H}_u = \frac{\partial \mathbf{g}_1}{\partial \dot{\mathbf{q}}} \qquad \text{and} \qquad \mathbf{e}_u = \mathbf{k}_1. \tag{2.8}$$

Being $\mathbf{H}_u$ a function of $\mathbf{q}$ only, joint velocities $\dot{\mathbf{q}}$ appear linearly in (2.6), which must have the form

$$\mathbf{g}(\mathbf{q}, \dot{\mathbf{q}}, t) = \mathbf{g}_2(\mathbf{q}) + \mathbf{H}_u(\mathbf{q})\dot{\mathbf{q}} + \mathbf{e}_u t = \mathbf{0}. \tag{2.9}$$

Again, differentiating (2.9) we obtain

$$\frac{\partial \mathbf{g}_2}{\partial \mathbf{q}}\dot{\mathbf{q}} + \dot{\mathbf{H}}_u\dot{\mathbf{q}} + \mathbf{H}_u\ddot{\mathbf{q}} + \mathbf{e}_u = \mathbf{0}. \tag{2.10}$$

The explicit expression of $\mathbf{c}_u$ in (2.3) is

$$\mathbf{c}_u = \dot{\mathbf{H}}_u\dot{\mathbf{q}} - \frac{1}{2}\Big(\frac{\partial}{\partial \mathbf{q}_u}(\dot{\mathbf{q}}^T\mathbf{H}\dot{\mathbf{q}})\Big)^T, \tag{2.11}$$

so that, comparing (2.3) with (2.10)

$$\frac{\partial \mathbf{g}_2}{\partial \mathbf{q}}\dot{\mathbf{q}} + \frac{1}{2}\Big(\frac{\partial}{\partial \mathbf{q}_u}(\dot{\mathbf{q}}^T\mathbf{H}\dot{\mathbf{q}})\Big)^T = \mathbf{0}. \tag{2.12}$$

It is possible to show [11] that (2.12) is equivalent to

$$\Big(\frac{\partial}{\partial \mathbf{q}_u}(\dot{\mathbf{q}}^T\mathbf{H}\dot{\mathbf{q}})\Big)^T = \mathbf{0} \quad \text{and} \quad \frac{\partial \mathbf{g}_2}{\partial \mathbf{q}} = \mathbf{0}, \tag{2.13}$$

i.e. the kinetic energy must not depend on the unactuated joint variables. Since $\mathbf{H}$ is symmetric this implies

$$\frac{\partial \mathbf{H}}{\partial q_{u,i}} = \mathbf{O}, \qquad i = m+1, \ldots, n, \tag{2.14}$$

with $\mathbf{O}$ being the $n \times n$ null matrix.

By equation (2.13), $\mathbf{g}_2$ reduces to a constant term $\mathbf{k}_2$. Thus, if (2.3) is partially integrable, its integral form is

$$\mathbf{g}(\mathbf{q}, \dot{\mathbf{q}}, t) = \mathbf{H}_u(\mathbf{q})\dot{\mathbf{q}} + \mathbf{e}_u t + \mathbf{k}_2 = \mathbf{0}, \tag{2.15}$$

in which $\mathbf{k}_2$ is a constant to be determined from the initial conditions of motion.

Equations (2.8) and (2.14) have been found to be necessary conditions for the partial integrability of (2.3). The following result shows that together they constitute also a sufficient condition.

Proposition 2.1

Constraint (2.3) is partially integrable if and only if:

(i) the gravitational torque $\mathbf{e}_u$ is constant;

(ii) the unactuated joint variables $\mathbf{q}_u$ do not appear in the manipulator inertia matrix.

If it exists, its partial integral form is given by (2.15).

Proof. The necessity has been proven above. To show the sufficiency, assume that (i) and (ii) hold. Equation (2.3) reduces to (see also (2.11))

$$\mathbf{H}_u\ddot{\mathbf{q}} + \dot{\mathbf{H}}_u\dot{\mathbf{q}} + \mathbf{e}_u = \mathbf{0}, \tag{2.16}$$

which is integrable to (2.15). ∎

2.2 Physical interpretation of Proposition 2.1

Conditions (i) and (ii) of Proposition 2.1 have an immediate physical interpretation. Letting U be the gravitational potential energy, it is $\mathbf{e}_u = \partial U/\partial \mathbf{q}_u$. Condition ($i$) is satisfied when $U = U(\mathbf{q}_a) + \mathbf{k}_1^T\mathbf{q}_u$, with $\mathbf{k}_1$ being a constant vector (and $\mathbf{k}_1 = \mathbf{e}_u$). For example, this holds for a free joint sliding along a vertical axis. Clearly, condition (i) is also satisfied in the particular case $\mathbf{e}_u = \mathbf{0}$, i.e. when the unactuated part of the robotic structure is contained in the horizontal plane or there is no gravitational field. Note that this case is particularly important, for example in space applications.

As for condition (ii), assume first that $\mathbf{e}_u = \mathbf{0}$. Then (2.14) is equivalent to the classical definition of *cyclic* coordinates, i.e. generalized coordinates which do not enter into the Lagrangian function L [13]. It is known that the *generalized momentum* $p_i = \partial L/\partial \dot{q}_i$ of a cyclic coordinate q_i is conserved. In our case

$$p_{u,i} = \frac{\partial L}{\partial \dot{q}_{u,i}} = \mathbf{h}_{u,i}^T\dot{\mathbf{q}} = \text{constant} \tag{2.17}$$

where $\mathbf{h}_{u,i}$ is the i-th column of the inertia submatrix $\mathbf{H}_u$ (compare with (2.15)). Hence, if $\mathbf{e}_u = \mathbf{0}$, constraint (2.3) can be partially integrated if and only if the unactuated joints coordinates are cyclic, and the integral form of the constraint expresses the corresponding generalized momentum conservation.

In the case $\mathbf{e}_u = \mathbf{k}_1 \neq \mathbf{0}$, it is $U = U(\mathbf{q}_a) + \mathbf{k}_1^T\mathbf{q}_u$ and the unactuated joints coordinates are not cyclic. The Lagrangian is

$$L = T - U = \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{H}\dot{\mathbf{q}} - U(\mathbf{q}_a) - \mathbf{k}_1^T\mathbf{q}_u, \tag{2.18}$$

Use of (2.14) in the Lagrange's equation $\dot{p}_i = \partial L/\partial q_i$ gives

$$p_{u,i} = -k_{1,i}t + p_{u,i}(0), \tag{2.19}$$

where $p_{u,i}(0)$ is the generalized momentum at $t=0$. In the case $\mathbf{e}_u \neq \mathbf{0}$, if the unactuated joints dynamics admit an integral form, this represents the linear decrease in time of the corresponding generalized momentum.

2.3 Conditions for complete integrability

So far, conditions have been given for the partial integrability of constraint (2.3). If they are satisfied, (2.3) can be integrated as (2.15). To decide whether the latter can be further integrated, consider the equation

$$\mathbf{H}_u(\mathbf{q})\dot{\mathbf{q}} = \mathbf{0}. \tag{2.20}$$

For any $\mathbf{q}$, (2.20) determines an m-dimensional linear subspace $\Delta(\mathbf{q})$, that is the null-space of matrix $\mathbf{H}_u$. The mapping $\Delta : \mathbf{q} \mapsto \Delta(\mathbf{q})$ is the *m-distribution* associated with the constraint (2.20). Frobenius Theorem [12] states that (2.20) is integrable if and only if Δ is *involutive*. If so, equation (2.15) is also integrable as

$$\mathbf{f}(\mathbf{q}, t) = \mathbf{f}_1(\mathbf{q}) + \frac{\mathbf{e}_u}{2}t^2 + \mathbf{k}_2 t + \mathbf{k}_3 = \mathbf{0}, \tag{2.21}$$

with $\partial \mathbf{f}_1/\partial \mathbf{q} = \mathbf{H}_u$, and $\mathbf{k}_2$, $\mathbf{k}_3$ are constant vectors depending on initial conditions. Summarizing

Proposition 2.2

Constraint (2.3) is completely integrable (holonomic) if and only if:

(i) is partially integrable;

(ii) the distribution Δ defined by (2.20) is involutive.

In this case, the integral form of (2.3) is given by (2.21).

Note that if $n = 2$ a constraint like (2.20) is *always* integrable, since Δ is 1-dimensional and thus involutive. This conclusion is consistent with the classical result in mechanics [13] stating that a 2-dof system with a cyclic coordinate is always integrable. In fact, the cyclicity of a joint coordinate implies that conditions (i) and (ii) of Proposition 2.1 are satisfied for that joint.

Proposition 2.2 places severe limitations on the class of underactuated robots for which constraint (2.3) is integrable, by requesting a special mechanical structure. As a result, the unactuated joint dynamics are a *second-order* nonholonomic constraint in the general case.

3. Examples

Consider a two-link robot with rotational joints in a horizontal plane, so that $\mathbf{e} = \mathbf{0}$ in the dynamic equation

$$\begin{bmatrix} a_1+2a_2\cos q_2 & a_3+a_2\cos q_2 \\ a_3+a_2\cos q_2 & a_3 \end{bmatrix}\ddot{\mathbf{q}} + a_2\sin q_2 \begin{bmatrix} -2\dot{q}_1\dot{q}_2-\dot{q}_2^2 \\ \dot{q}_1^2 \end{bmatrix} = \mathbf{u}, \tag{3.1}$$

where

$$\begin{aligned} a_1 &= m_1 l_{c1}^2 + J_1 + J_2 + m_2(l_1^2 + l_{c2}^2), \\ a_2 &= m_2 l_1 l_{c2}, \\ a_3 &= m_2 l_{c2}^2 + J_2. \end{aligned} \tag{3.2}$$

In (3.2), m_i, l_i, and J_i are respectively the length, mass and inertia moment (w.r.t. its center of mass) of the $i-$th link, while l_{ci} is the distance between the i-th joint axis and the center of mass of the $i-$th link.

Example 3.1 Suppose that the first joint is not actuated while the second is, so that $\mathbf{u} = (0\ u_2)^T$. Since q_1 does not enter in the manipulator inertia matrix (i.e. is a cyclic coordinate), condition *(ii)* of Proposition 2.1 is satisfied, and the first equation of (3.1)

$$(a_1{+}2a_2\cos q_2)\ddot{q}_1 + (a_3{+}a_2\cos q_2)\ddot{q}_2 - a_2\dot{q}_2(2\dot{q}_1 + \dot{q}_2)\sin q_2 = 0 \tag{3.3}$$

can be partially integrated according to (2.15) as

$$(a_1 + 2a_2\cos q_2)\dot{q}_1 + (a_3 + a_2\cos q_2)\dot{q}_2 + k_2 = 0 \tag{3.4}$$

as may be easily verified. Since $n = 2$, we expect that (3.4) is itself integrable. In the following, the special case $k_2 = 0$ will be considered for simplicity. For example, this is true when both initial joint velocities are zero. In this case (3.4) is integrable by separation as

$$q_1 = (\frac{a_1}{2} - a_3)\arctan\Big(\frac{\sqrt{a_1^2 - 4a_2^2}}{a_1 + 2a_2}\tan\frac{q_2}{2}\Big) + k_3. \tag{3.5}$$

Equation (3.5) shows that, for given initial conditions, there is a one-to-one mapping between q_1 and q_2 (a cyclic motion in q_2 yields a cyclic one in q_1). If $k_2 \neq 0$, the integral form of (3.5) can be found by exploiting the sufficiency proof of Frobenius Theorem [12].

Example 3.2 Now assume that the second joint is not actuated, while the first is. Condition (2.14) is violated, since q_2 appears in the manipulator inertia matrix. The unactuated joints dynamic equation is not even partially integrable, and hence it is nonholonomic.

Example 3.3 Again, assume that the second joint is not actuated, but suppose that $l_{c2} = 0$, i.e. the center of mass of the second link is located on the second joint axis (for example, this may be obtained by using counterweights). The inertia matrix in (3.1) is then constant, so that condition (2.14) is satisfied, and the second equation in (3.1) can be integrated twice to

$$q_1(t) + q_2(t) + c_1 t + c_2 = 0. \tag{3.6}$$

Equation (3.6) can be expressed in a more meaningful form by introducing the *absolute* joint coordinates $\theta_1 = q_1$, $\theta_2 = q_1 + q_2$, i.e. by defining joint variables w.r.t. the x axis. In fact, (3.6) becomes

$$\theta_2(t) - \dot{\theta}_2(0)\,t - \theta_2(0) = 0, \tag{3.7}$$

expressing the fact that the evolution of θ_2 is not affected by the input, and depends only on the initial conditions. Constraint (3.7) may be used to eliminate the variable θ_2 (or q_2) from the problem.

4. The control problem

We begin this section obtaining the state-space model of the underactuated robot. Henceforth, it will be assumed that $\mathbf{e}_u = \mathbf{0}$ in (2.3) (see the remark in Section 1). Define the state vector $\mathbf{x} \in I\!R^{2n}$ as $\mathbf{x}^T = (\mathbf{q}\,\dot{\mathbf{q}})^T = (\mathbf{x}_1\,\mathbf{x}_2)^T$. Letting $\mathbf{H}^{-1} = \mathbf{N}$, and partitioning $\mathbf{N}$ as $(\mathbf{N}_a\ \mathbf{N}_u)$ in accordance to the partition of $\mathbf{H}$, (2.1) gives

$$\begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{x}_2 \\ -\mathbf{N}(\mathbf{x}_1)\mathbf{c}(\mathbf{x}) \end{bmatrix} + \begin{bmatrix} \mathbf{O}_{n\times m} \\ \mathbf{N}_a(\mathbf{x}_1) \end{bmatrix}\mathbf{u}_a, \tag{4.1}$$

in which $\mathbf{O}_{n\times m}$ is the $n \times m$ null matrix. Differently from most nonholonomic systems in the literature, a nonzero drift term is present in (4.1), due to the problem formulation at a dynamical level.

All equilibrium positions for the system are *zero-velocity* points. As a preliminary step in the analysis of the control problem for underactuated manipulators, the linearization of system (4.1) around an equilibrium point $\mathbf{x}_e$ is computed as

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}\mathbf{v}, \tag{4.2}$$

where $\mathbf{z} = \mathbf{x} - \mathbf{x}_e$, and

$$\mathbf{A} = \begin{bmatrix} \mathbf{O}_{n\times n} & \mathbf{I} \\ \mathbf{O}_{n\times n} & \mathbf{O}_{n\times n} \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} \mathbf{O}_{n\times m} \\ \mathbf{N}_a(\mathbf{x}_e) \end{bmatrix}.$$

The controllability matrix has always rank equal to $2m$. Thus, the linearization of (4.1) around any equilibrium point has $2(n-m)$ uncontrollable modes, while all the eigenvalues of $\mathbf{A}$ are zero. Stated otherwise, we are in the presence of a typical *critical* problem of asymptotic stabilization. Therefore, the system cannot be asymptotically stabilized by a linear state feedback. It must be emphasized that the inclusion of gravity terms in the dynamic equation (2.3) would simplify the control problem, making the linearization around any equilibrium point controllable (see [8]); in this case even a linear controller will eventually stabilize the system.

System (4.1) might be asymptotically stabilizable by means of a nonlinear feedback. Brockett has established a necessary condition [6] for the existence of smooth stabilizing feedback laws for the generic nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$, with $\mathbf{x} \in I\!R^n$, $\mathbf{f}(\mathbf{x}_e, \mathbf{0}) = \mathbf{0}$, and $\mathbf{f}(\cdot,\cdot)$ continuously differentiable in a neighborhood of the equilibrium: the mapping $\gamma : I\!R^n \times I\!R^m \to I\!R^n$ defined by $\gamma : (\mathbf{x}, \mathbf{u}) \mapsto \mathbf{f}(\mathbf{x}, \mathbf{u})$ should be *onto* an open set containing the origin $\mathbf{O}$. For the system (4.1), this condition is satisfied if and only if the system

$$\epsilon = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix} = \begin{bmatrix} \mathbf{x}_2 \\ -\mathbf{N}\mathbf{c} \end{bmatrix} + \begin{bmatrix} \mathbf{O}_{n\times n} \\ \mathbf{N}_a \end{bmatrix}\mathbf{u}_a \tag{4.3}$$

is solvable for any ϵ 'near' $\mathbf{0}$. Now, let $\epsilon=(\mathbf{0}\ \epsilon_2)^T$. Then it must be $\mathbf{x}_2=\mathbf{0}$, which implies also $\mathbf{Nc}=\mathbf{0}$, so that system (4.3) reduces to $\epsilon_2=\mathbf{N}_a\mathbf{u}_a$ with $\epsilon_2\in I\!R^n$, $\mathbf{u}_a\in I\!R^m$, $\mathbf{N}_a : n\times m$. This is an overdetermined system, and in general admits no solution. Therefore, *smooth stabilization to a single equilibrium point is not possible* in the underactuated manipulator case.

The objective of asymptotic stabilization might still be achievable by nonsmooth feedback. However, results in the field of nonsmooth stabilization are quite recent, and systematic procedures to build nonsmooth stabilizing control laws exist only in special cases (e.g. for two-dimensional systems [14]). It is worth mentioning that the nonexistence of smooth stabilizing feedbacks is a common characteristic in many nonholonomic systems. This is for example the case of carts and space robots, in which stabilization has been obtained by nonsmooth strategies: e.g. the *Lie Bracket motions* of [15], or the *bidirectional approach* of [5]. On the other hand, different control objectives may be pursued, such as stabilization to *manifolds* of equilibrium points (as opposed to a single equilibrium position) or to trajectories (as long as they do not converge to a point: see for example [3]).

In the remainder, we will tackle system stabilization to a manifold of equilibria. In [16], Bloch and McClamroch have established conditions for the existence of solutions to this problem in the case of systems with *first order* nonholonomic constraints. Their approach is based on the use of a coordinate transformation in which the nonholonomic constraints have a trivial specification. A direct extension of this procedure to the case of systems with second-order nonholonomic constraints (like underactuated manipulators) does not seem to be possible.

5. Stabilization to equilibrium manifolds

In this section the two-link planar manipulator of Example 3.2 (i.e. with the second joint unactuated and $l_{c2}\neq 0$) will be considered for simplicity, but the following discussion can be extended to more general robotic structures [11]. The dynamic equations are

$$\begin{aligned}(a_1+2a_2\cos q_2)\ddot{q}_1+(a_3+a_2\cos q_2)\ddot{q}_2-4a_2\dot{q}_1\dot{q}_2\sin q_2&=u_1\\(a_3+a_2\cos q_2)\ddot{q}_1+a_3\ddot{q}_2-a_2\sin q_2\dot{q}_1^2&=0.\end{aligned}\tag{5.1}$$

A reasonable control objective is to bring the first joint to a given position q_{1d} with zero velocity, requesting the second joint to be at rest at the end of the movement; that is, the final position of the second joint is not specified. This may be interpreted as controlling the first link while 'rejecting' the mechanical disturbance introduced by the unactuated one. The equilibrium manifold to which the system should be stabilized is $M=\{(\mathbf{q},\dot{\mathbf{q}}) : q_1=q_{1d}, \dot{q}_1=0, \dot{q}_2=0\}$. We use the following stability definition [16]: a smooth manifold M of equilibrium points is *locally stable* if for any neighborhood $U\supset M$ there is a neighborhood V, with $U\supset V\supset M$, such that if $\mathbf{x}(0)\in V$, also $\mathbf{x}(t)\in U$ for all t. If $\mathbf{x}(t)\to M$ as $t\to\infty$, M is *asymptotically stable*.

In the following it will be shown that a simple PD controller on the first link position achieves the desired stabilization property. Let $\tilde{q}_1=q_1-q_{1d}$, and choose the input function u_1 as

$$u_1=-k_P\tilde{q}_1-k_D\dot{q}_1,\tag{5.2}$$

where k_P and k_D are positive real numbers. The effectiveness of (5.2) will be shown in two steps.

Proposition 5.1

With the choice of input (5.2), the equilibrium manifold $M=\{(\mathbf{q},\dot{\mathbf{q}}) : q_1=q_{1d}, \dot{q}_1=0, \dot{q}_2=0\}$ is locally stable. Also, $\tilde{q}_1$, $\dot{q}_1$ and $\dot{q}_2$ are bounded, and $\dot{q}_1\to 0$ as $t\to\infty$.

Proof. Define the *Lyapunov-like* function

$$V=\frac{1}{2}(k_P\tilde{q}_1^2+\dot{\mathbf{q}}^T\mathbf{H}\dot{\mathbf{q}})\geq 0.\tag{5.3}$$

Using equation (2.1), the derivative of V along the closed-loop system trajectories is obtained as

$$\dot{V}=k_P\tilde{q}_1\dot{q}_1+\dot{\mathbf{q}}^T(\mathbf{u}-\mathbf{C}\dot{\mathbf{q}})+\frac{1}{2}\dot{\mathbf{q}}^T\dot{\mathbf{H}}\dot{\mathbf{q}},\tag{5.4}$$

in which $\mathbf{u}=(u_1\,0)^T$, and u_1 is given by (5.2). It is always possible to choose matrix $\mathbf{C}$ so that matrix $\dot{\mathbf{H}}-2\mathbf{C}$ is antisymmetric. Thus

$$\dot{V}=-k_D\dot{q}_1^2\leq 0,\quad\text{and}\quad\dot{V}=0\Leftrightarrow\dot{q}_1=0.\tag{5.5}$$

This implies the stability of M. Also, (5.5) shows that $\tilde{q}_1$, $\dot{q}_1$ and $\dot{q}_2$ are bounded. To prove that $\dot{q}_1\to 0$ as $t\to\infty$, we make use of Barbalat's Lemma [17]: since V is lower bounded and $\dot{V}$ is negative semidefinite, the uniform continuity of $\dot{V}$ would yield $\dot{V}\to 0$ as $t\to\infty$. A sufficient condition for the uniform continuity of $\dot{V}$ is that $\ddot{V}$ is bounded. From (5.5) and (4.1)

$$\begin{aligned}\ddot{V}=&\frac{2k_D\dot{q}_1}{\Lambda}\big(a_2a_3\dot{q}_2\sin q_2(2\dot{q}_1+\dot{q}_2)\\&-a_2\dot{q}_1^2\sin q_2(a_3+a_2\cos q_2)-a_3(k_D\dot{q}_1+k_P\tilde{q}_1)\big),\end{aligned}\tag{5.6}$$

where Λ is the nonzero determinant of the inertia matrix. $\ddot{V}$ is bounded, since $\Lambda>0$, $\tilde{q}_1$, $\dot{q}_1$ and $\dot{q}_2$ are bounded, and terms involving q_2 are trigonometric. Uniform continuity of $\dot{V}$ follows, and $\dot{V}\to 0$ as $t\to\infty$. Finally, (5.5) shows that $\dot{q}_1\to 0$ as $t\to\infty$. ∎

Note that function (5.3) is not a proper Lyapunov function, being only semipositive definite in the state space. In particular, $V=0$ holds on the whole equilibrium manifold M. Therefore, given a positive real α, the set $\Omega_\alpha=\{\mathbf{x} : V(\mathbf{x})\leq\alpha\}$ is not bounded. This in turn implies that system trajectories are not bounded, and invariant set theorems cannot be invoked to prove convergence to the maximal invariant set contained in M. Convergence to the desired equilibrium manifold (and hence asymptotic stability) is guaranteed by

Proposition 5.2

Along the trajectories of the closed-loop system, $\tilde{q}_1\to 0$ and $\dot{q}_1$, $\dot{q}_2\to 0$ as $t\to\infty$.

Proof. By Proposition 5.1, $\dot{q}_1 \to 0$. As a preliminary step, it will be shown that also $\ddot{q}_1 \to 0$. Again, Barbalat's Lemma is used to draw this conclusion. Since $\dot{q}_1$ converges to a finite limit value as $t \to \infty$, uniform continuity of $\ddot{q}_1$ would imply that $\ddot{q}_1 \to 0$. The time derivative of $\ddot{q}_1$ is shown to be bounded by using similar considerations to those in the proof of Proposition 5.1. Consequently, $\ddot{q}_1 \to 0$ as $t \to \infty$. Thus, as $t \to \infty$, equations (5.1) yield also $\ddot{q}_2 \to 0$ and

$$k_2 \dot{q}_2^2 \sin q_2 + k_p \tilde{q}_1 \to 0. \tag{5.7}$$

By differentiating (5.7) twice w.r.t. time it follows that $\dot{q}_2 \to 0$, so that $\tilde{q}_1 \to 0$. ∎

Propositions 5.1 and 5.2 imply asymptotic stability of the equilibrium manifold M. However, the developed stability analysis offers no estimate of the convergence rate.

6. Conclusions

An analysis of underactuated manipulators from both the dynamic and control point of view has been presented. The nonholonomic nature of the constraint expressing the unactuated joints dynamics has been recognized in the general case, and conditions have been derived to identify special cases in which such constraint is integrable. Smooth feedback stabilization to a single equilibrium point is not possible, and a feedback scheme achieving stabilization to a manifold of equilibrium positions has been presented. Future work will address more demanding control objectives, giving up the requirement of smoothness for the feedback law.

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