Robotics I

Midterm classroom test – November 16, 2018

Exercise 1 [6 points]

The orientation of a rigid body is given in terms of the YXY Euler angles $(\alpha, \beta, \gamma) = (\pi/2, -\pi/4, \pi/4)$. This orientation is the result of a rotation around the unit vector $\boldsymbol{r} = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}^T$ (expressed in the absolute frame) by an angle $\eta = -30°$. Which was the initial orientation of the body? Is it uniquely defined? Express the solution (or solutions) for the initial orientation by a rotation matrix $\boldsymbol{R}$ and in terms of XYZ Roll-Pitch-Yaw angles (ψ, θ, ϕ) around fixed axes.

Exercise 2 [2 points]

The following 4×4 matrix is given:

$$\boldsymbol{M} = \begin{pmatrix} -0.7071 & 0.5 & -0.5 & -1 \\ -0.7071 & -0.5 & 0.5 & -1 \\ 0 & 0.7071 & 0.7071 & -0.7071 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Is it possible to generate $\boldsymbol{M}$ by a set of four Denavit-Hartenberg parameters (α, a, d, θ)? If so, provide these values. If not, explain why.

Exercise 3 [8 points]

Consider the 7R manipulator in Fig. 1, where Denavit-Hartenberg (DH) frames have been defined already (with axes $\boldsymbol{x}_i$ in red, axes $\boldsymbol{y}_i$ in green, and axes $\boldsymbol{z}_i$ in blue). Ten reference frames are shown in total, 8 of which are DH frames, plus one fixed frame attached with the base platform, and a last one attached with a generic tool.

- On the extra sheet provided separately [*to be returned with your name*], complete the table of DH parameters. Enter in the table only numerical values (expressed in [rad] or [m]), including those of the joint variables $\boldsymbol{q}$ in the configuration shown. In the drawings, all data are given in mm. Note the presence of offsets (equal to 45 mm) at the elbow of the arm.
- Provide numerically the transformation matrix ${}^B\boldsymbol{T}_0$ from the base frame to the DH frame 0.

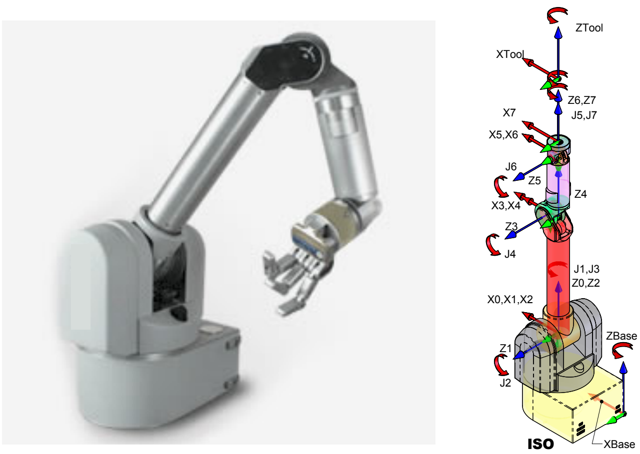

Figure 1: A 7R anthropomorphic manipulator.

Exercise 4 [6 points]

A revolute joint of a robot is actuated by a DC motor driven by a controlled voltage V_a in the armature circuit. We know that the motor is characterized by the following data, respectively for armature resistance, viscous friction, and current-to-torque gain parameters:

$$R_a = 0.309 \text{ [Ohm]}, \qquad F_m = 5 \cdot 10^{-2} \text{ [mNm/(rad/s)]}, \qquad k_t = 7.88 \text{ [mNm/A]}.$$

A harmonic drive reducer with a flexspline having 256 external teeth is used as a transmission element for moving the link. At steady state, the output axis on the link side is rotating at an angular speed $\omega = 180°/\text{s}$ in the counterclockwise direction.

- What is the value of the applied voltage V_a in this situation? Pay attention to the SI units!
- Which is the rotation speed ω_m (with sign, in [rad/s]) of the rotor of the motor?
- If an incremental encoder with quadrature detection is used on the motor side, how many bits should its internal counter use to obtain at least a resolution of $\epsilon = 10^{-3}$ [deg] on the link side? How many pulses/turn should the optical disk have? How many pulses would be counted in total by the counter in one second when the motor is rotating at the steady-state speed ω_m? Would the counter go up or down?

Exercise 5 [8 points]

The kinematics of a planar RP robot is defined by the following Tab. 1 of DH parameters:

i	α_i	a_i	d_i	θ_i
1	$-\pi/2$	$a_1 = 0.2$ [m]	0	q_1
2	$\pi/2$	0	q_2	π [rad]

Table 1: DH parameters for a planar RP robot.

We would like to solve an inverse kinematics problem for this robot using an iterative numerical method, either Newton or Gradient. The desired position the origin of the last frame in the plane $(\boldsymbol{x}_0, \boldsymbol{y}_0)$ is $\boldsymbol{p}_d = \begin{pmatrix} -2 & -3 \end{pmatrix}^T$ [m]. At some iteration k, the algorithm drives the robot from configuration $\boldsymbol{q}^k$ to $\boldsymbol{q}^{k+1}$, with

$$\boldsymbol{q}^k = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \text{ [rad, m]}, \qquad \boldsymbol{q}^{k+1} = \begin{pmatrix} -2.7742 \\ -0.6519 \end{pmatrix} \text{ [rad, m]}.$$

- Which is the solution method being used? Provide the symbolic expressions of each term in its general formula and their numerical values at the given iteration k.
- If the method converges, what is the expected solution $\boldsymbol{q}^*$? In this robot configuration, what will be the orientation of the last DH frame as expressed by the rotation matrix ${}^0\boldsymbol{R}_2(\boldsymbol{q}^*)$?
- In what configuration $\widehat{\boldsymbol{q}}$ would the Gradient method certainly stop with a non-zero position error for the above problem? What happens if we apply the Newton method there?

[240 minutes, open books]

Solution of Midterm Test

November 16, 2018

Exercise 1

The orientation of a rigid body expressed in terms a sequence of YXY Euler angles (α, β, γ) is represented by the rotation matrix

$$\begin{aligned}
\boldsymbol{R}_{YXY}(\alpha,\beta,\gamma) &= \boldsymbol{R}_Y(\alpha)\boldsymbol{R}_X(\beta)\boldsymbol{R}_Y(\gamma)\\
&= \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & 0 & \sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{pmatrix}\\
&= \begin{pmatrix} \cos\alpha\cos\gamma - \sin\alpha\cos\beta\sin\gamma & \sin\alpha\sin\beta & \cos\alpha\sin\gamma + \sin\alpha\cos\beta\cos\gamma \\ \sin\beta\sin\gamma & \cos\beta & -\sin\beta\cos\gamma \\ -\sin\alpha\cos\gamma - \cos\alpha\cos\beta\sin\gamma & \cos\alpha\sin\beta & -\sin\alpha\sin\gamma + \cos\alpha\cos\beta\cos\gamma \end{pmatrix}.
\end{aligned}$$

Thus, the final orientation of the considered body is specified with respect to the absolute reference frame as

$$^0\boldsymbol{R}_f = \boldsymbol{R}_{YXY}\left(\frac{\pi}{2}, -\frac{\pi}{4}, \frac{\pi}{4}\right) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{2}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{1}{2} \\ -\frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} -0.5 & -0.7071 & 0.5 \\ -0.5 & 0.7071 & 0.5 \\ -0.7071 & 0 & -0.7071 \end{pmatrix}.$$

Operatively, to obtain this result one can either evaluate numerically the symbolic matrix $\boldsymbol{R}_{YXY}$, or evaluate numerically the single elementary rotation matrices and then do their products.

The matrix associated to a rotation around an axis $\boldsymbol{r}$ by an angle η is given by

$$\boldsymbol{R}(\boldsymbol{r},\eta) = \boldsymbol{r}\boldsymbol{r}^T + \left(\boldsymbol{I} - \boldsymbol{r}\boldsymbol{r}^T\right)\cos\eta + \boldsymbol{S}(\boldsymbol{r})\sin\eta.$$

Thus, the rotation that changes the initial to the final orientation of the body is specified as

$$\begin{aligned}
^0\boldsymbol{R}_{i,f} &= \boldsymbol{R}\left(\begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}^T, -\frac{\pi}{6}\right)\\
&= \begin{pmatrix} \frac{\sqrt{3}+1}{3} & \frac{\sqrt{3}-1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{\sqrt{3}+1}{3} & \frac{\sqrt{3}-1}{3} \\ \frac{1-\sqrt{3}}{3} & -\frac{1}{3} & \frac{\sqrt{3}+1}{3} \end{pmatrix} = \begin{pmatrix} 0.9107 & 0.2440 & 0.3333 \\ -0.3333 & 0.9107 & 0.2440 \\ -0.2440 & -0.3333 & 0.9107 \end{pmatrix},
\end{aligned}$$

where the superscript 0 to $\boldsymbol{R}_{i,f}$ is there to indicate that the given rotation axis was expressed in the absolute reference frame (i.e., $^0\boldsymbol{r}$).

Since the two rotation matrices $^0\boldsymbol{R}_f$ and $^0\boldsymbol{R}_{if}$ are both defined with respect to the original reference frame, we have for their composition the product order

$$^0\boldsymbol{R}_f = {}^0\boldsymbol{R}_{i,f}\,{}^0\boldsymbol{R}_i.$$

Thus, the initial orientation expressed by the rotation matrix $^0\boldsymbol{R}_i$ is computed as

$$^0\boldsymbol{R}_i = {}^0\boldsymbol{R}_{i,f}^T\,{}^0\boldsymbol{R}_f = \begin{pmatrix} -0.1161 & -0.8797 & 0.4612 \\ -0.3416 & 0.4714 & 0.8131 \\ -0.9326 & -0.0632 & -0.3553 \end{pmatrix}.$$

This initial orientation is indeed uniquely specified. To express this solution using the XYZ Roll-Pitch-Yaw angles (ψ, θ, ϕ), we first compute symbolically the associated rotation matrix as

$$\begin{aligned}
\boldsymbol{R}_{XYZ}(\psi,\theta,\phi) &= \boldsymbol{R}_Z(\phi)\boldsymbol{R}_Y(\theta)\boldsymbol{R}_X(\psi) \\
&= \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{pmatrix} \\
&= \begin{pmatrix} \cos\phi\cos\theta & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi \\ \sin\phi\cos\theta & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi \\ -\sin\theta & \cos\theta\sin\psi & \cos\theta\cos\psi \end{pmatrix}.
\end{aligned}$$

Next, we solve the inverse representation problem for

$$\boldsymbol{R}_{XYZ}(\psi,\theta,\phi) = {}^0\boldsymbol{R}_i.$$

Denote by $R_{i,j}$ the elements of ${}^0\boldsymbol{R}_i$. Since $R_{3,2}^2 + R_{3,3}^2 \neq 0$, we are in a regular situation. Therefore, from

$$\cos\theta = \pm\sqrt{R_{3,2}^2 + R_{3,3}^2},$$

there are two solutions given by the angles (all given in [rad])

$$\begin{aligned}
\theta_1 &= \text{ATAN2}\left\{-R_{3,1}, \sqrt{R_{3,2}^2 + R_{3,3}^2}\right\} = 1.2016 \\
\psi_1 &= \text{ATAN2}\left\{\frac{R_{3,2}}{\sqrt{R_{3,2}^2 + R_{3,3}^2}}, \frac{R_{3,3}}{\sqrt{R_{3,2}^2 + R_{3,3}^2}}\right\} = -2.9657 \\
\phi_1 &= \text{ATAN2}\left\{\frac{R_{2,1}}{\sqrt{R_{2,1}^2 + R_{3,3}^2}}, \frac{R_{1,1}}{\sqrt{R_{3,2}^2 + R_{3,3}^2}}\right\} = -1.8985
\end{aligned}$$

and

$$\begin{aligned}
\theta_2 &= \text{ATAN2}\left\{-R_{3,1}, -\sqrt{R_{3,2}^2 + R_{3,3}^2}\right\} = 1.9400 \\
\psi_2 &= \text{ATAN2}\left\{\frac{R_{3,2}}{-\sqrt{R_{3,2}^2 + R_{3,3}^2}}, \frac{R_{3,3}}{-\sqrt{R_{3,2}^2 + R_{3,3}^2}}\right\} = 0.1759 \\
\phi_2 &= \text{ATAN2}\left\{\frac{R_{2,1}}{-\sqrt{R_{2,1}^2 + R_{3,3}^2}}, \frac{R_{1,1}}{-\sqrt{R_{3,2}^2 + R_{3,3}^2}}\right\} = 1.2431.
\end{aligned}$$

Exercise 2

The given 4×4 matrix $\boldsymbol{M}$ is indeed a homogeneous transformation matrix, since the upper left 3×3 block, say $\boldsymbol{R}$, is a rotation matrix (i.e., it is an orthonormal matrix, with determinant $= +1$). However, this is only a necessary condition for being able to express $\boldsymbol{R}$ only in terms of the two DH angular parameters α and θ. Therefore, we attempt directly to solve the matrix equation $\boldsymbol{A}(\alpha, a, d, \theta) = \boldsymbol{M}$, or

$$\begin{pmatrix} \cos\theta & -\cos\alpha\sin\theta & \sin\alpha\sin\theta & a\cos\theta \\ \sin\theta & \cos\alpha\cos\theta & -\sin\alpha\cos\theta & a\sin\theta \\ 0 & \sin\alpha & \cos\alpha & d \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -0.7071 & 0.5 & -0.5 & -1 \\ -0.7071 & -0.5 & 0.5 & -1 \\ 0 & 0.7071 & 0.7071 & -0.7071 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (1)$$

The following four values can be obtained (uniquely) from the analytic expressions

$$\alpha = \text{ATAN2}\{M_{3,2}, M_{3,3}\} = \text{ATAN2}\{0.7071, 0.7071\} = \frac{\pi}{4}$$
$$\theta = \text{ATAN2}\{M_{2,1}, M_{1,1}\} = \text{ATAN2}\{-0.7071, -0.7071\} = -\frac{3\pi}{4}$$
$$d = M_{3,4} = -0.7071 = -\frac{\sqrt{2}}{2}$$
$$a = M_{1,4}\cos\theta + M_{2,4}\sin\theta = (-1)\cdot(-0.7071) + (-1)\cdot(-0.7071) = \sqrt{2}.$$

It is easy to see that this set (α, a, d, θ) satisfies as identities also the remaining equations in (1).

Exercise 3

The robot shown in Fig. 1 is the Barrett Whole-Arm-Manipulator (WAM) with 7 revolute joints. The assigned frames comply with the standard Denavit-Hartenberg convention. The associated parameters are given in Tab. 2, with data expressed in [rad] or [m] and with the numerical values of the joint variables $\boldsymbol{q}$ taken in the configuration shown (the 'straight up' configuration is in fact the *zero* configuration for this arm). See also the compiled extra sheet at the end of the solution.

i	α_i	a_i	d_i	θ_i
1	$-\pi/2$	0	0	$q_1 = 0$
2	$\pi/2$	0	0	$q_2 = 0$
3	$-\pi/2$	0.045	0.55	$q_3 = 0$
4	$\pi/2$	-0.045	0	$q_4 = 0$
5	$-\pi/2$	0	0.3	$q_5 = 0$
6	$\pi/2$	0	0	$q_6 = 0$
7	0	0	0.06	$q_7 = 0$

Table 2: Parameters associated to the DH frames in Fig. 1.

From the sheet, one determines also the transformation matrix from base frame to DH frame 0 as

$$^B\boldsymbol{T}_0 = \begin{pmatrix} 1 & 0 & 0 & 0.220 \\ 0 & 1 & 0 & 0.140 \\ 0 & 0 & 1 & 0.346 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Exercise 4

The dynamic equations of a DC motor in the time domain are known to be

$$\begin{aligned} L_a \frac{di_a}{dt} &= V_a - R_a i_a - V_{emf}, \qquad \text{with } V_{emf} = k_v\,\omega_m, \qquad \text{(electrical balance)} \\ I_m \frac{\omega_m}{dt} &= T_m - F_m\omega_m - T_{load}, \qquad \text{with } T_m = k_t\,i_a. \qquad \text{(mechanical balance)} \end{aligned} \tag{2}$$

Assume no disturbance load, $T_{load} = 0$. When the rotor of the motor is rotating at a constant angular velocity[1] $\bar{\omega}_m$, we have the steady-state conditions

$$\bar{V}_a = R_a \bar{i}_a + k_v \bar{\omega}_m, \qquad \bar{T}_m = k_t \bar{i}_a = F_m \bar{\omega}_m, \tag{3}$$

which are obtained by setting to zero the two time derivatives in (2). The motor needs to apply a torque $\bar{T}_m$ (and thus deliver a steady-state armature current $\bar{i}_a$) to compensate for the energy dissipation due to the viscous friction term $F_m\bar{\omega}_m$ at constant angular velocity. The steady-state input voltage $\bar{V}_a$ will then balance the constant back emf $k_v\bar{\omega}_m$ and produce also the required current $\bar{i}_a$ flowing through the armature resistance R_a.

Special care should be taken for the numerical equivalence between the back emf coefficient k_v and the current-to-torque gain k_t of the motor. Based on the conservation of energy principle, we have

$$k_v \text{ [V/(rad/s)]} = k_t \text{ [Nm/A]}$$

only when using the indicated SI units. Therefore, in our case we will have

$$k_t = 7.88 \text{ [mNm/A]} = 7.88 \cdot 10^{-3} \text{ [Nm/A]} \qquad \Rightarrow \qquad k_v = 7.88 \cdot 10^{-3} \text{ [V/(rad/s)]}. \tag{4}$$

The harmonic drive (HD) has a reduction ratio $n > 1$ and transforms input angular velocities ω_m of the motor into output angular velocities ω of the link as

$$n = \frac{N_{flex}}{2} = \frac{256}{2} = 128, \qquad \omega_m = -n\,\omega, \tag{5}$$

where the minus sign denotes the inversion in the rotation direction due to the HD reducer.

When the link is rotating at steady state with an angular velocity $\bar{\omega} = 180°/\text{s} = \pi$ rad/s (positive, being a counterclockwise rotation), combining eqs. (3) and (5) and keeping into account the conversion (4), we obtain the numerical results

$$\begin{aligned}
\bar{\omega}_m &= -n\,\bar{\omega} = -128 \cdot \pi = -402.1239 \text{ [rad/s]} \quad \text{(clockwise!)} \\
\bar{i}_a &= \frac{F_m}{k_t}\,\bar{\omega}_m \left(= -\frac{F_m}{k_t}\,n\,\bar{\omega} \right) = -\frac{5 \cdot 10^{-2}}{7.88}\,128\,\pi = -2.5515 \text{ [A]}, \\
\bar{V}_a &= R_a \bar{i}_a + k_v \bar{\omega}_m \left(= -\left(\frac{R_a F_m}{k_t} + k_v \right) n\,\bar{\omega} \right) \\
&= -0.309 \cdot 2.5515 - 7.88 \cdot 10^{-3} \cdot 402.1239 = -(0.7885 + 3.1687) = -3.9572 \text{ [V]}.
\end{aligned}$$

With a desired resolution $\epsilon = 10^{-3}$ [deg] on the link side of the transmission, we need an output resolution of the incremental encoder on the motor side equal to

$$\epsilon_m = \frac{\Delta\theta_m}{\text{pulse}} = n \cdot \epsilon = 128 \cdot 10^{-3} \text{ [deg]}.$$

Thus, the internal counter of the incremental encoder should be able to count a number N_p of pulses/turn at least equal to

$$N_p = \left\lceil \frac{360}{\epsilon_m} \right\rceil = \left\lceil \frac{360}{128} \cdot 10^3 \right\rceil = 2813,$$

[1] An angular velocity is an angular speed with sign (positive if rotating CCW, as seen from the observation axis).

which requires a minimum number N_b of bits for the digital counter equal to

$$N_b = \lceil \log_2(N_p) \rceil = 12.$$

On the other hand, thanks to the quadrature electronics, the minimum number N_d of pulses/turn of the optical disk that guarantees the desired resolution is

$$N_d = \left\lceil \frac{N_p}{4} \right\rceil = 704.$$

With this incremental encoder, when the motor is spinning at the steady-state angular velocity $\bar{\omega}_m$, the total count of pulses by the digital counter in one second (without considering resets) would then be

$$\text{count} = \left\lfloor |\bar{\omega}_m| \cdot \frac{N_p}{2\pi} \right\rfloor = \left\lfloor 402.1239 \cdot \frac{2813}{2\pi} \right\rfloor = 180032.$$

The counter goes *down*, since the steady-state angular velocity of the motor is negative (the motor rotates CW, while the link rotates CCW having a positive angular velocity).

Exercise 5

The direct kinematics of the planar RP robot is computed from the parameters in Tab. 1, using the DH homogeneous transformation matrices (and keeping for the moment a_1 as a symbolic term):

$$\begin{aligned} {}^0\boldsymbol{A}_2(\boldsymbol{q}) \;=\; {}^0\boldsymbol{A}_1(q_1)\; {}^1\boldsymbol{A}_2(q_2) &= \begin{pmatrix} \cos q_1 & 0 & -\sin q_1 & a_1 \cos q_1 \\ \sin q_1 & 0 & \cos q_1 & a_1 \sin q_1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & q_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -\cos q_1 & -\sin q_1 & 0 & a_1 \cos q_1 - q_2 \sin q_1 \\ -\sin q_1 & \cos q_1 & 0 & a_1 \sin q_1 + q_2 \cos q_1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^0\boldsymbol{R}_2(q_1) & {}^0\boldsymbol{p}_2(\boldsymbol{q}) \\ \boldsymbol{0}^T & 1 \end{pmatrix}. \end{aligned} \tag{6}$$

The planar position mapping of interest is given by the first two components of ${}^0\boldsymbol{p}_2(\boldsymbol{q})$ in (6), i.e.,

$$\boldsymbol{p}(\boldsymbol{q}) = \begin{pmatrix} a_1 \cos q_1 - q_2 \sin q_1 \\ a_1 \sin q_1 + q_2 \cos q_1 \end{pmatrix}. \tag{7}$$

Differentiating (7) with respect to $\boldsymbol{q}$ yields the 2×2 Jacobian matrix

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{p}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -a_1 \sin q_1 - q_2 \cos q_1 & -\sin q_1 \\ a_1 \cos q_1 - q_2 \sin q_1 & \cos q_1 \end{pmatrix}, \tag{8}$$

which is nonsingular unless $\det \boldsymbol{J}(\boldsymbol{q}) = -q_2 = 0$.

The underlying inverse kinematics problem considered for this robot requires to solve two nonlinear equations, i.e.,

$$\boldsymbol{p}(\boldsymbol{q}) = \boldsymbol{p}_d \qquad \Rightarrow \qquad \begin{pmatrix} a_1 \cos q_1 - q_2 \sin q_1 \\ a_1 \sin q_1 + q_2 \cos q_1 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}, \tag{9}$$

through the use of an iterative numerical method. A generic iteration step toward the solution is presented, from a configuration $\boldsymbol{q}^k$ to $\boldsymbol{q}^{k+1}$.

Since the Jacobian at $\boldsymbol{q}^k$ is nonsingular ($q_2^k = 2 \neq 0$), the increment provided by the numerical method used at iteration k

$$\Delta\boldsymbol{q}^k = \boldsymbol{q}^{k+1} - \boldsymbol{q}^k = \begin{pmatrix} -2.7742 \\ -0.6519 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1.7742 \\ -2.6519 \end{pmatrix} \text{ [rad, m]} \tag{10}$$

could have been obtained in principle either by the Newton or by the Gradient method (the latter with some step size $\alpha_k > 0$).

To verify if the Newton method was used, we evaluate the relevant quantities at $\boldsymbol{q}^k$ (with $a_1 = 0.2$):

$$\boldsymbol{e}^k = \boldsymbol{p}_d - \boldsymbol{p}(\boldsymbol{q}^k) = \begin{pmatrix} -2 \\ -3 \end{pmatrix} - \begin{pmatrix} 1.7910 \\ 0.9123 \end{pmatrix} = \begin{pmatrix} -3.7910 \\ -3.9123 \end{pmatrix}, \qquad \|\boldsymbol{e}^k\| = 5.4477 \tag{11}$$

$$\boldsymbol{J}(\boldsymbol{q}^k) = \begin{pmatrix} -0.9123 & 0.8415 \\ 1.7910 & 0.5403 \end{pmatrix}. \tag{12}$$

It is easy to see that

$$\boldsymbol{q}^k + \boldsymbol{J}^{-1}(\boldsymbol{q}^k)\boldsymbol{e}^k = \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} -0.2702 & 0.4207 \\ 0.8955 & 0.4562 \end{pmatrix} \begin{pmatrix} -3.7910 \\ -3.9123 \end{pmatrix} = \begin{pmatrix} -1.6219 \\ -3.1795 \end{pmatrix} \neq \boldsymbol{q}^{k+1} = \begin{pmatrix} -2.7742 \\ -0.6519 \end{pmatrix}.$$

We conclude that the Newton method was not used. On the other hand, the increment given by the Gradient method is

$$\alpha_k \boldsymbol{J}^T(\boldsymbol{q}^k)\boldsymbol{e}^k = \alpha_k \begin{pmatrix} -0.9123 & 1.7910 \\ 0.8415 & 0.5403 \end{pmatrix} \begin{pmatrix} -3.7910 \\ -3.9123 \end{pmatrix} = \alpha_k \begin{pmatrix} -3.5484 \\ -5.3038 \end{pmatrix}.$$

We note that the following two equalities

$$\alpha_k \begin{pmatrix} -3.5484 \\ -5.3038 \end{pmatrix} = \begin{pmatrix} -1.7742 \\ -2.6519 \end{pmatrix} = \Delta\boldsymbol{q}^k$$

are simultaneously satisfied when selecting $\alpha_k = 0.5$ as step size. Thus, the Gradient method was used at iteration k.

The displacement $\Delta\boldsymbol{q}^k$ obtained by the Gradient method leads to a new Cartesian position error $\boldsymbol{e}^{k+1}$ that has a smaller norm than the previous $\boldsymbol{e}^k$ in (11):

$$\boldsymbol{e}^{k+1} = \boldsymbol{p}_d - \boldsymbol{p}(\boldsymbol{q}^{k+1}) = \begin{pmatrix} -2 \\ -3 \end{pmatrix} - \begin{pmatrix} -0.4208 \\ 0.5366 \end{pmatrix} = \begin{pmatrix} -1.5792 \\ -3.5366 \end{pmatrix}, \quad \|\boldsymbol{e}^{k+1}\| = 3.8731 < \|\boldsymbol{e}^k\|. \tag{13}$$

The method is thus converging at this stage, although it may eventually require a reduction of the step size in order to avoid the missing of a solution.

Indeed, we can also determine all solutions to the inverse kinematics problem (9) in a closed form. For this, consider again the two nonlinear equations (7) of the direct kinematics. Squaring each equation and summing yields after simplifications

$$p_x^2 + p_y^2 = a_1^2 + q_2^2,$$

and so

$$q_2^{a,b} = \pm\sqrt{p_x^2 + p_y^2 - a_1^2}. \tag{14}$$

The two solutions (14) for the prismatic joint are real and distinct iff $\|\boldsymbol{p}\|^2 = p_x^2 + p_y^2 > a_1^2$ and collapse into the same one for $\|\boldsymbol{p}\|^2 = a_1^2$. This is in fact a singular case, and provides $q_2 = 0$ as the unique solution. For $\|\boldsymbol{p}\|^2 < a_1^2$, the point in the plane $(\boldsymbol{x}_0, \boldsymbol{y}_0)$ is out of the workspace (it belongs to an inner circle of radius $|a_1| \geq 0$).

Assume now that $a_1 \neq 0$ and that $\boldsymbol{p}$ belongs to the (primary) workspace of the RP robot. For each solution (14), i.e., two in the regular case or only one, $q_2 = 0$, in the singular case, reorganize the direct kinematics as a linear system in the unknowns $\sin q_1$ and $\cos q_1$:

$$\begin{pmatrix} -q_2^{a,b} & a_1 \\ a_1 & q_2^{a,b} \end{pmatrix} \begin{pmatrix} \sin q_1 \\ \cos q_1 \end{pmatrix} = \begin{pmatrix} p_x \\ p_y \end{pmatrix}.$$

A unique solution for q_1 can always be found for each value entered as q_2. We obtain

$$q_1^{a,b} = \text{ATAN2}\left\{a_1 p_y - q_2^{a,b} p_x, a_1 p_x + q_2^{a,b} p_y\right\}. \tag{15}$$

Using the problem data ($\boldsymbol{p} = \boldsymbol{p}_d$ and $a_1 = 0.2$), equations (14) and (15) provide the two solutions

$$\boldsymbol{q}^a = \begin{pmatrix} 2.6091 \\ 3.6 \end{pmatrix}, \qquad \boldsymbol{q}^b = \begin{pmatrix} -0.6435 \\ -3.6 \end{pmatrix} \quad \text{[rad, m]}.$$

At this stage, there is no simple argument that helps us in identifying which inverse kinematic solution would be reached by the Gradient iterative method when continuing its evolution from $\boldsymbol{q}^{k+1}$. Instead of guessing, we just provide the requested orientation of the last DH frame in the two cases, as given by ${}^0\boldsymbol{R}_2(q_1)$ in eq. (6):

$${}^0\boldsymbol{R}_2(q_1^a) = \begin{pmatrix} 0.8615 & -0.5077 & 0 \\ -0.5077 & -0.8615 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad {}^0\boldsymbol{R}_2(q_1^b) = \begin{pmatrix} -0.8 & 0.6 & 0 \\ 0.6 & 0.8 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

We determine next the configurations $\widehat{\boldsymbol{q}}$ at which the Gradient method would certainly stop with a non-zero position error for the problem (9) at hand. This requires the Jacobian $\boldsymbol{J}(\widehat{\boldsymbol{q}})$ in (8) to be singular and the position error vector $\widehat{\boldsymbol{e}} = \boldsymbol{p}_d - \boldsymbol{p}(\widehat{\boldsymbol{q}})$ to belong to the null space of $\boldsymbol{J}^T(\widehat{\boldsymbol{q}})$. Therefore, rewrite the Jacobian transpose, the direct kinematics, and the position error in the singularity $\widehat{q}_2 = q_2 = 0$:

$$\boldsymbol{J}_0^T(q_1) = \boldsymbol{J}^T(\boldsymbol{q})|_{q_2=0} = \begin{pmatrix} -a_1 \sin q_1 & a_1 \cos q_1 \\ -\sin q_1 & \cos q_1 \end{pmatrix},$$

$$\boldsymbol{p}_0(q_1) = \boldsymbol{p}(\boldsymbol{q})|_{q_2=0} = \begin{pmatrix} a_1 \cos q_1 \\ a_1 \sin q_1 \end{pmatrix}, \qquad \boldsymbol{e}_0(q_1) = \boldsymbol{e}(\boldsymbol{q})|_{q_2=0} = \boldsymbol{p}_d - \boldsymbol{p}_0 = \begin{pmatrix} -2 - a_1 \cos q_1 \\ -3 - a_1 \sin q_1 \end{pmatrix}.$$

The null space of $\boldsymbol{J}_0^T$ is spanned by a single basis vector

$$\ker\left\{\boldsymbol{J}_0^T(q_1)\right\} = \beta \begin{pmatrix} \cos q_1 \\ \sin q_1 \end{pmatrix}, \qquad \forall \beta.$$

In order to find a suitable value of q_1 such that $\boldsymbol{e}_0 \in \ker\left\{\boldsymbol{J}_0^T\right\}$, we consider the simple linear system in the unknowns $\sin q_1$ and $\cos q_1$, parametrized by the scalar β:

$$\begin{pmatrix} -2 - a_1 \cos q_1 \\ -3 - a_1 \sin q_1 \end{pmatrix} = \beta \begin{pmatrix} \cos q_1 \\ \sin q_1 \end{pmatrix} \quad \Rightarrow \quad (\beta + a_1) \begin{pmatrix} \cos q_1 \\ \sin q_1 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}.$$

This leads to two solutions (depending on the arbitrary sign —positive or negative— that the factor $(\beta + a_1)$ can assume):

$$\hat{q}_1^a = \text{ATAN2}\{-3, -2\} = -2.1588, \qquad \hat{q}_1^b = \text{ATAN2}\{3, 2\} = 0.9828 \quad \text{[rad]}.$$

It is left to the reader to check that in both cases $\boldsymbol{J}^T(\hat{\boldsymbol{q}})\boldsymbol{e}(\hat{\boldsymbol{q}}) = \boldsymbol{0}$ (even if $\boldsymbol{e}(\hat{\boldsymbol{q}}) \neq \boldsymbol{0}$), resulting in a stopping condition for the Gradient method. It is also very informative at this point to sketch a picture of the robot arm in the configuration $\hat{\boldsymbol{q}}$, and draw the error $\hat{\boldsymbol{e}}$ associated to the considered positioning task for the end effector.

When such a situation is encountered, one can force the Gradient method to restart in many possible ways, e.g., by slightly perturbing the current robot configuration so that the position error $\boldsymbol{e}$ exits the null space of $\boldsymbol{J}^T$ or, even better, by momentarily rotating the actual error $\boldsymbol{e}$ (multiplying it by a skew-symmetric matrix $\boldsymbol{K}_s$) so as to obtain the same effect. On the other hand, in a singular configuration (or very close to it) we can never apply the Newton method —at least, not as such.

* * * * *