Robotics 1

July 13, 2026

Exercise 1

Consider the RRP spatial robot in Fig. 1. The robot has an offset at the shoulder. The base frame should be placed on the ground.

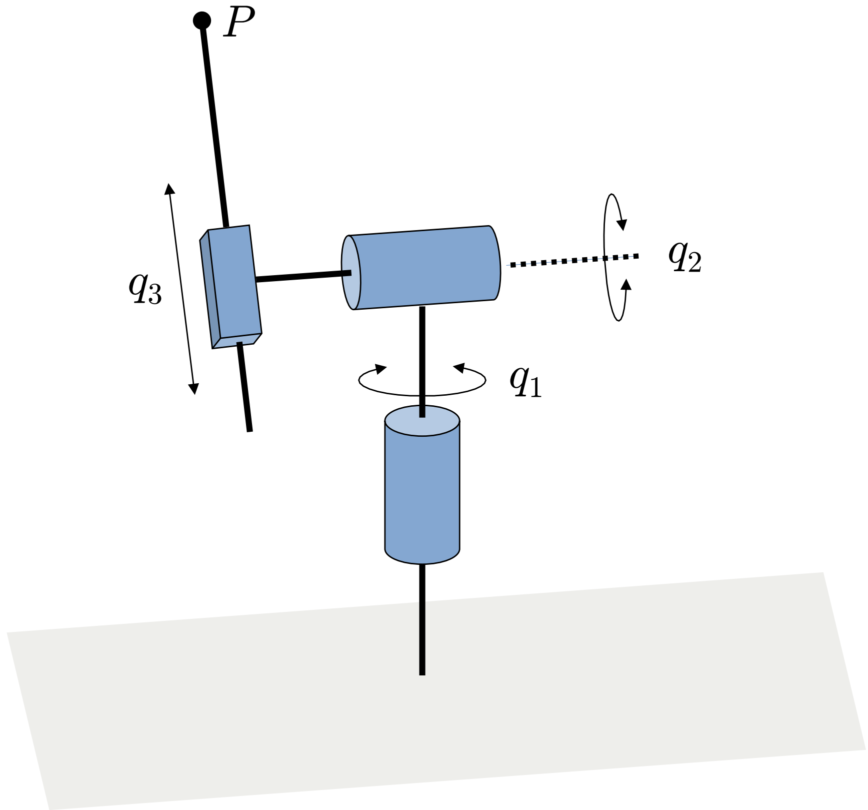

Figure 1: An RRP spatial robot

- **a)** Assign a set of frames according to the standard Denavit-Hartenberg (DH) convention so that all constant DH parameters are *nonnegative*. Place the origin of the last DH frame at point P. Draw the frames directly on the figure. Provide the corresponding table of parameters.
- **b)** Compute the direct kinematics $\boldsymbol{p} = \boldsymbol{k}(\boldsymbol{q})$ of point P and the associated analytic Jacobian $\boldsymbol{J}(\boldsymbol{q})$ in symbolic form.
- **c)** For the symbolic expression of the rotation matrix $\boldsymbol{R} = {}^0\boldsymbol{R}_3(\boldsymbol{q})$ of the last frame, solve the inverse problem in terms of XYZ angles (ψ, θ, ϕ) of the RPY-type. Is it possible to encounter a singular situation for this specific problem?
- **d)** Determine *all* singular configurations $\boldsymbol{q}_s$ of the Jacobian $\boldsymbol{J}(\boldsymbol{q})$. For each case, evaluate the rank and give a basis for the null space of $\boldsymbol{J}(\boldsymbol{q}_s)$.
- **e)** In a singular configuration $\boldsymbol{q}_s$, provide a velocity $\boldsymbol{v} \in \mathbb{R}^3$ of point P that is admissible. Find then a joint velocity $\dot{\boldsymbol{q}}_s \in \mathbb{R}^3$ with minimum norm that realizes $\boldsymbol{v}$.
- **f)** For a given $\boldsymbol{p} = \boldsymbol{p}_d$, determine all inverse kinematics solutions $\boldsymbol{q} = \boldsymbol{k}^{-1}(\boldsymbol{p}_d)$ in closed form, taking into account the singular cases.

Exercise 2

Determine a polynomial joint trajectory $\boldsymbol{q}(t)$ that realizes a *coordinated* motion in minimum time T of two robot joints from $\boldsymbol{q}(0) = (0, -\pi/2)$ to $\boldsymbol{q}(T) = (\pi/4, \pi/4)$ [rad], with boundary conditions $\dot{\boldsymbol{q}}(0) = \ddot{\boldsymbol{q}}(0) = \dot{\boldsymbol{q}}(T) = \ddot{\boldsymbol{q}}(T) = \mathbf{0}$ and under the acceleration bounds

$$|\ddot{q}_1(t)| \leq A_1 = 6 \qquad |\ddot{q}_2(t)| \leq A_2 = 10 \text{ [rad/s}^2\text{]} \qquad \text{for } t \in [0, T].$$

Provide the minimum time T and the maximum absolute velocities reached by the two joints, i.e. $V_i = \max_{t\in[0.T]} |\dot{q}_i(t)|$, for $i = 1, 2$. Sketch (at least) the joint acceleration profiles.

[180 minutes (3 hours); open books]

Solution

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Exercise 1

a) A correct assignment of DH frames for the RRP spatial robot is shown in Fig. 2. In the corresponding Tab. 1, all constant parameters are nonnegative as required.

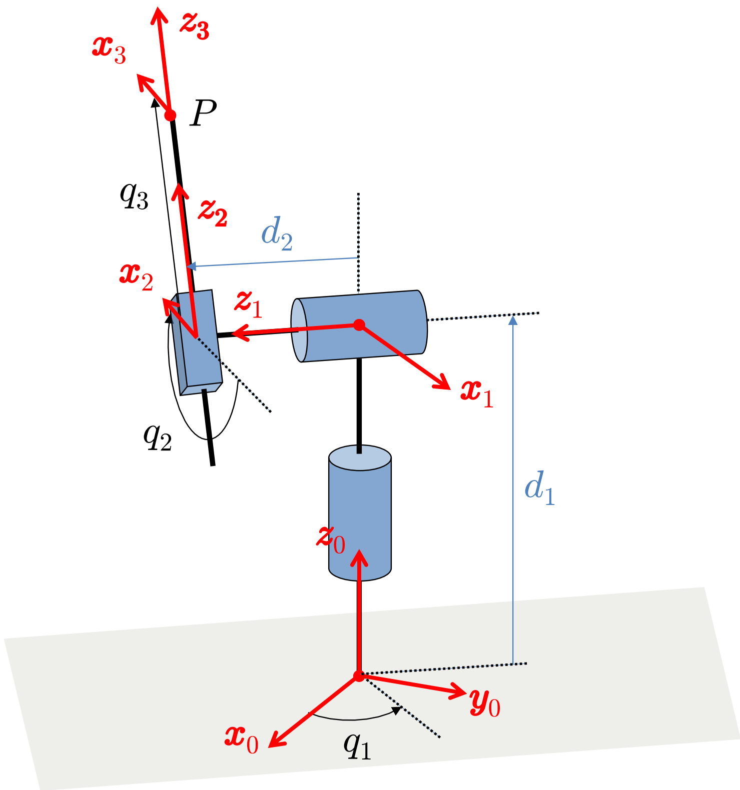

Figure 2: DH frames for the RRP robot in Fig. 1

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	$d_1 > 0$	q_1
2	$\pi/2$	0	$d_2 > 0$	q_2
3	0	0	q_3	0

Table 1: DH parameters for the frame assignment in Fig. 2

In the configuration shown in Fig. 2, the signs of the joint variables are $q_1 > 0$, $q_2 < 0$ and $q_3 > 0$.

b) Based on Tab. 1, the direct kinematics gives the position of point $P = O_3$ as

$$\boldsymbol{p} = \begin{pmatrix} d_2 \sin q_1 + q_3 \cos q_1 \sin q_2 \\ -d_2 \cos q_1 + q_3 \sin q_1 \sin q_2 \\ d_1 - q_3 \cos q_2 \end{pmatrix} = \boldsymbol{k}(\boldsymbol{q}) \tag{1}$$

and the orientation of the last DH frame as

$$\boldsymbol{R} = \begin{pmatrix} \cos q_1 \cos q_2 & \sin q_1 & \cos q_1 \sin q_2 \\ \sin q_1 \cos q_2 & -\cos q_1 & \sin q_1 \sin q_2 \\ \sin q_2 & 0 & -\cos q_2 \end{pmatrix} = {}^0\boldsymbol{R}_3(\boldsymbol{q}) = {}^0\boldsymbol{R}_2(\boldsymbol{q}). \tag{2}$$

Differentiating (1) with respect to time, we obtain the velocity $\boldsymbol{v} \in \mathbb{R}^3$ of point P and the Jacobian matrix $\boldsymbol{J}(\boldsymbol{q})$ as

$$\boldsymbol{v} = \frac{\partial \boldsymbol{k}}{\partial \boldsymbol{q}}\,\dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} \qquad \boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} d_2 \cos q_1 - q_3 \sin q_1 \sin q_2 & q_3 \cos q_1 \cos q_2 & \cos q_1 \sin q_2 \\ d_2 \sin q_1 + q_3 \cos q_1 \sin q_2 & q_3 \sin q_1 \cos q_2 & \sin q_1 \sin q_2 \\ 0 & q_3 \sin q_2 & -\cos q_2 \end{pmatrix}. \tag{3}$$

For rank analysis, it may be convenient to express the Jacobian in the rotated frame 1

$${}^1\boldsymbol{J}(\boldsymbol{q}) = {}^0\boldsymbol{R}_1^T(q_1)\,\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} \cos q_1 & \sin q_1 & 0 \\ 0 & 0 & 1 \\ \sin q_1 & -\cos q_1 & 0 \end{pmatrix} \boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} d_2 & q_3 \cos q_2 & \sin q_2 \\ 0 & q_3 \sin q_2 & -\cos q_2 \\ -q_3 \sin q_2 & 0 & 0 \end{pmatrix},$$

or even in frame 2 (for this, we can use directly the matrix $\boldsymbol{R}$ in (2))

$${}^2\boldsymbol{J}(\boldsymbol{q}) = {}^0\boldsymbol{R}_2^T(q_1, q_2)\,\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} d_2 \cos q_2 & q_3 & 0 \\ -q_3 \sin q_2 & 0 & 0 \\ d_2 \sin q_2 & 0 & 1 \end{pmatrix}.$$

c) The RPY-type orientation matrix obtained by the XYZ sequence of angles (ψ, θ, ϕ) around *fixed* axes is

$$\begin{aligned} \boldsymbol{R}_{RPY} &= \boldsymbol{R}_Z(\phi)\boldsymbol{R}_Y(\theta)\boldsymbol{R}_X(\psi) \\ &= \begin{pmatrix} \cos\phi\cos\theta & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi \\ \sin\phi\cos\theta & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi \\ -\sin\theta & \cos\theta\sin\psi & \cos\theta\cos\psi \end{pmatrix}. \end{aligned} \tag{4}$$

Comparing (4) with (2) it can be recognized that, provided the elements R_{ij} of $\boldsymbol{R}$ satisfy

$$R_{32}^2 + R_{33}^2 = \cos^2 q_2 \neq 0 \qquad \Rightarrow \qquad q_2 \neq \pm\frac{\pi}{2},$$

the matrix equation $\boldsymbol{R}_{RPY} = \boldsymbol{R}$ has the two regular solutions

$$(\psi, \theta, \phi) = (\pi, -q_2, q_1) \qquad (\psi, \theta, \phi) = (0, q_2 + \pi, q_1 - \pi). \tag{5}$$

The two identites can be easily verified by direct substitution of the symbolic values from (5) in (4).

When $q_2 = \pm\pi/2$, matrix $\boldsymbol{R}$ becomes

$$\boldsymbol{R}\,|_{q_2 = \pm\pi/2} = \begin{pmatrix} 0 & \sin q_1 & \pm\cos q_1 \\ 0 & -\cos q_1 & \pm\sin q_1 \\ \pm 1 & 0 & 0 \end{pmatrix}$$

and only the angle $\theta = \mp\pi/2$ in the RPY sequence is determined (singularity of the RPY representation), while there is an infinity of solutions for the pair (ψ, ϕ), either with $\phi - \psi = q_1 + \pi$ (for $\theta = \pi/2$) or with $\phi + \psi = q_1 + \pi$ (for $\theta = -\pi/2$).

d) The determinant of the Jacobian in (3) is

$$\det \boldsymbol{J}(\boldsymbol{q}) \left(= \det {}^1\boldsymbol{J}(\boldsymbol{q}) = \det {}^2\boldsymbol{J}(\boldsymbol{q})\right) = q_3^2 \sin q_2,$$

so that the singular configurations $\boldsymbol{q}_s$ of the robot have $q_3 = 0$ and/or $q_2 = \{0, \pi\}$.

Substituting $q_3 = 0$ in (3) with $q_2 \neq \{0, \pi\}$ gives

$$\boldsymbol{J}_a = \boldsymbol{J}(\boldsymbol{q}_s)|_{q_3=0} = \begin{pmatrix} d_2 \cos q_1 & 0 & \cos q_1 \sin q_2 \\ d_2 \sin q_1 & 0 & \sin q_1 \sin q_2 \\ 0 & 0 & -\cos q_2 \end{pmatrix} \quad \text{rank}\{\boldsymbol{J}_a\} = 2 \quad \mathcal{N}(\boldsymbol{J}_a) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{6}$$

Moreover, when $q_2 = \pm\pi/2$, we have

$$\boldsymbol{J}_b = \boldsymbol{J}(\boldsymbol{q}_s)|_{q_2=\pm\pi/2, q_3=0} = \begin{pmatrix} d_2 \cos q_1 & 0 & \pm\cos q_1 \\ d_2 \sin q_1 & 0 & \pm\sin q_1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{rank}\{\boldsymbol{J}_b\} = 1$$

and

$$\mathcal{N}(\boldsymbol{J}_b) = \text{span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ \mp d_2 \end{pmatrix} \right\}. \tag{7}$$

Substituting $q_2 = 0$ in (3) with $q_3 \neq 0$ gives

$$\boldsymbol{J}_c = \boldsymbol{J}(\boldsymbol{q}_s)|_{q_2=0} = \begin{pmatrix} d_2 \cos q_1 & q_3 \cos q_1 & 0 \\ d_2 \sin q_1 & q_3 \sin q_1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{rank}\{\boldsymbol{J}_c\} = 2 \quad \mathcal{N}(\boldsymbol{J}_c) = \begin{pmatrix} -q_3 \\ d_2 \\ 0 \end{pmatrix}$$

and similarly for $q_2 = \pi$ and $q_3 \neq 0$

$$\boldsymbol{J}_d = \boldsymbol{J}(\boldsymbol{q}_s)|_{q_2=\pi} = \begin{pmatrix} d_2 \cos q_1 & -q_3 \cos q_1 & 0 \\ d_2 \sin q_1 & -q_3 \sin q_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{rank}\{\boldsymbol{J}_d\} = 2 \quad \mathcal{N}(\boldsymbol{J}_d) = \begin{pmatrix} q_3 \\ d_2 \\ 0 \end{pmatrix}.$$

Finally, when $q_3 = 0$ and $q_2 = 0$ or $q_2 = \pi$ are considered together, one has

$$\boldsymbol{J}_e = \boldsymbol{J}(\boldsymbol{q}_s)|_{q_2=\{0,\pi\}, q_3=0} = \begin{pmatrix} d_2 \cos q_1 & 0 & 0 \\ d_2 \sin q_1 & 0 & 0 \\ 0 & 0 & \mp 1 \end{pmatrix} \quad \text{rank}\{\boldsymbol{J}_e\} = 2 \quad \mathcal{N}(\boldsymbol{J}_e) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

As a result, the only situation in which there is a double loss of rank for the Jacobian $\boldsymbol{J}(\boldsymbol{q})$ is when $(q_2, q_3) = (\pm\pi/2, 0)$ —case b) with (7).

e) Consider now for instance the singular case in (6). An admissible velocity of point P can be generated by assigning an arbitrary value to the joint velocity $\dot{\boldsymbol{q}}$ and multiplying it by the Jacobian evaluated at the chosen singular configuration $\boldsymbol{q}_s$. Choose, e.g., $\boldsymbol{q}_s^* = (\pi/4, \pi/4, 0)$ [rad] and $\dot{\boldsymbol{q}}^* = (1, 1, 1)$ [rad/s]. Note that $\|\dot{\boldsymbol{q}}^*\| = \sqrt{3}$. We have

$$\boldsymbol{v} = \boldsymbol{J}(\boldsymbol{q}_s^*)\, \dot{\boldsymbol{q}}^* = \begin{pmatrix} \sqrt{2}\, d_2/2 & 0 & 0.5 \\ \sqrt{2}\, d_2/2 & 0 & 0.5 \\ 0 & 0 & -\sqrt{2}/2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} (1 + \sqrt{2}\, d_2)/2 \\ (1 + \sqrt{2}\, d_2)/2 \\ -\sqrt{2}/2 \end{pmatrix}. \tag{8}$$

The same admissible velocity $\boldsymbol{v}$ in (8) can also be obtained by a joint velocity with lower norm. In particular, the pseudoinverse solution[1]

$$\dot{\boldsymbol{q}}_s = \boldsymbol{J}^{\#}(\boldsymbol{q}_s^*)\,\boldsymbol{v} = \begin{pmatrix} \sqrt{2}/(2d_2) & \sqrt{2}/(2d_2) & 1/d_2 \\ 0 & 0 & 0 \\ 0 & 0 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} (1+\sqrt{2}\,d_2)/2 \\ (1+\sqrt{2}\,d_2)/2 \\ -\sqrt{2}/2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \tag{9}$$

is the joint velocity with minimum norm (e.g., $\|\dot{\boldsymbol{q}}_s\| = \sqrt{2} < \|\dot{\boldsymbol{q}}^*\|$) realizing the given admissible $\boldsymbol{v}$. The obtained solution $\dot{\boldsymbol{q}}_s$ should not come to a surprise: being the second column of the singular Jacobian $\boldsymbol{J}(\boldsymbol{q}_s^*)$ zero, suggests already that no velocity $\dot{q}_2$ could contribute to an admissible $\boldsymbol{v}$.

A similar analysis can be conducted for the other singular cases.

f) Let a desired position $\boldsymbol{p}_d = (p_{dx}, p_{dy}, p_{dz})$ of point P be given. From the direct kinematics (1) with $\boldsymbol{p} = \boldsymbol{p}_d$ one has

$$p_{dx}^2 + p_{dy}^2 + (p_{dz} - d_1)^2 = d_2^2 + q_3^2$$

and so

$$q_3 = \pm\sqrt{p_{dx}^2 + p_{dy}^2 + (p_{dz} - d_1)^2 - d_2^2}. \tag{10}$$

For each solution $q_3 \neq 0$, we have

$$\cos q_2 = \frac{d_1 - p_{dz}}{q_3} \qquad \sin^2 q_2 = \frac{p_{dx}^2 + p_{dy}^2 - d_2^2}{q_3^2}$$

and thus the two solutions

$$q_2 = \text{ATAN2}\{\pm\sqrt{\frac{p_{dx}^2 + p_{dy}^2 - d_2^2}{q_3^2}},\, \frac{d_1 - p_{dz}}{q_3}\}. \tag{11}$$

The arguments inside the square roots in (10) and (11) should be non-negative in order to obtain real values for q_3 and, respectively, q_2. Therefore, when

$$p_{dx}^2 + p_{dy}^2 < d_2^2, \tag{12}$$

the desired position $\boldsymbol{p}_d \in \mathbb{R}^3$ is out of the primary robot workspace (i.e., it is too close to the axis of joint 1) and there is no solution to the inverse kinematics problem. On the other hand, when (12) is violated, also the argument of the square root in (10) is nonnegative. Note that the presence of such a ‘hole’ in the robot workspace is due to the shoulder offset $d_2 > 0$.

For each of the four pairs (q_2^+, q_3^+), (q_2^-, q_3^+), (q_2^+, q_3^-), and (q_2^-, q_3^-), the solution for q_1 is unique. It is obtained by solving the linear system

$$\begin{pmatrix} q_3 \sin q_2 & d_2 \\ -d_2 & q_3 \sin q_2 \end{pmatrix} \begin{pmatrix} \cos q_1 \\ \sin q_1 \end{pmatrix} = \begin{pmatrix} p_{dx} \\ p_{dy} \end{pmatrix}$$

in the unknowns $\sin q_1$ and $\cos q_1$. The determinant of this system is $d_2^2 + q_3^2 \sin q_2^2 > 0$. Therefore,

$$q_1 = \text{ATAN2}\{d_2\, p_{dx} + q_3 \sin q_2\, p_{dy},\, q_3 \sin q_2\, p_{dx} - d_2\, p_{dy}\}. \tag{13}$$

[1] The pseudoinverse in (9) can be computed explicitly. Modulo an exchange of columns, for a block partitioned square matrix $\boldsymbol{J}$ with a set of columns $\boldsymbol{J}_1$ having full column rank and a remaining set of zero columns, we have

$$\boldsymbol{J} = \begin{pmatrix} \boldsymbol{J}_1 & \boldsymbol{0} \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{J}^{\#} = \begin{pmatrix} \boldsymbol{J}_1^{\#} \\ \boldsymbol{0}^T \end{pmatrix} = \begin{pmatrix} (\boldsymbol{J}_1^T \boldsymbol{J}_1)^{-1} \boldsymbol{J}_1^T \\ \boldsymbol{0}^T \end{pmatrix}.$$

When $q_3 = 0$, i.e., when

$$p_{dx}^2 + p_{dy}^2 + (p_{dz} - d_1)^2 - d_2^2 = 0,$$

we are in a singular case for the inverse kinematics problem. This occurs when point P should be placed at a distance d_2 from the origin O_1 (at the robot shoulder). Then, the angle q_2 is undefined (infinite solutions), while for the first joint we have the single solution (being $d_2 > 0$)

$$q_1 = \mathrm{ATAN2}\{p_{dx}, -p_{dy}\}. \tag{14}$$

Exercise 2

For each joint, we use the doubly normalized quintic polynomial

$$q_i(t) = q_i(0) + \Delta q_i \left(10\tau^3 - 15\tau^4 + 6\tau^5\right) \qquad \Delta q_i = q_i(T) - q_i(0) \qquad \tau = \frac{t}{T} \in [0,1] \qquad i = 1, 2. \tag{15}$$

The first and second time derivatives of (15) are

$$\dot{q}_i(t) = \frac{30\Delta q_i}{T}\left(\tau^2 - 2\tau^3 + \tau^4\right) \tag{16}$$

and

$$\ddot{q}_i(t) = \frac{60\Delta q_i}{T^2}\left(\tau - 3\tau^2 + 2\tau^3\right) = \frac{60\Delta q_i}{T^2}\tau\left(1 - 3\tau + 2\tau^2\right) = \frac{60\Delta q_i}{T^2}\tau\,(1-\tau)\,(1-2\tau)\,. \tag{17}$$

To determine the maxima of $|\ddot{q}_i(t)|$ in the closed interval $[0, T]$, since $\ddot{q}_i(0) = \ddot{q}_i(T) = 0$, we need to find the roots of

$$\dddot{q}_i(t) = \frac{60\Delta q_i}{T^3}\left(1 - 6\tau + 6\tau^2\right) = 0 \qquad \Rightarrow \qquad \tau_{1,2} = 0.5 \pm \frac{1}{\sqrt{12}} = \{0.2113, 0.7887\}.$$

Substituting these roots in (17), with $t_i = \tau_i T$ $(i = 1, 2)$, we obtain the two acceleration extremals

$$\ddot{q}_i(t_1) = \frac{60\Delta q_i}{T^2}\,0.0962 \qquad \ddot{q}_i(t_2) = -\frac{60\Delta q_i}{T^2}\,0.0962 = -\ddot{q}_i(t_1).$$

Thus, acceleration feasibility implies

$$|\ddot{q}_i(t_1)| \le A_i \qquad i = 1, 2 \qquad \Rightarrow \qquad T^2 = \max\left\{T_1^2, T_2^2\right\} = \max\left\{\frac{5.7735\,\Delta q_1}{A_1}, \frac{5.7735\,\Delta q_2}{A_2}\right\}.$$

With the data $\Delta q_1 = \pi/4$ and $\Delta q_2 = 3\pi/4$ [rad], $A_1 = 6$ and $A_2 = 10$ [rad/s^2], we obtain

$$T = \max\left\{T_1, T_2\right\} = \max\{0.8693, 1.1663\} = 1.1663 \text{ s} = T_2. \tag{18}$$

Coordination is thus achieved by uniformly slowing down the fastest joint, in this case joint 1 by a factor $k = 1.1663/0.8693 = 1.3416$. With respect to its independent minimum motion time $T_1 = 0.8693$, the velocity of joint 1 will be scaled down by $1/k = 0.7454$, whereas its acceleration by $1/k^2 = 0.5556$ (i.e., by 55%).

The stationary points of the velocity of the two joints are evaluated when their accelerations are zero. Using the factorization in the last identity of (17), the roots of the cubic polynomials are found at $\tau = 0$, $\tau = 1$ $(t = T)$, and $\tau = 0.5$ $(t = T/2)$. Since $\dot{\boldsymbol{q}}(0) = \dot{\boldsymbol{q}}(T) = \boldsymbol{0}$, the maximum

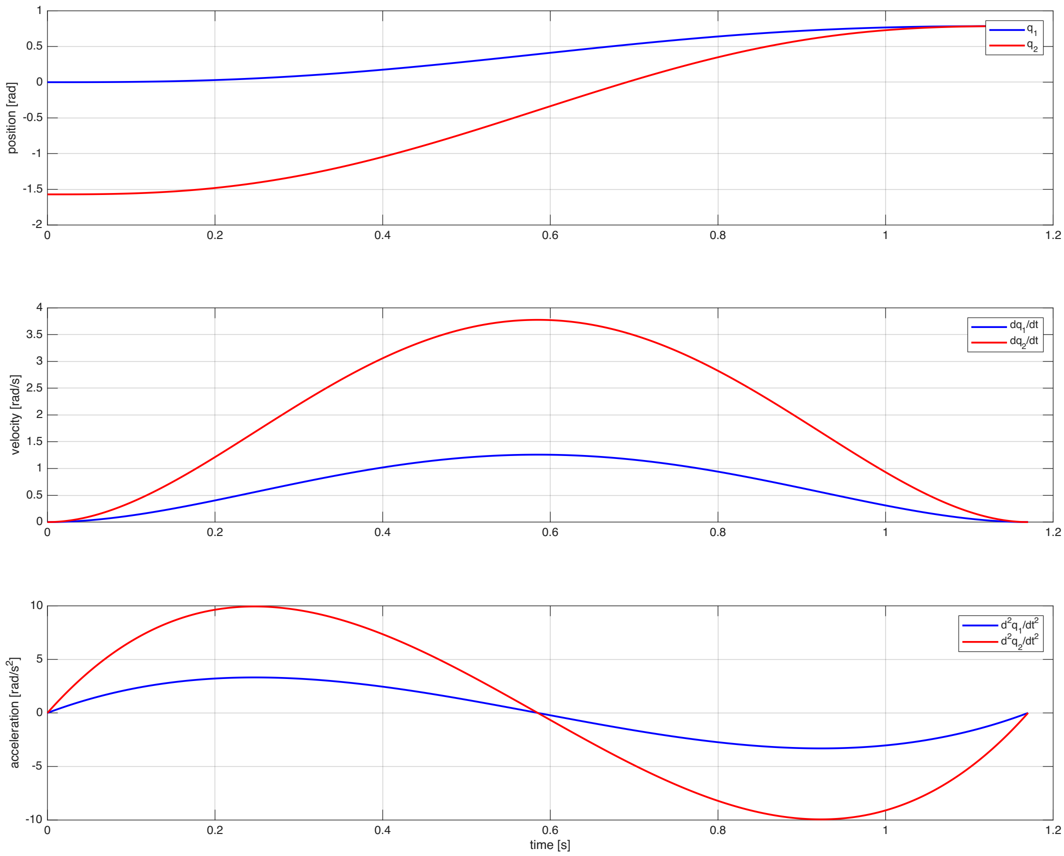

Figure 3: Position, velocity, and acceleration of the two coordinated joint trajectories [joint 1 = blue, joint 2 = red]

absolute velocity of each joint occurs at the trajectory midpoint. Evaluating there eq. (16), we finally obtain

$$V_1 = |\dot{q}_1(T/2)| = \frac{1.875}{1.1663}\,\frac{\pi}{4} = 1.2625 \text{ rad/s} \qquad V_2 = |\dot{q}_2(T/2)| = \frac{1.875}{1.1663}\,\frac{3\pi}{4} = 3.7878 \text{ rad/s}. \tag{19}$$

Figure 3 shows the time evolution of the two joint trajectories. It can be seen that the maximum acceleration of joint 1 is about the half of its bound $A_1 = 6$ rad/s^2, while the acceleration of joint 2 reaches its maximum in absolute value $A_2 = 10$ rad/s^2 at $t_1 = 0.2465$ s and $t_2 = 0.9199$ s.

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