Robotics 1

June 12, 2024

Exercise 1

Let an initial and a final orientation be specified by the two matrices

$$\boldsymbol{R}_i = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad \boldsymbol{R}_f = \begin{pmatrix} \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ \sqrt{2}/2 & 0 & -\sqrt{2}/2 \\ 0 & 1 & 0 \end{pmatrix}.$$

Using the ZXZ Euler angles $\boldsymbol{\phi} = (\alpha, \beta, \gamma)$, generate a trajectory $\boldsymbol{\phi}(t)$, with $t = [0, T]$, which interpolates these two orientations in $T = 1.5$ s, with initial and final angular velocity given by

$$\boldsymbol{\omega}_i = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \boldsymbol{\omega}_f = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{[rad/s]}.$$

Provide the value of the resulting angular velocity $\boldsymbol{\omega}(t)$ at the midtime $t = T/2$. Does the found trajectory $\boldsymbol{\phi}(t)$ cross any representation singularity? Is it the unique solution to this problem in the class of interpolating trajectories using ZXZ Euler angles?

Exercise 2

Table 1 contains the Denavit-Hartenberg (D-H) parameters of a robot with three revolute joints.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	0	θ_1
2	0	$a_2 > 0$	0	θ_2
3	0	$a_3 > 0$	0	θ_3

Table 1: D-H parameters of a 3R robot

- Sketch a skeleton of this robot and of its reachable workspace.
- Assign the frames to the robot links according to the above table.
- Compute the 3×3 Jacobian matrix $\boldsymbol{J}(\boldsymbol{\theta})$ associated to the velocity of the origin of the last D-H frame and determine all its singularities.
- In a singular configuration $\boldsymbol{\theta}_s$, let $\boldsymbol{J}_s = \boldsymbol{J}(\boldsymbol{\theta}_s)$; find a basis for the subspaces $\mathcal{N}(\boldsymbol{J}_s)$ and $\mathcal{R}(\boldsymbol{J}_s)$ and provide an interpretation in terms of robot motion.

Exercise 3

Consider a digital camera with a number of pixel $W \times H = 720 \times 524$ on the image plane and having a lens with focus $f = 8$ mm. Each pixel is square with size $d = 7\mu$m. Assuming a pinhole camera model, which are the horizontal and vertical spatial resolutions (expressed in mm) of points on a plane parallel to the image plane and placed at a distance $L = 2$ m from the lens center? What is the angular field of view (in degrees) on the horizontal plane of this camera-lens system?

Exercise 4

For a 3R planar robot with unit length links, can you define a circular path of diameter $d = 1$ m for its end-effector position $\boldsymbol{p} \in \mathbb{R}^2$ so that the robot will certainly trace such path without crossing a singular configuration? If so, provide an example. If not, explain why.

Exercise 5

Consider a point-to-point path planning problem for a 2R planar robot with unit length links. The robot should move its end-effector between the two Cartesian positions $\boldsymbol{p}_i = (0.6, -0.4)$ and $\boldsymbol{p}_f = (1, 1)$ [m]. Moreover, the Cartesian path should have tangent direction at the start and at the end specified respectively by the vectors $\boldsymbol{p}'_i = (-2, 0)$ and $\boldsymbol{p}'_f = (2, 2)$, where $\boldsymbol{p}' = d\boldsymbol{p}/ds$.

- Define a solution path $\boldsymbol{q}(s)$ directly in the joint space.
- Within the chosen class of interpolating functions, how many solution paths exists? Does any of these paths cross a kinematic singularity?
- On the chosen solution path, define a rest-to-rest timing law $s(t)$ that completes the motion in $T = 3$ s and has continuous acceleration $\ddot{s}(t)$ in the (open) time interval $(0, T)$.
- What is the value of the resulting joint velocity $\dot{\boldsymbol{q}}(t)$ at the midtime $t = T/2$?
- What is the value of the resulting end-effector velocity $\boldsymbol{v}(t) = \dot{\boldsymbol{p}}(t)$ at the midtime $t = T/2$?

[240 minutes (4 hours); open books]

Solution

June 12, 2024

Exercise 1

Using the inverse relationships for the representation of orientation with the ZXZ Euler angles $\boldsymbol{\phi} = (\alpha, \beta, \gamma)$, each of the two rotation matrices $\boldsymbol{R}_i$ and $\boldsymbol{R}_f$ can be transformed in a pair of solution triples, being the regularity condition $R_{13}^2 + R_{23}^2 \neq 0$ on the elements R_{ij} of both matrices satisfied. Therefore, there are four possible combinations for the boundary conditions of the interpolation problem when using this minimal representation of orientation.

The regularity condition is equivalent to having $\sin\beta \neq 0$, or $\beta \neq \{0, \pm\pi\}$, in the triple. Thus, each pair of solutions is characterized by a positive or a negative value for β, depending on the choice of the sign in the four-quadrant arctangent function that is used to compute this value. As a result, a trajectory that interpolates the initial and final values of β, in the usual domain $(-\pi, \pi]$ of definition, will avoid crossing a singularity *if and only if* both values β_i and β_f will have the same sign. Moreover, the amplitude of the needed change for the Euler angles is likely to be smaller when $\beta(t)$ does not change sign along the trajectory. These remarks reduce the interesting combinations of boundary conditions to two; the following solution will present just one of them.

From

$$\beta = \text{atan2}\left\{+\sqrt{R_{13}^2 + R_{23}^2}, R_{33}\right\} \qquad \alpha = \text{atan2}\left\{\frac{R_{31}}{\sin\beta}, \frac{R_{32}}{\sin\beta}\right\} \qquad \gamma = \text{atan2}\left\{\frac{R_{13}}{\sin\beta}, -\frac{R_{23}}{\sin\beta}\right\},$$

one obtains

$$\boldsymbol{\phi}_i = \begin{pmatrix} \alpha_i \\ \beta_i \\ \gamma_i \end{pmatrix} = \begin{pmatrix} \pi/2 \\ \pi/2 \\ -\pi/2 \end{pmatrix} \qquad \boldsymbol{\phi}_f = \begin{pmatrix} \alpha_f \\ \beta_f \\ \gamma_f \end{pmatrix} = \begin{pmatrix} \pi/4 \\ \pi/2 \\ 0 \end{pmatrix}.$$

The interpolating trajectory has to satisfy also boundary conditions on the initial and final angular velocity. The transformation between $\dot{\boldsymbol{\phi}}$ and $\boldsymbol{\omega}$ is found to be

$$\boldsymbol{\omega} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \boldsymbol{T}(\boldsymbol{\phi})\,\dot{\boldsymbol{\phi}} = \begin{pmatrix} 0 & \cos\alpha & \sin\alpha\sin\beta \\ 0 & \sin\alpha & -\cos\alpha\sin\beta \\ 1 & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix}.$$

In fact, since the rotation matrix associated to the ZXZ Euler angles is $\boldsymbol{R} = \boldsymbol{R}_z(\alpha)\boldsymbol{R}_x(\beta)\boldsymbol{R}_z(\gamma)$, the three individual contributions of $\dot{\alpha}$, $\dot{\beta}$ and $\dot{\gamma}$ to the angular velocity $\boldsymbol{\omega}$ are given by

$$\boldsymbol{\omega}_{\dot{\alpha}} = \boldsymbol{z}\,\dot{\alpha} \qquad \boldsymbol{\omega}_{\dot{\beta}} = \boldsymbol{R}_z(\alpha)\,\boldsymbol{x}\,\dot{\beta} \qquad \boldsymbol{\omega}_{\dot{\gamma}} = \boldsymbol{R}_z(\alpha)\boldsymbol{R}_x(\beta)\,\boldsymbol{z}\,\dot{\gamma},$$

which build the three columns of $\boldsymbol{T}(\boldsymbol{\phi})$ (above, we used $\boldsymbol{z} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$ and $\boldsymbol{x} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$). Evaluating $\boldsymbol{T}$ with the obtained $\boldsymbol{\phi}_i$ and $\boldsymbol{\phi}_f$, one has

$$\boldsymbol{T}(\boldsymbol{\phi}_i) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \boldsymbol{T}(\boldsymbol{\phi}_f) = \begin{pmatrix} 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 1 & 0 & 0 \end{pmatrix},$$

and thus

$$\dot{\boldsymbol{\phi}}_i = \boldsymbol{T}^{-1}(\boldsymbol{\phi}_i)\,\boldsymbol{\omega}_i = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \dot{\boldsymbol{\phi}}_f = \boldsymbol{T}^{-1}(\boldsymbol{\phi}_f)\,\boldsymbol{\omega}_f = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

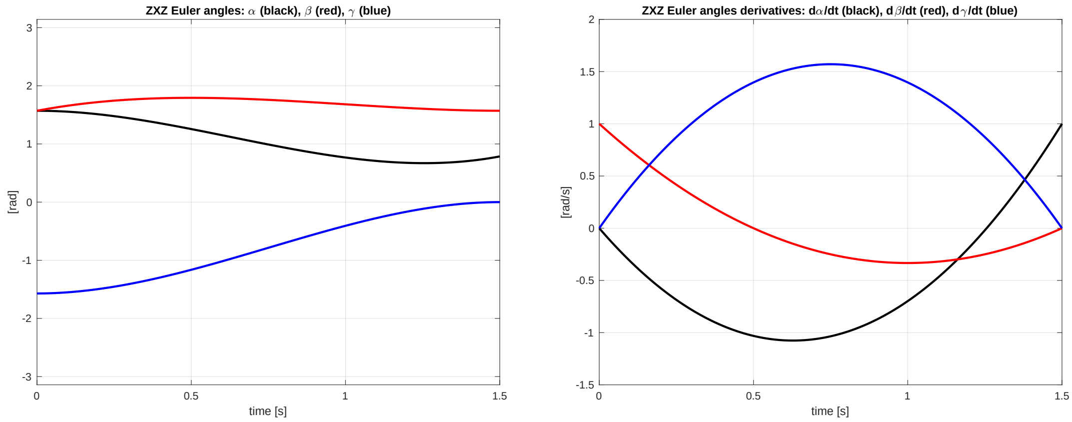

Figure 1: The components of the interpolation trajectory $\boldsymbol{\phi}(t)$ and of its derivative $\dot{\boldsymbol{\phi}}(t)$.

The interpolation problem has the four boundary conditions

$$\boldsymbol{\phi}(0) = \boldsymbol{\phi}_i \qquad \dot{\boldsymbol{\phi}}(0) = \dot{\boldsymbol{\phi}}_i \qquad \boldsymbol{\phi}(T) = \boldsymbol{\phi}_f \qquad \dot{\boldsymbol{\phi}}(T) = \dot{\boldsymbol{\phi}}_f,$$

with $T = 1.5$ s, and can be solved in a unique way when using a cubic polynomial. In its normalized form with $\tau = t/T$, the interpolating trajectory for $\tau \in [0, 1]$ is

$$\boldsymbol{\phi}(\tau) = \boldsymbol{\phi}_i + \left(\dot{\boldsymbol{\phi}}_i T\right)\tau + \left(3\,\Delta\boldsymbol{\phi} - (\dot{\boldsymbol{\phi}}_f + 2\dot{\boldsymbol{\phi}}_i)T\right)\tau^2 + \left(-2\,\Delta\boldsymbol{\phi} + (\dot{\boldsymbol{\phi}}_f + \dot{\boldsymbol{\phi}}_i)T\right)\tau^3,$$

where

$$\Delta\boldsymbol{\phi} = \boldsymbol{\phi}_f - \boldsymbol{\phi}_i = \begin{pmatrix} -\pi/4 \\ 0 \\ \pi/2 \end{pmatrix}.$$

Substituting the numerical data, the trajectories of the Euler angles for $\tau \in [0, 1]$ are (in [rad]):

$$\begin{aligned} \alpha(\tau) &= 1.5708 - 3.8562\,\tau^2 + 3.0708\,\tau^3 \\ \beta(\tau) &= 1.5708 + 1.5\,\tau - 3\,\tau^2 + 1.5\,\tau^3 \\ \gamma(\tau) &= -1.5708 + 4.7124\,\tau^2 - 3.1416\,\tau^3. \end{aligned}$$

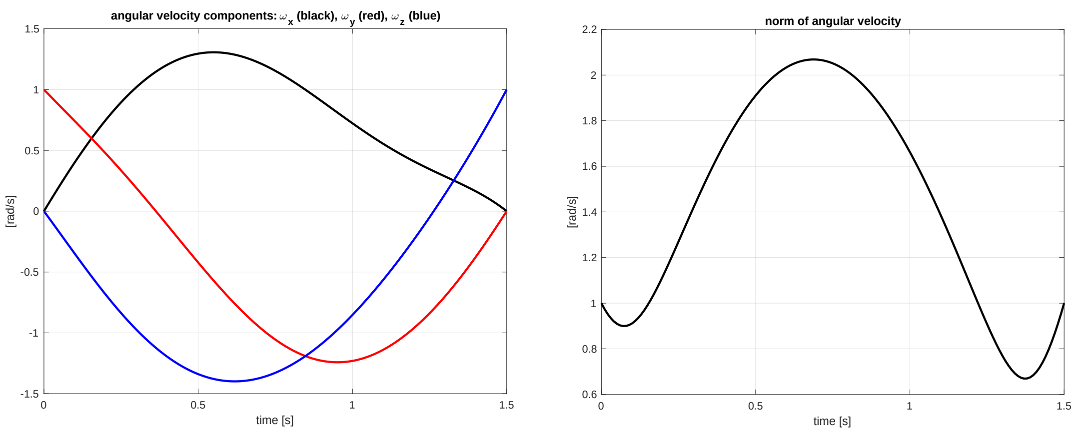

Figure 2: The components of the angular velocity vector $\boldsymbol{\omega}(t)$ and its norm.

The time evolutions of the three components of the trajectory $\boldsymbol{\phi}(t)$ and of its derivative $\dot{\boldsymbol{\phi}}(t)$ are shown in Fig. 1. The angular velocity along the trajectory is computed as $\boldsymbol{\omega}(t) = \boldsymbol{T}(\boldsymbol{\phi}(t))\,\dot{\boldsymbol{\phi}}(t)$ and is shown in Fig. 2, together with its norm. At $t = T/2 = 0.75$ s, it is

$$\boldsymbol{\omega}(T/2) = \begin{pmatrix} 1.1537 \\ -1.0551 \\ -1.3282 \end{pmatrix} \text{ [rad/s]}.$$

Exercise 2

A possible structure described by Tab. 1 is a 3R spatial robot, as sketched in Fig. 3 with the associated D-H frames. Since the size/length of the robot base is irrelevant for the kinematics, the first joint could also be placed on the ground; this is also the meaning of having $a_1 = d_1 = 0$ in the D-H table.

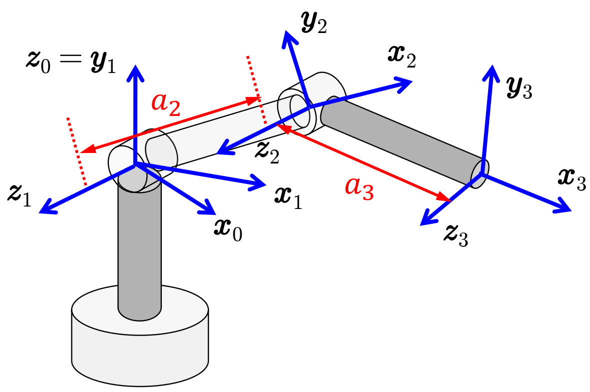

Figure 3: A 3R spatial robot with D-H frame assignment according to Tab. 1.

The reachable workspace is a sphere centered at the robot shoulder (i.e., at $O_0 = O_1$) and with radius $R = a_2 + a_3 > 0$. If $a_2 \neq a_3$, an inner sphere of radius $r = |a_2 - a_3| > 0$ is removed from the workspace.

The direct kinematics in position is

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{\theta}) = \begin{pmatrix} c_1\,(a_2 c_2 + a_3 c_{23}) \\ s_1\,(a_2 c_2 + a_3 c_{23}) \\ a_2 s_2 + a_3 s_{23} \end{pmatrix}.$$

The associated 3×3 Jacobian is

$$\boldsymbol{J}(\boldsymbol{\theta}) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\theta}} = \begin{pmatrix} -s_1\,(a_2 c_2 + a_3 c_{23}) & -c_1\,(a_2 s_2 + a_3 s_{23}) & -a_3 c_1 s_{23} \\ c_1\,(a_2 c_2 + a_3 c_{23}) & -s_1\,(a_2 s_2 + a_3 s_{23}) & -a_3 s_1 s_{23} \\ 0 & a_2 c_2 + a_3 c_{23} & a_3 c_{23} \end{pmatrix}.$$

To simplify the study of the singularities of $\boldsymbol{J}$, it is convenient to express the Jacobian in the rotated frame 1, namely

$${}^1\boldsymbol{J}(\boldsymbol{\theta}) = \boldsymbol{R}_1^T(\theta_1)\,\boldsymbol{J}(\boldsymbol{\theta}) = \begin{pmatrix} c_1 & s_1 & 0 \\ 0 & 0 & 1 \\ s_1 & -c_1 & 0 \end{pmatrix} \boldsymbol{J}(\boldsymbol{\theta}) = \begin{pmatrix} 0 & -\,(a_2 s_2 + a_3 s_{23}) & -a_3 s_{23} \\ 0 & a_2 c_2 + a_3 c_{23} & a_3 c_{23} \\ -\,(a_2 c_2 + a_3 c_{23}) & 0 & 0 \end{pmatrix}.$$

Thus, for the determinant we have

$$\det \boldsymbol{J}(\boldsymbol{\theta}) = \det {}^1\boldsymbol{J}(\boldsymbol{\theta}) = a_2 a_3\, s_3\, (a_2 c_2 + a_3 c_{23}),$$

and singularities occur when:

a) $s_3 = 0$: the forearm is fully stretched ($\theta_3 = 0$) or fully folded ($\theta_3 = \pm\pi$);

b) $a_2 c_2 + a_3 c_{23} = 0$: being this equal to $\sqrt{p_x^2 + p_y^2} = 0$, the end-effector is located on the $\boldsymbol{z}_0$ axis;

c) both above situations hold: the robot is stretched or folded vertically along the $\boldsymbol{z}_0$ axis.

Consider for instance case a) and set $\boldsymbol{\theta}_s = (\theta_1, \theta_2, 0)$. Then

$$\boldsymbol{J}_s = \boldsymbol{J}(\boldsymbol{\theta}_s) = \begin{pmatrix} -(a_2+a_3)\, s_1 c_2 & -(a_2+a_3)\, c_1 s_2 & -a_3 c_1 s_2 \\ -(a_2+a_3)\, c_1 c_2 & -(a_2+a_3)\, s_1 s_2 & -a_3 s_1 s_2 \\ 0 & (a_2+a_3)\, c_2 & a_3 c_2 \end{pmatrix}.$$

It is easy to find a basis for each of the two requested subspaces:

$$\mathcal{N}(\boldsymbol{J}_s) = \begin{pmatrix} 0 \\ -a_3 \\ a_2+a_3 \end{pmatrix} \qquad \mathcal{R}(\boldsymbol{J}_s) = \operatorname{span}\left\{ \begin{pmatrix} s_1 c_2 \\ c_1 c_2 \\ 0 \end{pmatrix}, \begin{pmatrix} -c_1 s_2 \\ -s_1 s_2 \\ c_2 \end{pmatrix} \right\}.$$

As a result, in this singular configuration, the robot end effector will remain at rest when the first joint does not move while the second and third joints move in opposite directions with instantaneous velocities scaled by the relative factor $a_3/(a_2+a_3)$ (motion in the *null space*). If the two links have unit length ($a_1 = a_2 = 1$), then joint 3 will move instantaneously at twice the (opposite) velocity of joint 2.

On the other hand, to better visualize the *range space* motion at $\boldsymbol{\theta}_s$, consider the rotated Jacobian in the chosen singular configuration

$${}^1\boldsymbol{J}_s = {}^1\boldsymbol{J}(\boldsymbol{\theta}_s) = \begin{pmatrix} 0 & -(a_2+a_3)\, s_2 & -a_3 s_2 \\ 0 & (a_2+a_3)\, c_2 & a_3 c_2 \\ -(a_2+a_3)\, c_2 & 0 & 0 \end{pmatrix}.$$

Then

$$\mathcal{R}({}^1\boldsymbol{J}_s) = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ c_2 \end{pmatrix}, \begin{pmatrix} -s_2 \\ c_2 \\ 0 \end{pmatrix} \right\}.$$

Therefore, when observing the end-effector motion in frame $(\boldsymbol{x}_1, \boldsymbol{y}_1, \boldsymbol{z}_1)$ with the robot in this singular configuration, an instantaneous velocity of the first joint produces only a velocity in the $\boldsymbol{z}_1$ (horizontal) direction; the second and third joint can move the end effector only along a single direction in the plane $(\boldsymbol{x}_1, \boldsymbol{y}_1)$. If the two links have unit length and we set $\theta_2 = 0$, a unit velocity of joint 1 gives an end-effector velocity ${}^1\boldsymbol{v}_e = (0, 0, 2)$ [m/s]; a unit velocity of joint 2 (or of joint 3) gives ${}^1\boldsymbol{v}_e = (0, 2, 0)$ [m/s] (or ${}^1\boldsymbol{v}_e = (0, 1, 0)$ [m/s]).

Exercise 3

Let a point ${}^cP = (X, Y, Z)$ be expressed in the camera frame, which is placed on the physical image plane of the camera, i.e., behind the lens and at a focal distance f. The pinhole model maps point P into the image plane point $p = (u, v)$ (horizontal and vertical coordinates) as

$$u = u_0 - f\,\frac{X}{Z} \qquad v = v_0 - f\,\frac{Y}{Z}.$$

where (u_0, v_0) corresponds to the center point of the image on the lens axis (the origin of the camera coordinates is usually at the top-left of the image plane). Accordingly, a displacement $\mathbf{\Delta} = (\Delta X, \Delta Y)$ of point P on the plane at a distance $Z = L + f$ from the camera frame will result in variations of the coordinates of the point p on the image plane given by

$$\Delta u = -f\,\frac{\Delta X}{L+f} \qquad \Delta v = -f\,\frac{\Delta Y}{L+f}.$$

Therefore, being the physical size of the square pixels $\Delta u_{\min} = \Delta v_{\min} = 7\mu\text{m} = 7 \cdot 10^{-3}$ mm, the spatial resolution of a point P in the plane at $Z = L + f$ (namely, the minimum displacement detectable by the camera) is

$$\Delta X_{min} (= \Delta Y_{min}) = \Delta u_{\min}\,\frac{L+f}{f} = 7 \cdot 10^{-3} \cdot \frac{2008}{8} = 1.757 \text{ mm},$$

equal in the horizontal and vertical case.

The physical size of the sensor is $Wd \times Hd = 720 \cdot 7 \times 524 \cdot 7\,\mu\text{m} = 5.040 \times 3.668$ mm. The angular field of view (FOV) on the horizontal plane of this camera-lens system, expressed in degrees, is

$$\text{FOV}_H = 2 \arctan \frac{Wd/2}{f}\,[\text{rad}] = \frac{360°}{\pi} \cdot \arctan \frac{Wd/2}{f} = 114.59\,[°/\text{rad}] \cdot \arctan \frac{2.520}{8} \simeq 36°.$$

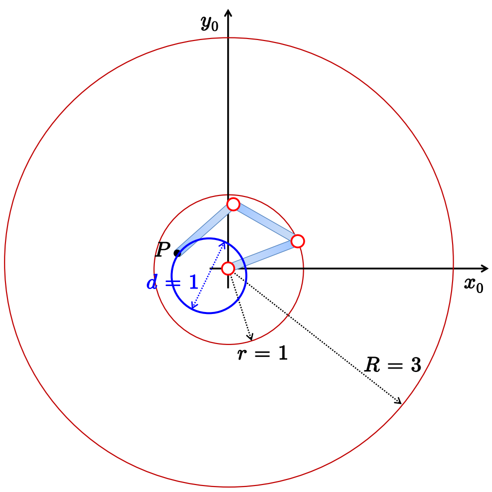

Figure 4: A circular path traced without singularities by a 3R planar robot with unit length links.

Exercise 4

The answer is yes. With reference to Fig. 4, a 3R planar robot is in a singular configuration for a task involving only its end-effector position (with a 2×3 Jacobian) when the three links are stretched or folded along a single radial direction from the robot base, i.e., when $\theta_2 = \{0, \pm\pi\}$ and $\theta_3 = \{0, \pm\pi\}$. When the links have unit length, these singular configurations correspond to the external boundary of the workspace (a circle of radius $R = 3$ m that the robot end effector can reach only in the fully stretched and singular configuration) and to a circle of radius $r = 1$ m inside the workspace (where the robot is in singularity only if folded along a radial direction). Therefore, a circular path of diameter $d = 1$ m (i.e., of radius $d/2 = 0.5$ m) for the end-effector can be defined safely (and in many ways!) inside the circle of unit radius, and certainly no singular configurations will be encountered.

Exercise 5

Being the given two Cartesian positions $\boldsymbol{p}_i$ and $\boldsymbol{p}_f$ strictly inside the reachable workspace of the 2R planar robot (a circle of radius $r_{\max} = 2$, whereas $\|\boldsymbol{p}_i\| = 0.7211$ and $\|\boldsymbol{p}_f\| = 1.4142$), they do not correspond to singular configurations ($q_2 = 0$ or $\pm\pi$). Thus, there are two regular solutions to the inverse kinematics problem for these positions. We choose in both cases the *elbow-down* (viz. *right-arm*) solution, namely

$$\boldsymbol{q}_i = \begin{pmatrix} -1.7899 \\ 2.4039 \end{pmatrix} \qquad \boldsymbol{q}_f = \begin{pmatrix} 0 \\ \pi/2 \end{pmatrix} \quad \text{[rad]}.$$

By doing so, we choose one out of the four possible combinations of boundary conditions for the interpolating joint path to be defined (and, correspondingly, four classes of trajectories). Moreover, choosing the initial and final inverse solutions in the same class will guarantee that interpolating joint paths that have no wandering (i.e., remain between the initial and final joint values) will not cross a singular configuration.[1] On the other hand, when choosing the two combinations with values of the second joint of opposite signs, a singular configuration will certainly be encountered.

In order to invert the Cartesian tangent data, we need to compute the 2×2 analytical Jacobian $\boldsymbol{J}$ of the robot. In fact, for the derivatives with respect to the path parameter s, one has

$$\boldsymbol{p}' = \frac{d\boldsymbol{p}}{ds} = \boldsymbol{J}(\boldsymbol{q})\,\frac{d\boldsymbol{q}}{ds} = \boldsymbol{J}(\boldsymbol{q})\,\boldsymbol{q}',$$

just like for the transformation between joint velocity $\dot{\boldsymbol{q}}$ and end-effector velocity $\dot{\boldsymbol{p}}$ (which are derivatives of the position $\boldsymbol{p}$ with respect to time t in the two spaces). Being

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -\sin q_1 - \sin(q_1+q_2) & -\sin(q_1+q_2) \\ \cos q_1 + \cos(q_1+q_2) & \cos(q_1+q_2) \end{pmatrix},$$

we have

$$\boldsymbol{J}(\boldsymbol{q}_i) = \begin{pmatrix} 0.4000 & -0.5761 \\ 0.6000 & 0.8174 \end{pmatrix} \qquad \boldsymbol{J}(\boldsymbol{q}_f) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$$

and thus

$$\boldsymbol{q}'_i = \boldsymbol{J}^{-1}(\boldsymbol{q}_i)\,\boldsymbol{p}'_i = \begin{pmatrix} -2.4305 \\ 1.7841 \end{pmatrix} \qquad \boldsymbol{q}'_f = \boldsymbol{J}^{-1}(\boldsymbol{q}_f)\,\boldsymbol{p}'_f = \begin{pmatrix} 2 \\ -4 \end{pmatrix} \quad \text{[rad]}.$$

The interpolation problem for a geometric path $\boldsymbol{q}(s)$, with $s \in [0,1]$, has the four boundary conditions

$$\boldsymbol{q}(0) = \boldsymbol{q}_i \qquad \boldsymbol{q}'(0) = \boldsymbol{q}'_i \qquad \boldsymbol{q}(1) = \boldsymbol{q}_f \qquad \boldsymbol{q}'(1) = \boldsymbol{q}'_f,$$

and can be solved using the cubic polynomial

$$\boldsymbol{q}(s) = \boldsymbol{q}_i + \boldsymbol{q}'_i\,s + \left(3\,\Delta\boldsymbol{q} - (\boldsymbol{q}'_f + 2\boldsymbol{q}'_i)\right)s^2 + \left(-2\,\Delta\boldsymbol{q} + (\boldsymbol{q}'_f + \boldsymbol{q}'_i)\right)s^3,$$

where

$$\Delta\boldsymbol{q} = \boldsymbol{q}_f - \boldsymbol{q}_i = \begin{pmatrix} 1.7899 \\ -0.8331/2 \end{pmatrix} \quad \text{[rad]}.$$

[1] The situation is somewhat similar to Exercise 1.

Substituting the numerical data, the paths of the two joints for $s \in [0, 1]$ are (in [rad]):

$$q_1(s) = -1.7899 - 2.4305\, s + 8.2308\, s^2 - 4.0104\, s^3$$
$$q_2(s) = 2.4039 + 1.7841\, s - 2.0674\, s^2 - -0.5498\, s^3.$$

The two components of the joint path $\boldsymbol{q}(s)$ and of its derivative (tangent) $\boldsymbol{q}'(s)$ are shown in Fig. 5. Since $q_2(t)$ never crosses 0 (or $\pm\pi$), the robot does not encounter a singular configuration. Figure 6 shows the Cartesian path corresponding to $\boldsymbol{q}(s)$.

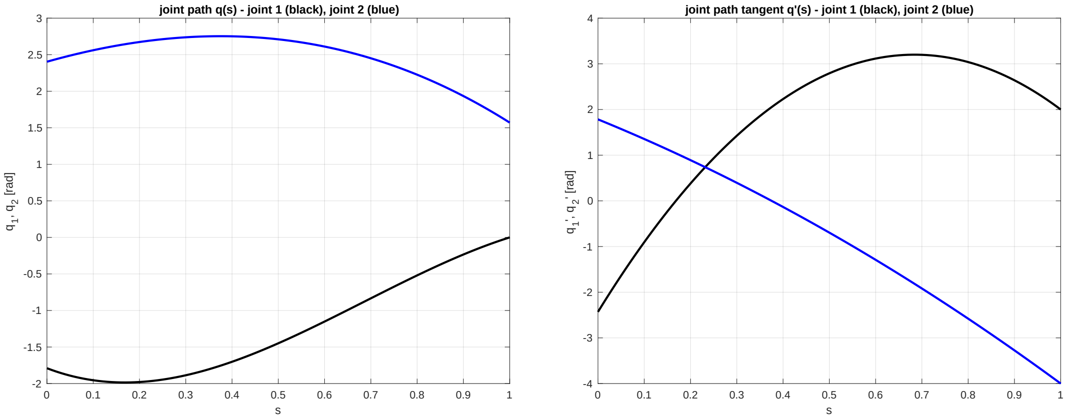

Figure 5: The components of the joint path $\boldsymbol{q}(s)$ and of its derivative $\boldsymbol{q}'(s)$.

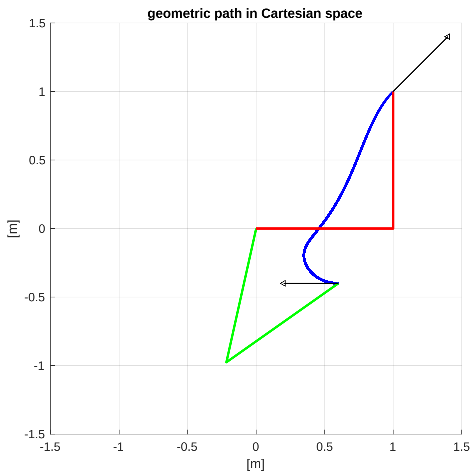

Figure 6: The Cartesian path $\boldsymbol{p}(s)$ (in blue) corresponding to $\boldsymbol{q}(s)$. Initial (green) and final (red) robot postures and initial and final tangent directions to the Cartesian path are also shown.

To obtain a smooth rest-to-rest motion (strictly) inside the motion interval $[0, T]$, with $T = 3$ s, one can use a cubic polynomial as timing law — see Fig. 7:

$$s(\tau) = 3\,\tau^2 - 2\,\tau^3 \qquad \tau = \frac{t}{T} \qquad \Rightarrow \qquad s(t) = \frac{1}{3}\,t^2 - \frac{2}{27}\,t^3 \qquad t \in [0, 3].$$

Combining the parametrized path $\boldsymbol{q}(s)$ with the timing law $s(t)$ yields the desired trajectory $\boldsymbol{q}(t)$ shown in Fig. 8. When comparing this with Fig. 5, one can appreciate the modulation of the joint path motion obtained through the timing law — in particular, the obtained zero initial and final velocities (horizontal tangents to $q_1(t)$ and $q_2(t)$ at $t = 0$ and $t = 3$ s) in face of nonzero tangents at the path boundaries $s = 0$ and $s = 1$.

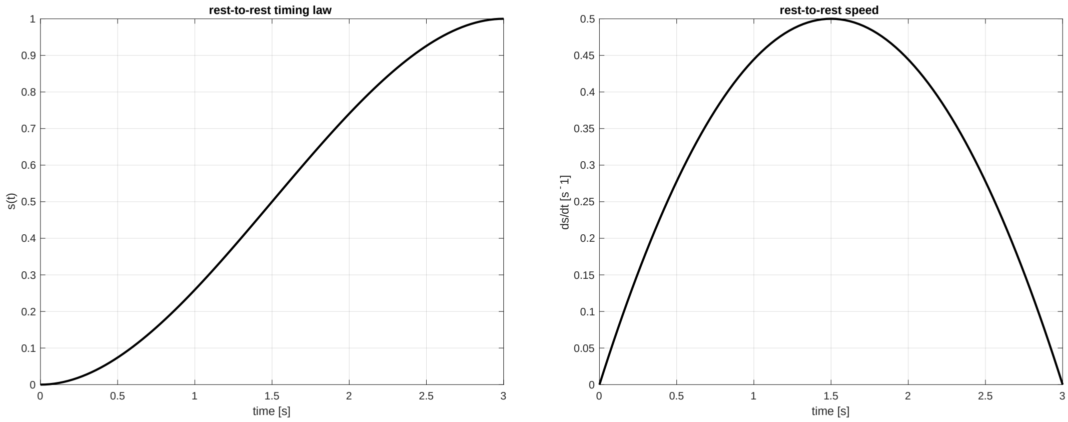

Figure 7: The rest-to-rest timing law $s(t)$ and its speed $\dot{s}(t)$.

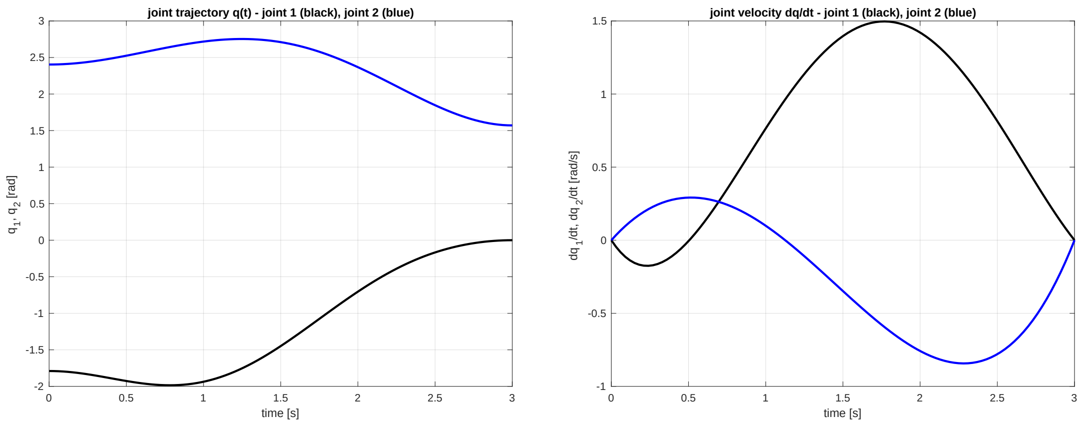

Figure 8: The two components of the resulting joint trajectory $\boldsymbol{q}(t)$ and velocity $\dot{\boldsymbol{q}}(t)$.

The requested values at the trajectory midtime $t = T/2 = 1.5$ s (or $\tau = 0.5$) are computed as follows. For the midtime configuration $\boldsymbol{q}_{\rm mid}$:

$$s(t = 1.5) = s(\tau = 0.5) = 0.5 \quad \Rightarrow \quad \boldsymbol{q}_{\rm mid} = \boldsymbol{q}(t = 1.5) = \boldsymbol{q}(s = 0.5) = \begin{pmatrix} -1.4488 \\ 2.7103 \end{pmatrix} \quad [\text{rad}];$$

for the midtime joint velocity $\dot{\boldsymbol{q}}_{\rm mid}$:

$$\dot{\boldsymbol{q}}_{\rm mid} = \dot{\boldsymbol{q}}(t = T/2) = \boldsymbol{q}'(s = 0.5)\,\dot{s}(t = T/2) = \begin{pmatrix} 2.7925 \\ -0.6956 \end{pmatrix} \cdot 0.5 = \begin{pmatrix} 1.3963 \\ -0.3478 \end{pmatrix} \quad [\text{rad/s}];$$

finally, for the midtime end-effector velocity $\dot{\boldsymbol{p}}_{\rm mid}$:

$$\dot{\boldsymbol{p}}_{\rm mid} = \boldsymbol{J}(\boldsymbol{q}_{\rm mid})\,\dot{\boldsymbol{q}}_{\rm mid} = \begin{pmatrix} 0.0400 & -0.9526 \\ 0.4260 & 0.3043 \end{pmatrix} \begin{pmatrix} 1.3963 \\ -0.3478 \end{pmatrix} = \begin{pmatrix} 0.3872 \\ 0.4890 \end{pmatrix} \quad [\text{m/s}].$$

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