



## ***Robotics 1***

# **Statics and force transformations**

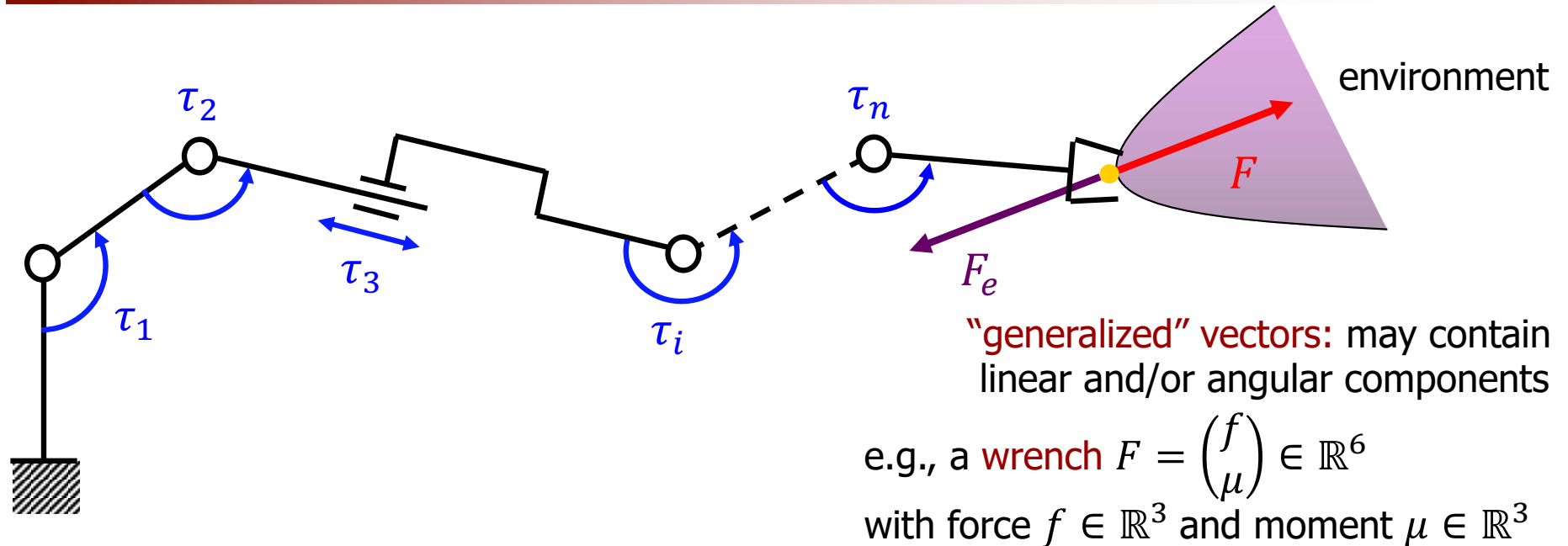
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AUTOMATICA E GESTIONALE ANTONIO RUBERTI



**SAPIENZA**  
UNIVERSITÀ DI ROMA

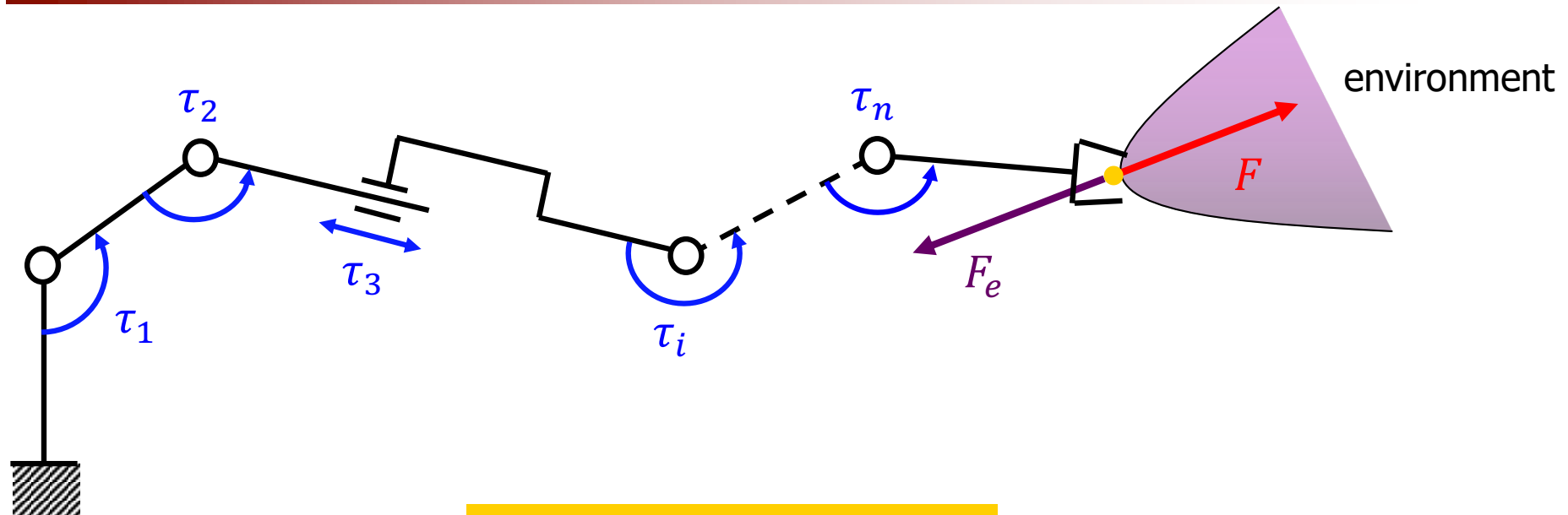
# Generalized forces and torques



- $\tau$  = forces/torques exerted **by the motors** at the robot joints
- $F$  = **equivalent** (generalized) forces/torques exerted by the robot end-effector
- $F_e$  = forces/torques exerted **by the environment** at the end-effector
- principle of action and reaction:  $F_e = -F$

*reaction from environment is **equal and opposite** to the robot action on it*

# Transformation of forces – Statics

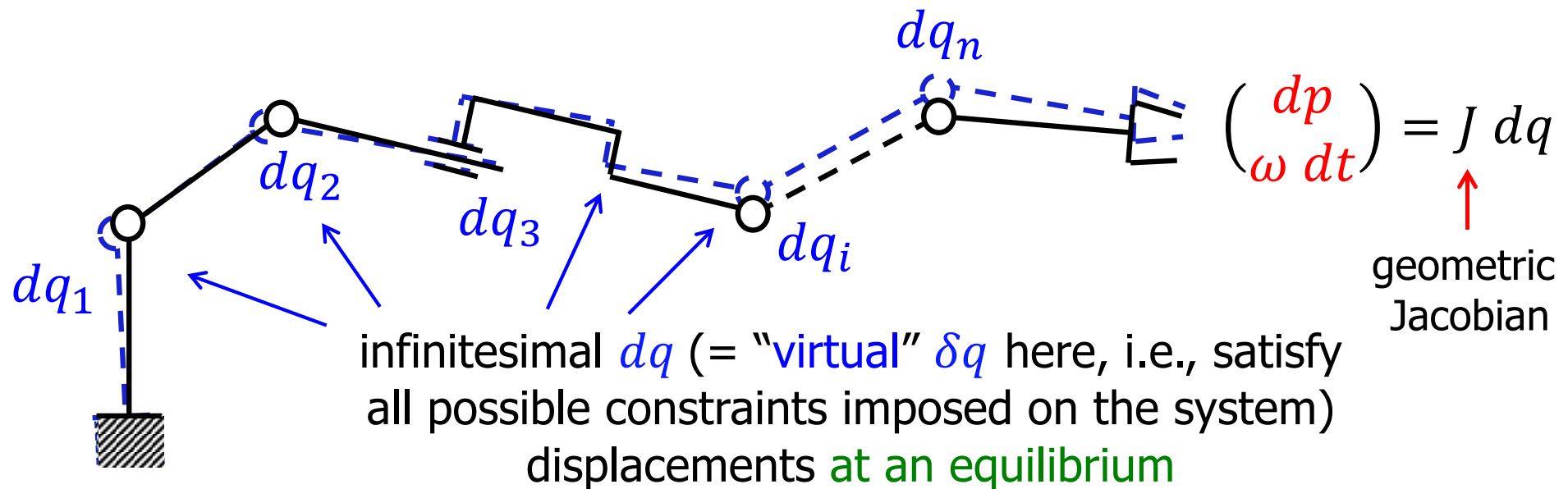


in a given configuration

- what is the transformation between  $F$  at robot end-effector and  $\tau$  at joints?  
in **static equilibrium** conditions (i.e., **no motion**):
- what  $F$  will be exerted on environment by a  $\tau$  applied at the robot joints?
- what  $\tau$  at the joints will balance a  $F_e (= -F)$  exerted by the environment?

all equivalent formulations

# Virtual displacements and works

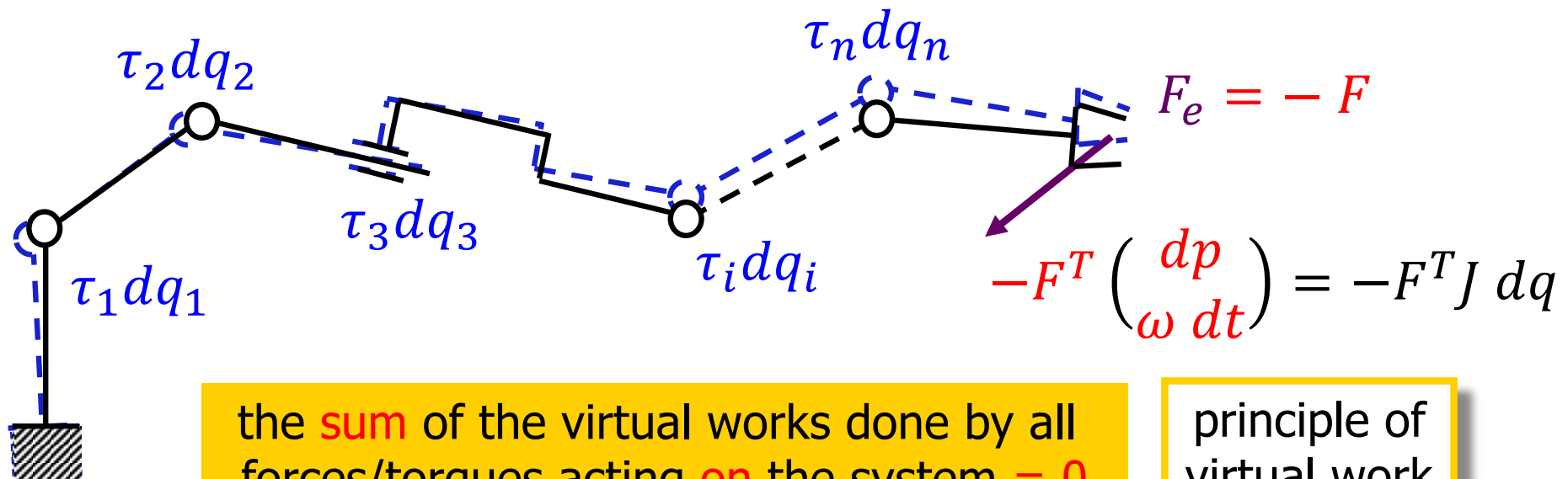


- ➔
- without kinetic energy variation (zero acceleration)
  - without dissipative effects (zero velocity)

the **virtual work** is the work done by all forces/torques acting **on** the system for a given virtual displacement

*Hint: one of the advantages of working with (D-H) joint variables is that they are **already free of equality constraints** (= generalized coordinates)*

# Principle of virtual work



$$\tau^T dq - F^T \begin{pmatrix} dp \\ \omega dt \end{pmatrix} = \tau^T dq - F^T J dq = 0 \quad \boxed{\forall dq}$$

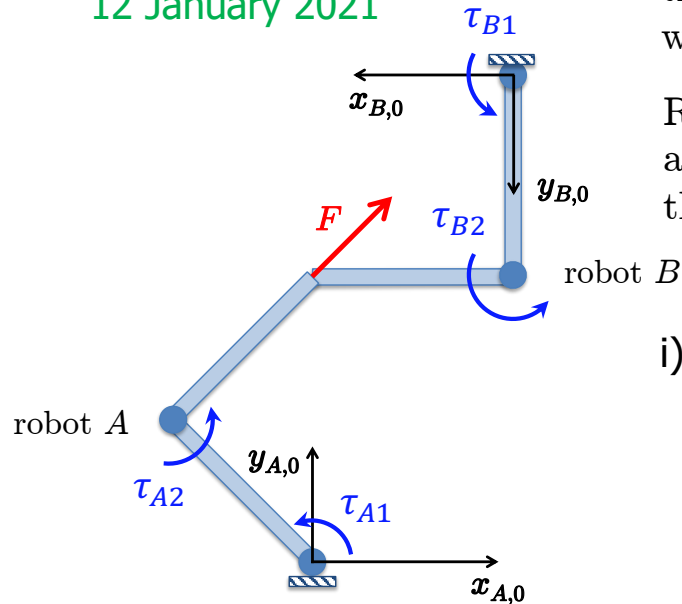
$$\Rightarrow \boxed{\tau = J^T(q)F}$$



# Exercise on static balance

## whiteboard ...

Q#7 in Robotics 1 exam,  
12 January 2021



Two 2R planar robots,  $A$  and  $B$ , having unitary link lengths are in their D-H configurations  $\mathbf{q}_A = (3\pi/4, -\pi/2)$ ,  $\mathbf{q}_B = (\pi/2, -\pi/2)$  [rad] w.r.t. their base frames, as in figure (no gravity!).

Robot  $A$  pushes against robot  $B$  with a force  $\mathbf{F} \in \mathbb{R}^2$  of norm  $\|\mathbf{F}\| = 10$  [N], as in figure. Compute the joint torques  $\boldsymbol{\tau}_A \in \mathbb{R}^2$  and  $\boldsymbol{\tau}_B \in \mathbb{R}^2$  (both in [Nm]) that keep the two robots in equilibrium.

**solution**

i) evaluate the task Jacobians of the two robots ( $\dot{\mathbf{q}}_A \rightarrow \mathbf{v}_A$  and  $\dot{\mathbf{q}}_B \rightarrow \mathbf{v}_B$ )

$$\mathbf{J}_A(\mathbf{q}_A) = \begin{pmatrix} -\sin q_1 - \sin(q_1 + q_2) & -\sin(q_1 + q_2) \\ \cos q_1 + \cos(q_1 + q_2) & \cos(q_1 + q_2) \end{pmatrix} \Big|_{\mathbf{q}=\mathbf{q}_A} = \begin{pmatrix} -\sqrt{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} \end{pmatrix}$$

$$\mathbf{J}_B(\mathbf{q}_B) = \begin{pmatrix} -\sin q_1 - \sin(q_1 + q_2) & -\sin(q_1 + q_2) \\ \cos q_1 + \cos(q_1 + q_2) & \cos(q_1 + q_2) \end{pmatrix} \Big|_{\mathbf{q}=\mathbf{q}_B} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$$

ii) express the exchanged force in the proper frame(s) ...

$${}^A\mathbf{F}_A = \|\mathbf{F}\| \cdot \begin{pmatrix} \cos(q_1 + q_2) \\ \sin(q_1 + q_2) \end{pmatrix} \Big|_{\mathbf{q}=\mathbf{q}_A} = 10 \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix} \text{ [N]}$$

$${}^B\mathbf{F}_B = {}^B\mathbf{R}_A {}^A\mathbf{F}_B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} (-{}^A\mathbf{F}_A) = {}^A\mathbf{F}_A = 10 \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix} \text{ [N]}$$

planar rotation matrix  $\in SO(2)$

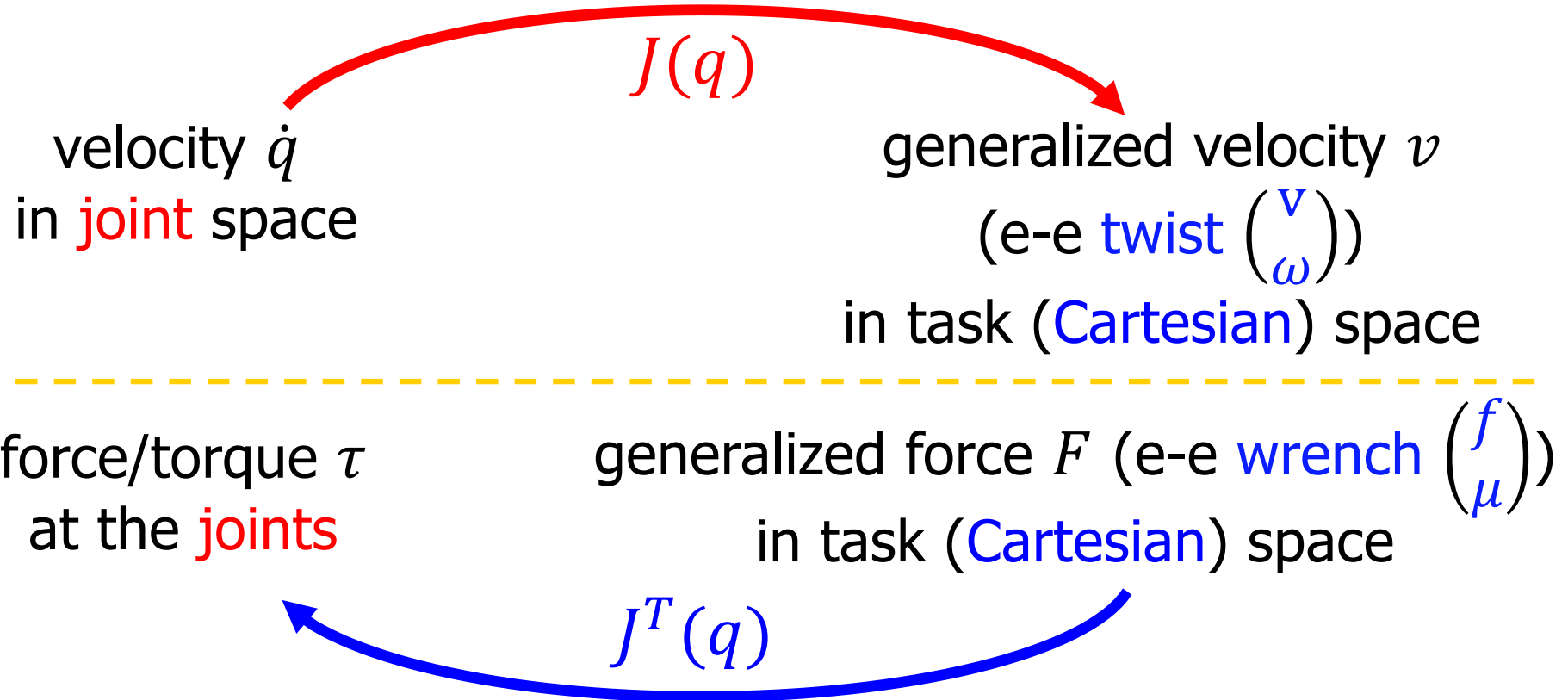
iii) ... and compute the torque for each robot by the virtual work principle

$$\boldsymbol{\tau}_A = \mathbf{J}_A^T(\mathbf{q}_A) {}^A\mathbf{F}_A = \begin{pmatrix} -10 \\ 0 \end{pmatrix} \text{ [Nm]}$$

$$\boldsymbol{\tau}_B = \mathbf{J}_B^T(\mathbf{q}_B) {}^B\mathbf{F}_B = \begin{pmatrix} 0 \\ 5\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 7.0711 \end{pmatrix} \text{ [Nm]}.$$



# Duality between velocity and force



same singular configurations of  
**velocity map** and **force map**

$$\rho(J) = \rho(J^T)$$

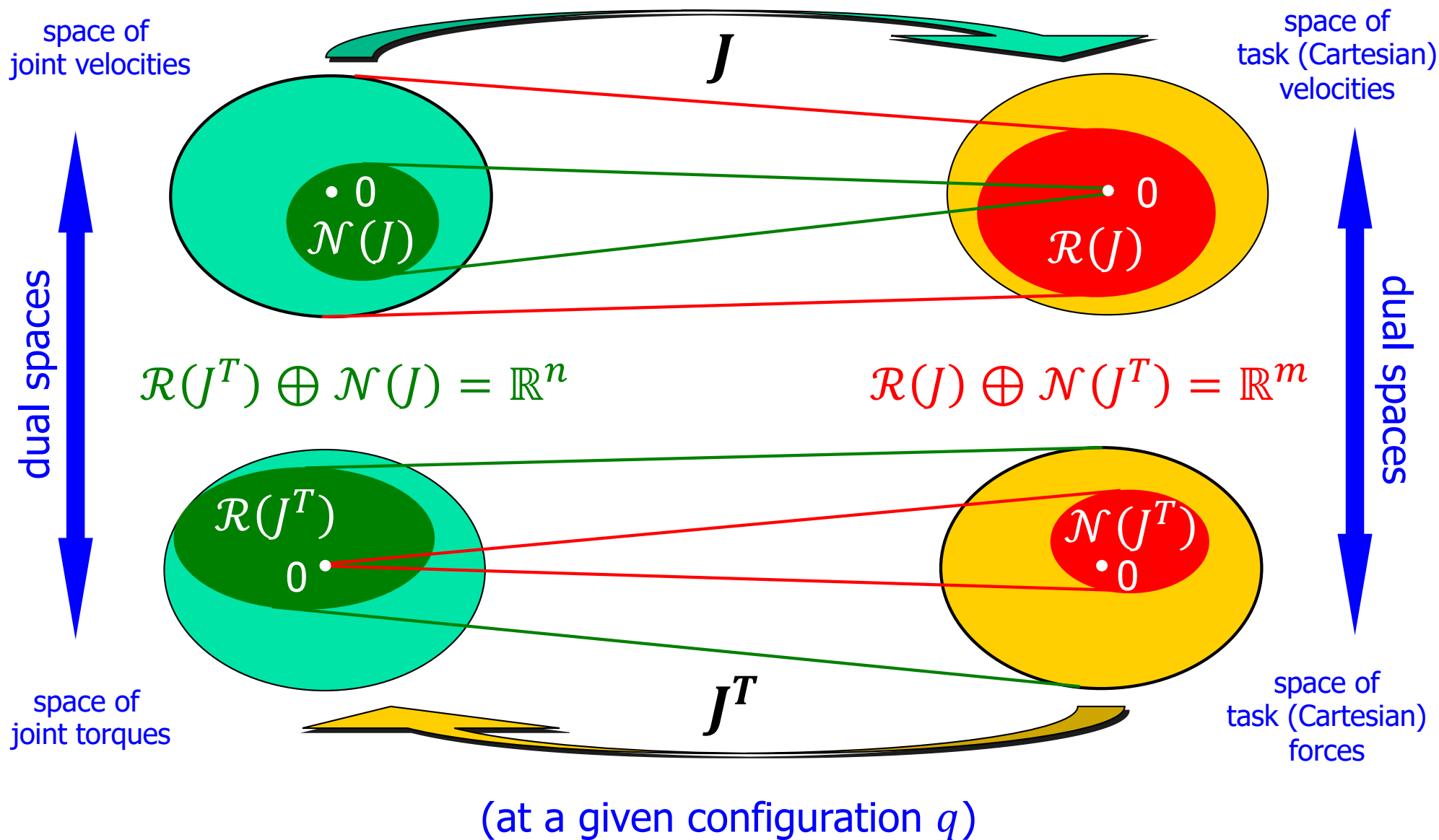
**note:** beware of orthogonality!  $\Rightarrow$  **only** between twists and wrenches

duplicate slide  
from block 11



# Robot Jacobian

decomposition in linear subspaces and duality





# Dual subspaces of velocity and force

## summary of definitions



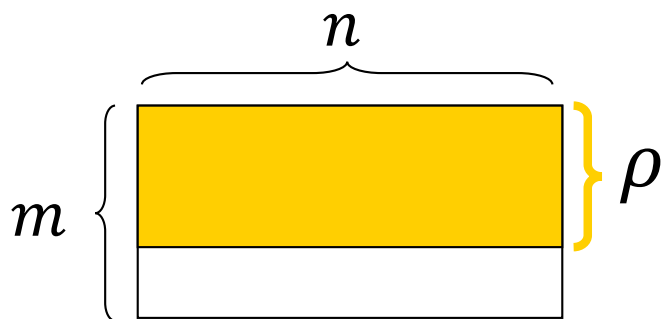
$$\begin{aligned}\mathcal{R}(J) &= \{v \in \mathbb{R}^m : \exists \dot{q} \in \mathbb{R}^n, J\dot{q} = v\} \\ \mathcal{N}(J^T) &= \{F \in \mathbb{R}^m : J^T F = 0\} \\ \mathcal{R}(J) \oplus \mathcal{N}(J^T) &= \mathbb{R}^m\end{aligned}$$

$$\begin{aligned}\mathcal{R}(J^T) &= \{\tau \in \mathbb{R}^n : \exists F \in \mathbb{R}^m, J^T F = \tau\} \\ \mathcal{N}(J) &= \{\dot{q} \in \mathbb{R}^n : J\dot{q} = 0\} \\ \mathcal{R}(J^T) \oplus \mathcal{N}(J) &= \mathbb{R}^n\end{aligned}$$

# Velocity and force singularities

list of possible cases

$$\rho = \text{rank}(J) = \text{rank}(J^T) \leq \min(m, n)$$



1.  $\rho = m$

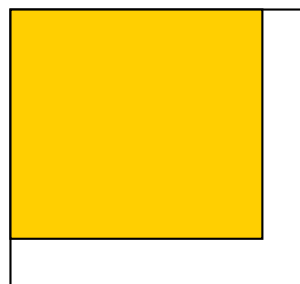
$$\exists \dot{q} \neq 0 : J\dot{q} = 0$$

$$\mathcal{N}(J^T) = \{0\}$$

2.  $\rho < m$

$$\exists \dot{q} \neq 0 : J\dot{q} = 0$$

$$\exists F \neq 0 : J^T F = 0$$



1.  $\det J \neq 0$

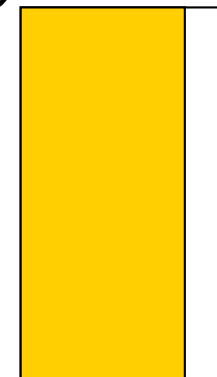
$$\mathcal{N}(J) = \{0\}$$

$$\mathcal{N}(J^T) = \{0\}$$

2.  $\det J = 0$

$$\exists \dot{q} \neq 0 : J\dot{q} = 0$$

$$\exists F \neq 0 : J^T F = 0$$



1.  $\rho = n$

$$\mathcal{N}(J) = \{0\}$$

$$\exists F \neq 0 : J^T F = 0$$

2.  $\rho < n$

$$\exists \dot{q} \neq 0 : J\dot{q} = 0$$

$$\exists F \neq 0 : J^T F = 0$$



# Singularity analysis

planar 2R arm with  
link lengths  $l_1$  and  $l_2$

$$J(q) = \begin{pmatrix} -(l_1 s_1 + l_2 s_{12}) & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{pmatrix} \quad \det J(q) = l_1 l_2 s_2$$

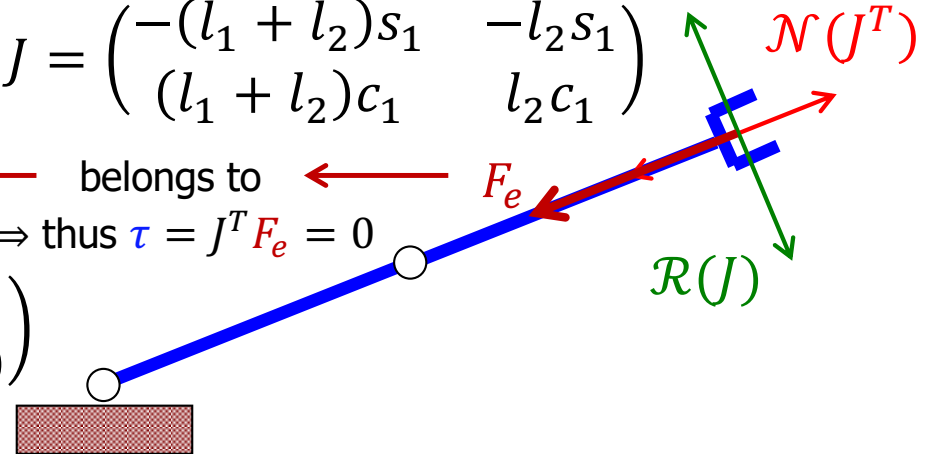
singularity at  $q_2 = 0$  (arm stretched)  $\Rightarrow J = \begin{pmatrix} -(l_1 + l_2)s_1 & -l_2 s_1 \\ (l_1 + l_2)c_1 & l_2 c_1 \end{pmatrix}$

$$\mathcal{R}(J) = \alpha \begin{pmatrix} -s_1 \\ c_1 \end{pmatrix}$$

$$\mathcal{N}(J^T) = \beta \begin{pmatrix} c_1 \\ s_1 \end{pmatrix}$$

belongs to  $\Rightarrow$  thus  $\tau = J^T F_e = 0$

$$\mathcal{R}(J^T) = \gamma \begin{pmatrix} l_1 + l_2 \\ l_2 \end{pmatrix} \quad \mathcal{N}(J) = \delta \begin{pmatrix} l_2 \\ -(l_1 + l_2) \end{pmatrix}$$



singularity at  $q_2 = \pi$  (arm folded)  $\Rightarrow J = \begin{pmatrix} (l_2 - l_1)s_1 & l_2 s_1 \\ -(l_2 - l_1)c_1 & -l_2 c_1 \end{pmatrix}$

$\mathcal{R}(J)$  and  $\mathcal{N}(J^T)$  as above

$$\mathcal{R}(J^T) = \gamma \begin{pmatrix} l_2 - l_1 \\ l_2 \end{pmatrix} \quad (\text{for } l_1 = l_2: \gamma \begin{pmatrix} 0 \\ 1 \end{pmatrix}) \quad \mathcal{N}(J) = \delta \begin{pmatrix} l_2 \\ -(l_2 - l_1) \end{pmatrix} \quad (\text{for } l_1 = l_2: \delta \begin{pmatrix} 1 \\ 0 \end{pmatrix})$$

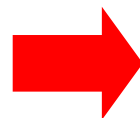
Cartesian forces  $F_e$  applied at the end-effector of the 2R arm along the stretched or folded direction need NO joint torques  $\tau$  for balance



# Force manipulability

- in a given configuration, evaluate how effective is the **transformation** between joint torques and end-effector forces
  - “how easily” can the end-effector apply generalized forces (or balance applied ones) in the various directions of the task space
  - in singular configurations, there are directions in the task space where external forces are **balanced without the need of any joint torque**
- we consider all end-effector forces that can be applied (or balanced) by choosing joint torque vectors of **unit norm**

$$\tau^T \tau = 1$$



$$F^T J J^T F = 1$$

same **directions** of the principal axes of the velocity ellipsoid, but with semi-axes of **inverse lengths**



task **force**  
manipulability **ellipsoid**





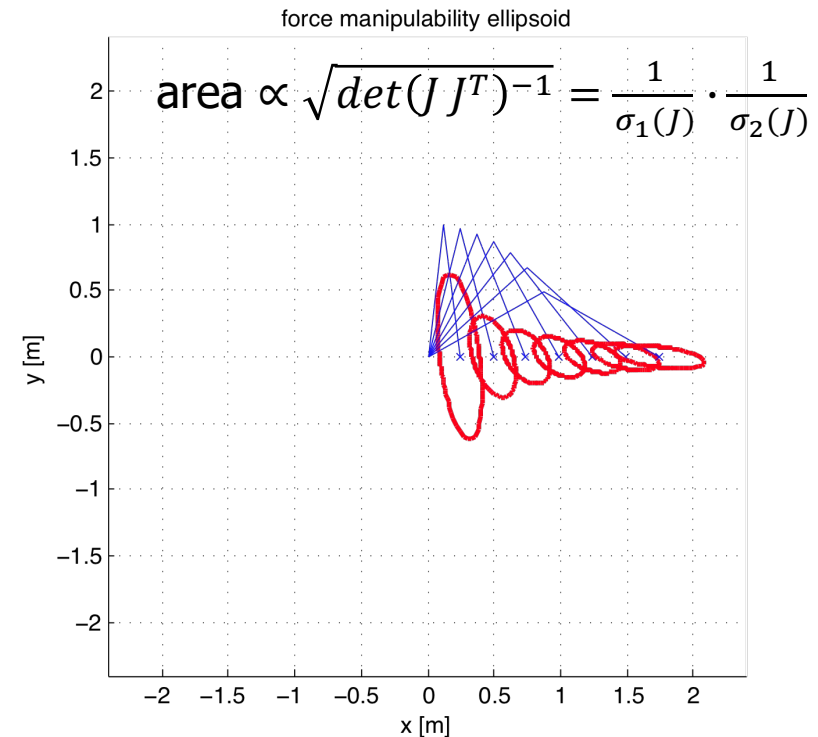
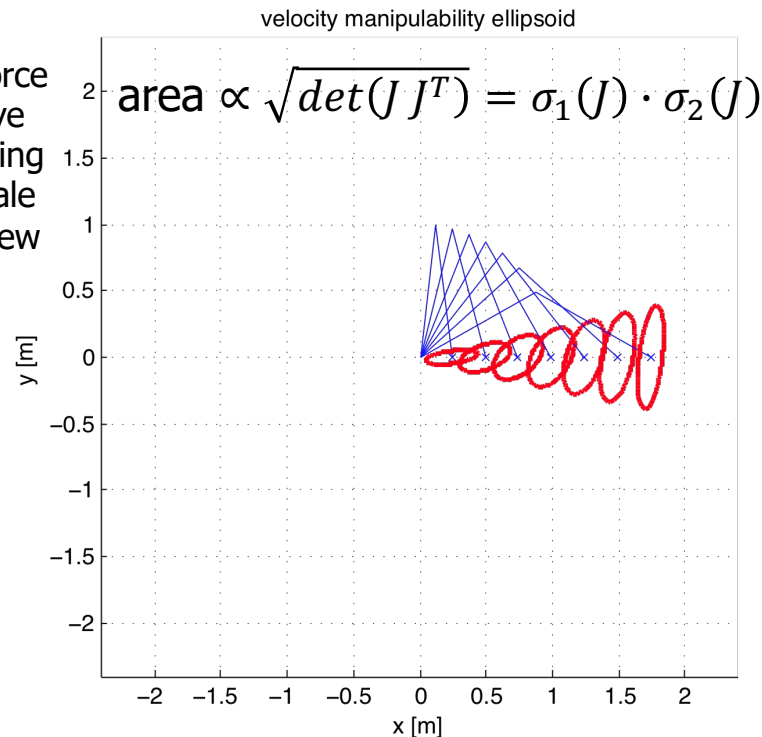
# Velocity and force manipulability

## dual comparison of actuation vs. control

### planar 2R arm with unitary links

note:

velocity and force ellipsoids have been drawn using a different scale for a better view



Cartesian **actuation** task (joint-to-task **high transformation** ratio):  
preferred velocity (or force) directions are those where the ellipsoid **stretches**

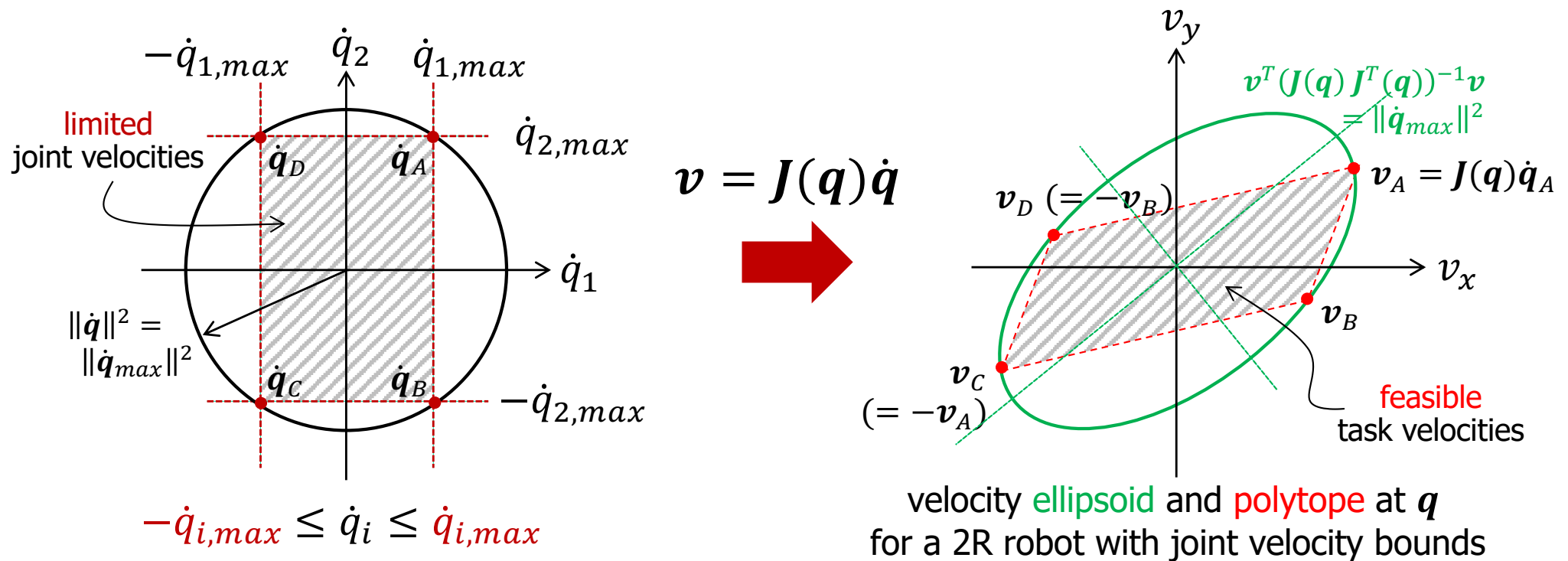


Cartesian **control** task (**low transformation** ratio = **high resolution**):  
preferred velocity (or force) directions are those where the ellipsoid **shrinks**

# Ellipsoids and polytopes

## manipulability versus task limits due to bounds

- **manipulability**: instantaneous capability of moving the end-effector (or of resisting to task forces) in different directions
- **task limits**: maximum velocity (or static balanced force) achievable in different task directions in the presence of **joint velocity bounds**



- a **polytope** is the convex hull of a set of  $p$  points in a ( $n$ -dimensional) Euclidean space
- linear maps transform polytopes into polytopes



# Velocity and force transformations

- same reasoning made for relating end-effector to joint forces/torques (virtual work principle + static equilibrium) used also for transforming forces and torques applied at different places of a rigid body and/or expressed in different reference frames

transformation among twists (generalized velocities)

$$\begin{bmatrix} {}^A v_A \\ {}^A \omega \end{bmatrix} = \begin{bmatrix} {}^A R_B & -{}^A R_B S({}^B r_{BA}) \\ 0 & {}^A R_B \end{bmatrix} \begin{bmatrix} {}^B v_B \\ {}^B \omega \end{bmatrix} = J_{BA} \begin{bmatrix} {}^B v_B \\ {}^B \omega \end{bmatrix}$$

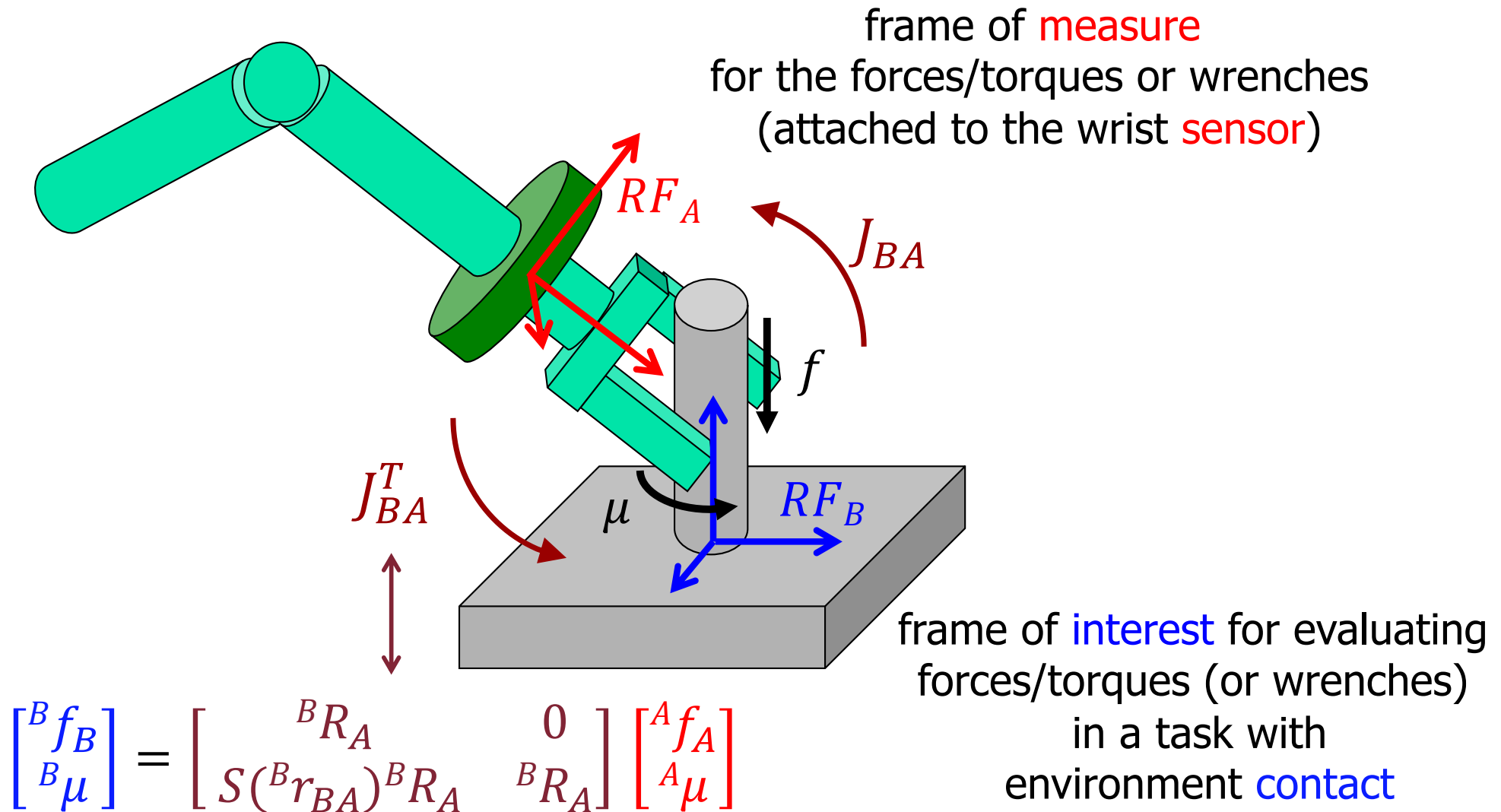


$$\begin{bmatrix} {}^B f_B \\ {}^B \mu \end{bmatrix} = J_{BA}^T \begin{bmatrix} {}^A f_A \\ {}^A \mu \end{bmatrix} = \begin{bmatrix} {}^B R_A & 0 \\ S({}^B r_{BA}) {}^B R_A & {}^B R_A \end{bmatrix} \begin{bmatrix} {}^A f_A \\ {}^A \mu \end{bmatrix}$$

transformation among wrenches (generalized forces)

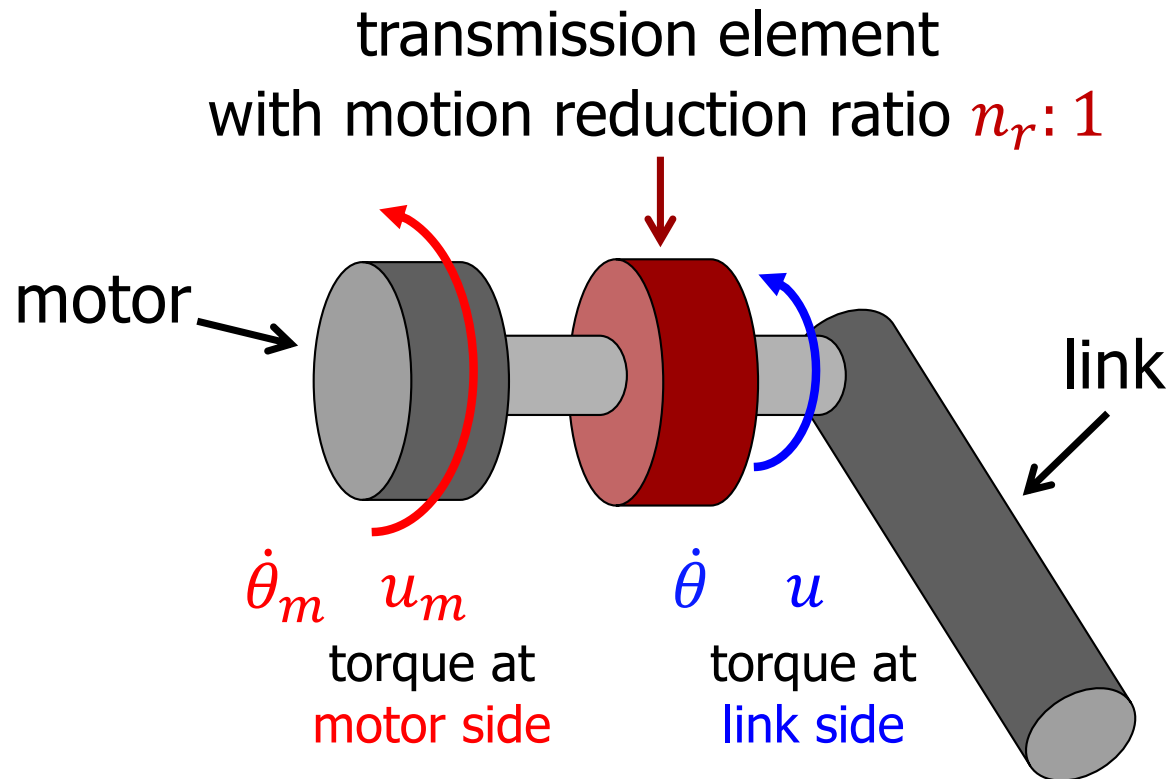
for skew-symmetric matrices, it is:  $-S^T(r) = S(r)$

# Example: 6D force/torque sensor





# Example: Gear reduction at joints



one of the simplest applications  
of the principle of virtual work:

$$P_m = u_m \dot{\theta}_m = u \dot{\theta} = P$$

$$\begin{aligned} \dot{\theta}_m &= n_r \dot{\theta} \\ u &= n_r u_m \end{aligned}$$

here,  $J = J^T = \frac{1}{n_r}$  (a scalar!)  
[from motor to link]