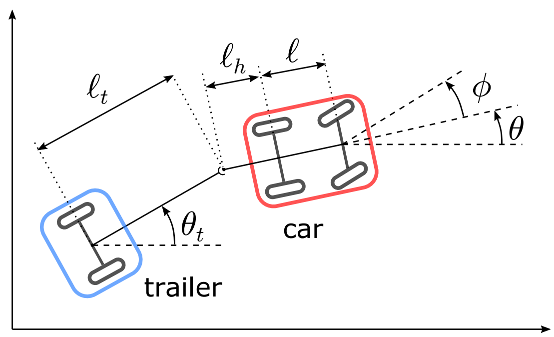
Autonomous and Mobile Robotics
Solution of Midterm Class Test, 2023/2024

Solution of Problem 1

Assimilate all two-wheel axles to a single wheel located at the axle midpoint. The vehicle has then three wheels: the car front wheel, the car rear wheel, and the trailer wheel.

Let (x, y) be the Cartesian coordinates of the (contact point of the) car rear wheel. A convenient choice of generalized coordinates is $\boldsymbol{q} = (x, y, \theta, \phi, \theta_t)$ (see figure), i.e., a set of generalized coordinates for the car plus the absolute orientation of the trailer.



The kinematic constraints acting on the robot are the following (one pure rolling condition for each wheel):

$$
\begin{aligned}
\dot{x}\sin\theta - \dot{y}\cos\theta &= 0\\
\dot{x}_f\sin(\theta+\phi) - \dot{y}_f\cos(\theta+\phi) &= 0\\
\dot{x}_t\sin\theta_t - \dot{y}_t\cos\theta_t &= 0,
\end{aligned}
$$

where (x_f, y_f) and (x_t, y_t) are the Cartesian coordinates of the car front wheel and the trailer wheel, respectively. Being

$$
\begin{aligned}
x_f &= x + \ell\cos\theta\\
y_f &= y + \ell\sin\theta
\end{aligned}
$$

and

$$
\begin{aligned}
x_t &= x - \ell_h\cos\theta - \ell_t\cos\theta_t && (1)\\
y_t &= y - \ell_h\sin\theta - \ell_t\sin\theta_t, && (2)
\end{aligned}
$$

it is easy to obtain the following expression for the Pfaffian kinematic constraints

$$
\begin{aligned}
\dot{x}\sin\theta - \dot{y}\cos\theta &= 0\\
\dot{x}\sin(\theta+\phi) - \dot{y}\cos(\theta+\phi) - \dot{\theta}\,\ell\cos\phi &= 0\\
\dot{x}\sin\theta_t - \dot{y}\cos\theta_t + \ell_h\,\dot{\theta}\cos(\theta-\theta_t) + \ell_t\,\dot{\theta}_t &= 0,
\end{aligned}
$$

or, in matrix form

$$
\begin{pmatrix}
\sin\theta & -\cos\theta & 0 & 0 & 0\\
\sin(\theta+\phi) & -\cos(\theta+\phi) & -\ell\cos\phi & 0 & 0\\
\sin\theta_t & -\cos\theta_t & \ell_h\cos(\theta-\theta_t) & 0 & \ell_t
\end{pmatrix}
\begin{pmatrix}
\dot{x}\\ \dot{y}\\ \dot{\theta}\\ \dot{\phi}\\ \dot{\theta}_t
\end{pmatrix}
= \boldsymbol{A}^T(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{0}.
$$

The submatrix consisting of the first two rows and the first four columns of $\boldsymbol{A}^T$ coincides with the constraint matrix for the car. A basis $\{\boldsymbol{g}_1, \boldsymbol{g}_2\}$ for the 2-dimensional null space of $\boldsymbol{A}^T$ can then be easily written by augmenting the input vector fields $\boldsymbol{g}_1^{\rm car}$, $\boldsymbol{g}_2^{\rm car}$ of the car with suitable fifth elements. The fifth element of $\boldsymbol{g}_2$ is obviously 0, whereas the fifth element α of $\boldsymbol{g}_1$ can be found by imposing

$$\begin{pmatrix} \sin\theta_t & -\cos\theta_t & \ell_h\cos(\theta-\theta_t) & 0 & \ell_t \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \\ (\tan\phi)/\ell \\ 0 \\ \alpha \end{pmatrix} = 0.$$

One easily obtains

$$\boldsymbol{g}_1(\boldsymbol{q}) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ (\tan\phi)/\ell \\ 0 \\ \dfrac{\sin(\theta-\theta_t)}{\ell_t} - \dfrac{\ell_h\cos(\theta-\theta_t)\tan\phi}{\ell\,\ell_t} \end{pmatrix} \qquad \boldsymbol{g}_2(\boldsymbol{q}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

The kinematic control system is then[1]

$$\dot{\boldsymbol{q}} = \boldsymbol{g}_1(\boldsymbol{q})\,v + \boldsymbol{g}_2(\boldsymbol{q})\,\omega,$$

where v and ω are respectively the driving and the steering velocity of the car.

The last part of the problem deals with the particular case $\ell_h = 0$. To prove that (x_t, y_t) are flat outputs, we must show that the state $\boldsymbol{q}$ and inputs v, ω can be reconstructed algebraically from x_t, y_t and their time derivatives. Let us start by the reconstruction formula for θ_t:

$$\theta_t = \arctan \dot{y}_t/\dot{x}_t,$$

which is a consequence of the pure rolling constraint for the trailer wheel. Then, setting $\ell_h = 0$ in eqs. (1–2) we obtain the reconstruction formulas for x, y

$$x = x_t + \ell_t\cos\theta_t = x_t + \ell_t\cos(\arctan\dot{y}_t/\dot{x}_t) \tag{3}$$
$$y = y_t + \ell_t\sin\theta_t = y_t + \ell_t\sin(\arctan\dot{y}_t/\dot{x}_t), \tag{4}$$

and by differentiating these we can easily derive the reconstruction formula for v, since

$$v = \pm\sqrt{\dot{x}^2 + \dot{y}^2}.$$

Similarly, the reconstruction formulas for the remaining state variables θ, ϕ and control input ω can be easily found by considering that

$$\theta = \arctan\dot{y}/\dot{x}$$
$$\phi = \arctan\ell\,\dot{\theta}/v$$

and that $\omega = \dot{\phi}$.

[1]This kinematic model is associated to the choice of $\boldsymbol{q}$ made at the beginning. A different choice (e.g., $\boldsymbol{q}' = (x, y, \theta, \phi, \delta)$, with $\delta = \theta - \theta_t$) would have led to a different model, although equivalent via a change of coordinates.

Solution of Problem 2

The kinematic model of the rear-wheel drive bicycle with acceleration inputs is readily written as

$$\begin{aligned}
\dot{x} &= v\cos\theta \\
\dot{y} &= v\sin\theta \\
\dot{\theta} &= v\,\frac{\tan\phi}{\ell} \\
\dot{\phi} &= \omega \\
\dot{v} &= a_v \\
\dot{\omega} &= a_\omega,
\end{aligned}$$

with the usual meaning of symbols. Note that in this model the state consists of the configuration $\boldsymbol{q} = (x, y, \theta, \phi)$ augmented with the driving and steering velocity v and ω.

From simple geometry, the coordinates of point P are computed as

$$\begin{aligned}
y_1 &= x + \ell\cos\theta + b\cos(\theta+\phi) \\
y_2 &= y + \ell\sin\theta + b\sin(\theta+\phi).
\end{aligned}$$

To perform feedback linearization for these outputs, we start by computing their first-order time derivatives. Using the equations of the kinematic model, one gets:

$$\begin{aligned}
\dot{y}_1 &= \left(\cos\theta - \tan\phi\left(\sin\theta + b\sin(\theta+\phi)/\ell\right)\right)\,v - b\sin(\theta+\phi)\,\omega \\
\dot{y}_2 &= \left(\sin\theta + \tan\phi\left(\cos\theta + b\cos(\theta+\phi)/\ell\right)\right)\,v + b\cos(\theta+\phi)\,\omega,
\end{aligned}$$

or, in matrix form,

$$\dot{\boldsymbol{y}} = \boldsymbol{T}(\boldsymbol{q})\begin{pmatrix} v \\ \omega \end{pmatrix}, \tag{5}$$

where

$$\boldsymbol{T}(\boldsymbol{q}) = \begin{pmatrix} t_{11}(\boldsymbol{q}) & t_{12}(\boldsymbol{q}) \\ t_{21}(\boldsymbol{q}) & t_{22}(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} \cos\theta - \tan\phi\left(\sin\theta + b\sin(\theta+\phi)/\ell\right) & -b\sin(\theta+\phi) \\ \sin\theta + \tan\phi\left(\cos\theta + b\cos(\theta+\phi)/\ell\right) & b\cos(\theta+\phi). \end{pmatrix}$$

Since the control inputs a_v, a_ω do not appear in (5), we differentiate again w.r.t. time, obtaining

$$\ddot{\boldsymbol{y}} = \boldsymbol{T}(\boldsymbol{q})\begin{pmatrix} a_v \\ a_\omega \end{pmatrix} + \dot{\boldsymbol{T}}(\boldsymbol{q})\begin{pmatrix} v \\ \omega \end{pmatrix}.$$

We can now let[2]

$$\begin{pmatrix} a_v \\ a_\omega \end{pmatrix} = \boldsymbol{T}^{-1}(\boldsymbol{q})\left(\boldsymbol{u} - \dot{\boldsymbol{T}}(\boldsymbol{q})\begin{pmatrix} v \\ \omega \end{pmatrix}\right), \tag{6}$$

obtaining thus a second-order linear mapping between the output vector $\boldsymbol{y}$ and the new input vector $\boldsymbol{u} = (u_1, u_2)$:

$$\ddot{\boldsymbol{y}} = \boldsymbol{u}.$$

Globally exponential tracking of the desired trajectory $\boldsymbol{y}_d(t)$ is then guaranteed by choosing $\boldsymbol{u}$ as

$$\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \ddot{y}_{1d} + k_{p1}(y_{1d} - y_1) + k_{d1}(\dot{y}_{1d} - \dot{y}_1) \\ \ddot{y}_{2d} + k_{p2}(y_{2d} - y_2) + k_{d2}(\dot{y}_{2d} - \dot{y}_2) \end{pmatrix} \tag{7}$$

as long as the control gains k_{p1}, k_{d1}, k_{p2}, k_{d2} are positive.

[2]One may verify that matrix $\boldsymbol{T}(\boldsymbol{q})$ is always invertible, since its determinant is $b/\cos\phi$.

Note the following points.

- The final expression of the original control inputs a_v, a_ω is found by plugging (7) in (6). Measurements of the configuration $\boldsymbol{q}$ as well as of the additional state variables v, ω are needed to carry out this computation.

- The elements of $\dot{\boldsymbol{T}}(\boldsymbol{q})$ can be computed in closed form. For example, we have

$$\dot{t}_{12}(\boldsymbol{q}) = \frac{\partial\, t_{12}}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}} = -b\cos(\theta+\phi)(\dot{\theta}+\dot{\phi}) = -b\cos(\theta+\phi)(v\,\frac{\tan\phi}{\ell}+\omega),$$

  a quantity which can be computed at each instant of time because it only depends on state variables.

Solution of Problem 3

Using Euler integration, a discrete-time version of the kinematic model of Problem 2 is written as

$$\begin{aligned}
x_{k+1} &= x_k + v_k \cos\theta_k \, T_s \\
y_{k+1} &= y_k + v_k \sin\theta_k \, T_s \\
\theta_{k+1} &= \theta_k + v_k \frac{\tan\phi_k}{\ell} T_s \\
\phi_{k+1} &= \phi_{k+1} + \omega_k \, T_s \\
v_{k+1} &= v_k + a_{v,k} \, T_s \\
\omega_{k+1} &= \omega_k + a_{\omega,k} \, T_s .
\end{aligned}$$

This motion model is assumed to be perturbed by a white gaussian noise with zero mean and known covariance.

As for the measurement model, we have a total of four measurements coming from the sensors at each sampling instant. The first two are the range and bearing of the landmark

$$\begin{pmatrix} y_{1k} \\ y_{2k} \end{pmatrix} = \begin{pmatrix} \sqrt{(x_{f,k} - x_l)^2 + (y_{f,k} - x_l)^2} \\ \operatorname{atan2}(y_l - y_{f,k}, x_l - x_{f,k}) - \phi_k - \theta_k \end{pmatrix},$$

where $x_f = x + \ell \cos\theta$ and $y_f = y + \ell \sin\theta$ are the Cartesian coordinates of the front wheel. The third measurement is the rotation of the rear wheel around the horizontal wheel axis during the sampling interval

$$y_{3k} = \Delta\alpha = \frac{v_k \, T_s}{r},$$

where r is the radius of the wheel.[3] The fourth and last measurement is the rotation of the front wheel around the vertical wheel axis during the sampling interval

$$y_{4k} = \Delta\phi = \omega_k \, T_s .$$

The measurement model is therefore

$$\boldsymbol{y} = \begin{pmatrix} y_{1k} \\ y_{2k} \\ y_{3k} \\ y_{4k} \end{pmatrix}$$

with the previous formulas providing the expression of each component as a function of the state variables. This model is also assumed to be perturbed by a white gaussian noise with zero mean and known covariance.

The rest of the solution is straightforward: linearize the motion and measurement models (note that the last three equations of the former and the last two of the latter are already linear) and then write the EKF equations.

In the prediction stage, the acceleration inputs a_v and a_ω will be needed: one can either use the nominal values coming from the control module, or estimate them numerically using the two encoder readings, based on the fact that $a_v = \dot{v}$ and $a_\omega = \dot{\omega}$. In the latter case, the encoders will be used both in the prediction and in the correction stage.

[3]Here, we are using $\Delta s = v_k \, T_s = r \, \Delta\alpha$, where Δs is the traveled distance.

Solution of Problem 4

(a) FALSE. It is also necessary to describe the 3D orientation of the quadruped body. The configuration space is therefore $I\!R^3 \times SO(3) \times (SO(2))^{N_1+N_2}$.

(b) TRUE. Assume $h(\boldsymbol{q}) = 0$ is the geometric constraint. Then, its continued satisfaction implies $dh/dt = \partial h/\partial \boldsymbol{q}\, \dot{\boldsymbol{q}} = 0$, i.e., a kinematic constraint that is integrable.

(c) TRUE. If the front wheels had the exact same orientation, their zero motion lines would be parallel and no instantaneous center of rotation would exist in general (unless the car was traveling in a straight line); as a consequence, the vehicle would slip. To avoid this effect, the front wheels of a car do not have the same exact orientation when turning, thanks to a particular device called *Ackermann steering*.

(d) FALSE. All state variables can be reconstructed from the flat outputs. In the case of the car-like robot, both θ and ϕ can be expressed as a function of x, y and their derivatives. Therefore, the initial and final values of θ and ϕ generate a total of 4 boundary conditions for the interpolation.

(e) FALSE. In general, the Cartesian trajectory must be such that the associated state trajectory computed via the flatness reconstruction formulas is continuous, because jumps in the configuration variables cannot be executed by the robot. In the case of the unicycle, this condition boils down to θ being continuous along the trajectory; therefore, Cartesian trajectories with sharp corners are not allowed.