Autonomous and Mobile Robotics
Solution of Midterm Class Test, 2022/2023

Solution of Problem 1

(a) The robot configuration space is the 3-dimensional manifold $\mathcal{C} = SO(2) \times I\!R \times SO(2)$.

(b) Angular momentum conservation is a Pfaffian constraint:

$$\begin{pmatrix} m(d+\ell)^2 & 0 & J \end{pmatrix} \begin{pmatrix} \dot{\phi} \\ \dot{\ell} \\ \dot{\theta} \end{pmatrix} = 0 \qquad \text{i.e.,} \quad \boldsymbol{a}^T(\boldsymbol{q})\dot{\boldsymbol{q}} = 0 \tag{1}$$

A basis for $\mathcal{N}(\boldsymbol{a}^T(\boldsymbol{q}))$ consists of the following vector fields

$$\boldsymbol{g}_1(\boldsymbol{q}) = \begin{pmatrix} 1 \\ 0 \\ -\frac{m(d+\ell)^2}{J} \end{pmatrix} \qquad \boldsymbol{g}_2(\boldsymbol{q}) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

and the corresponding kinematic model is

$$\begin{aligned} \dot{\phi} &= u_1 \\ \dot{\ell} &= u_2 \\ \dot{\theta} &= -\frac{m(d+\ell)^2}{J}\, u_1 \end{aligned} \tag{2}$$

This kinematic model is the most appropriate, since u_1 and u_2 are clearly the velocity inputs of the two leg joints (revolute and prismatic, respectively), which are in fact directly actuated.

(c) From a local viewpoint, mobility is obviously restricted by the angular momentum conservation constraint. In fact, at any $\boldsymbol{q}$ the generalized velocity $\dot{\boldsymbol{q}}$ must belong to the null space of $\boldsymbol{a}^T(\boldsymbol{q})$, i.e., a 2-dimensional subspace of the tangent space $I\!R^3$.

To investigate global mobility, we study the controllability of the kinematic model. One easily finds

$$\boldsymbol{g}_3(\boldsymbol{q}) = [\boldsymbol{g}_1, \boldsymbol{g}_2]\,(\boldsymbol{q}) = \begin{pmatrix} 0 \\ 0 \\ \frac{2m(d+\ell)}{J} \end{pmatrix}$$

where the third component is never zero being $\ell \geq 0$. Therefore, we have

$$\text{rank}\,(\boldsymbol{g}_1(\boldsymbol{q}) \;\; \boldsymbol{g}_2(\boldsymbol{q}) \;\; \boldsymbol{g}_3(\boldsymbol{q})) = 3$$

and the kinematic model is controllable. This means that the conservation of angular momentum is a nonholonomic constraint for this robot. We can then conclude that global mobility is not restricted, as the robot can reach any configuration in $\mathcal{C}$.

(d) In our kinematic model, the dynamics of ℓ is a simple integrator. Therefore, choosing the velocity input of the prismatic joint as

$$u_2 = k_\ell(h - d - \ell) \qquad k_\ell > 0 \tag{3}$$

will guarantee exponential convergence of ℓ to $h - d$, and therefore of the total leg length to h.

As for θ, we can obtain the same situation with the following input transformation:

$$u_1 = -\frac{J}{m(d+\ell)^2}\, v_1$$

which is always invertible. With this choice, the dynamics of θ becomes

$$\dot{\theta} = -\frac{m(d+\ell)^2}{J}\, u_1 = v_1$$

Exponential convergence of θ to zero is obtained by choosing the new input as

$$v_1 = -k_\theta\,\theta \qquad k_\theta > 0$$

The original velocity input for the revolute joint is finally computed as

$$u_1 = k_\theta\, \frac{J}{m(d+\ell)^2}\, \theta \tag{4}$$

Note that implementation of the feedback control law (3–4) only requires measurements of ℓ and θ (not ϕ).

(e) If $d = 0$, the kinematic model (2) becomes

$$\begin{aligned} \dot{\phi} &= u_1 \\ \dot{\ell} &= u_2 \\ \dot{\theta} &= -\frac{m\,\ell^2}{J}\, u_1 \end{aligned} \tag{5}$$

This model is differentially flat with flat outputs ϕ, θ. In fact, from the last equation of (5) we can write

$$\ell^2 = -\frac{J\,\dot{\theta}}{m\,u_1} = -\frac{J\,\dot{\theta}}{m\,\dot{\phi}}$$

from which we get the reconstruction formula[1] for ℓ:

$$\ell = \sqrt{-\frac{J}{m}\,\frac{\dot{\theta}}{\dot{\phi}}}$$

As for reconstruction of the inputs, we have immediately

$$u_1 = \dot{\phi}$$

[1] Note that constraint (1) implies that $\dot{\theta}$ and $\dot{\phi}$ always have opposite sign, so the formula is well-posed.

while computing the time derivative of ℓ as given by the reconstruction formula we obtain

$$u_2 = \dot{\ell} = \frac{J(\dot{\theta}\,\ddot{\phi} - \ddot{\theta}\,\dot{\phi})}{2\,\dot{\phi}\sqrt{-m\,J\dot{\phi}\,\dot{\theta}}}$$

For path planning, it is convenient to consider the geometric version of the kinematic model (5):

$$\begin{aligned} \phi' &= \tilde{u}_1 \\ \ell' &= \tilde{u}_2 \\ \theta' &= -\frac{m\,\ell^2}{J}\,\tilde{u}_1 \end{aligned} \tag{6}$$

where the prime symbol ($'$) denotes differentiation with respect to the path parameter s and the tilde symbol ($\tilde{}$) is used to distinguish the geometric inputs from the original velocity inputs.

Since the flat outputs are ϕ and θ, an algorithm for path planning is the following:

1. Let $\phi(s) = as + b$, with $s \in [0,1]$. Compute a and b that satisfy $\phi(0) = \phi_i$ and $\phi(1) = \phi_f$. The associated geometric input is $\tilde{u}_1(s) = \phi'(s) = a$, for any s.
2. Let $\theta(s) = cs^3 + ds^2 + es + f$, with $s \in [0,1]$. Compute c, d, e and f that satisfy $\theta(0) = \theta_i$, $\theta(1) = \theta_f$ and the additional boundary conditions

$$\begin{aligned} \theta'(0) &= -\frac{m\,\ell_i^2}{J}\,a \\ \theta'(1) &= -\frac{m\,\ell_f^2}{J}\,a \end{aligned}$$

   where we have used the last equation of (6) and the fact that $\tilde{u}_1(s) = a$.
3. Reconstruct the remaining state variable $\ell(s)$ and the second geometric input $\tilde{u}_2(s)$ using the previous reconstruction formulas.

Solution of Problem 2

A localization system for the hopping robot can be designed using the EKF. To this end, first use Euler method to derive the following discrete-time motion model from the continuous-time model (2)

$$\begin{aligned}
\phi_{k+1} &= \phi_k + T_s\, u_{1,k} \\
\ell_{k+1} &= \ell_k + T_s\, u_{2,k} \\
\theta_{k+1} &= \theta_k - T_s\, \frac{m(d+\ell_k)^2}{J}\, u_{1,k}
\end{aligned}$$

where T_s is the sampling interval. This model is assumed to be perturbed by a white gaussian noise with zero mean and known covariance.

The measurements of the joint encoders, respectively denoted by $\Delta\phi$ and $\Delta\ell$, will be used to reconstruct the actual values of $u_{1,k} = \Delta\phi_k/T_s$ and $u_{2,k} = \Delta\ell_k/T_s$.

The only exteroceptive measurement is γ. A simple geometric computation gives

$$\tan(\theta+\gamma) = \frac{z_b - z}{x_z - x}$$

leading to the following measurement model:

$$y_k = \gamma_k = \arctan\frac{z_b - z_k}{x_z - x_k} - \theta_k$$

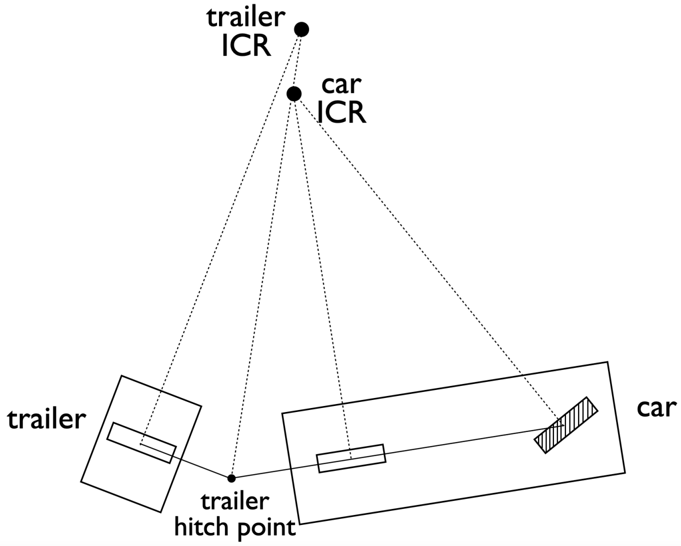
with the first term in the right-hand side consisting of known quantities (the location of the beacon is known while x_k, z_k are made available by the external vision system). Also this model is assumed to be perturbed by a white gaussian noise with zero mean and known covariance.

The rest of the solution is straightforward: linearize the motion and measurement models (the first is nonlinear, the second is actually affine in the state) and then write the EKF equations. In the block scheme, the joint encoders will be used in the prediction stage of the filter, while the bearing sensor will be used to compute the innovation in the correction stage.

Solution of Problem 3

(a) FALSE. The Lie bracket itself is aligned with the zero motion line, but the total final displacement is not. In general, the displacement after a Lie Bracket maneuver is given by a quadratic term in ϵ, which is aligned with the Lie bracket, *plus higher-order terms in* ϵ. In the case of a unicycle, the higher-order terms are not zero. This can be easily proven graphically by drawing the motion of the robot during the maneuver (drive forward with $v = 1$ for ϵ seconds, then steer in the CCW direction with $\omega = 1$ for ϵ seconds, then drive backwards with $v = -1$ for ϵ seconds, and finally steer in the CW direction with $\omega = -1$ for ϵ seconds).

(b) TRUE. The orientation of the front and rear wheels of the car will determine the ICR for the car, around which all points of the car, including the trailer hitch point, are instantaneously rotating. This allows to determine the position of the ICR for the trailer as shown in the figure. Note that the trailer hitch point is simultaneously rotating around both ICRs.



(c) TRUE. In fact, the two speeds are the same only if the bicycle is moving in a straight line. In any other case, both wheels move instantaneously along arcs of circles centered at the ICR, with the radius of front wheel circle always larger than the radius of the rear wheel. Therefore, the front wheel must move at a higher speed.

(d) FALSE. Although computing the error only requires measurements of x and y, θ is still needed for the feedback transformation, both in the static and in the dynamic controllers.

(e) FALSE. It is used because V is positive *semidefinite* rather than *definite*.