Control Systems - January 12, 2026

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1) Consider the system $P(s)$ and the static controller $C(s) = K_c$ with

$$P(s) = \frac{-3(s+2)}{(s+6)^3}.$$

1. Study with the root locus the stability of the closed loop system in terms of K_c.
2. Compute the impulse response for the closed loop complementary sensitivity function when $K_c = 36$.

2) Given the interconnected system shown in Fig. 1 with

$$H(s) = \frac{s+1}{(s+10)^2}, \qquad F(s) = \frac{s}{s+1}, \qquad G(s) = \frac{1}{s+10}.$$

Design a controller in a feedback control scheme such that:

1. the steady state response to a reference $r = t\delta_{-1}(t)$ is in magnitude smaller equal to 0.01,
2. the crossover frequency is $\omega_c^* = 10$ rad/s and the phase margin is at least 30°.

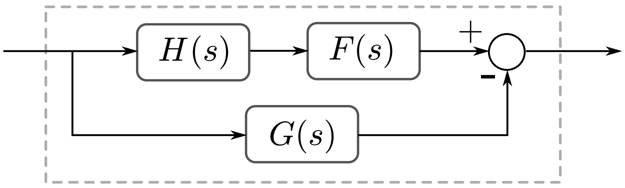

Figure 1: Interconnected system

3) Let the plant be represented by the following differential equations

$$\begin{aligned}
\dot{x}_1 &= x_1 + x_2 - 4\,x_3 + 2\,u \\
\dot{x}_2 &= 2\,x_1 - 4\,x_3 + 2\,u \\
\dot{x}_3 &= x_1 - 3\,x_3 + u \\
y &= x_2 - x_3
\end{aligned}$$

1. Design, if possible, a state feedback which assigns the closed loop eigenvalues $\{-1, -2, -3\}$.
2. Is it possible to design a state observer? Justify your response.
3. What will the dimension of the transfer function be? Compute it.
4. Is the plant stabilizable from the output with a simple gain (in a feedback control scheme)?

4) Consider the loop function

$$L(s) = K\frac{s-10}{(s-100)^2}.$$

1. Study the stability of the closed loop system in terms of K.
2. Confirm the results through an analysis of the Nyquist plot.

Sol 1A) First we need to put the loop function $L(s) = K_c P(s)$ in the form

$$L(s) = K_c P(s) = -3\,K_c \frac{s+2}{(s+6)^3} = K \frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)} = K \frac{s+2}{(s+6)^3}$$

this means we set $K = -3\,K_c$ and draw the root locus with respect to K. The positive root locus for K will correspond to the negative root locus for K_c (scaled differently). Subsequently we will translate the results in terms of K_c. We have $n = 3$ and $m = 1$, so $n - m = 2$ asymptotes for the positive (and similarly for the negative root locus) with a center of asymptotes $s_0 = (-18+2)/2 = -8$. We know that we will have a singular point of multiplicity 3 in $s_1^* = -6$ (open loop multiple pole) with alternating positive and negative branches. The pole polynomial is given by

$$p(s, K) = (s+6)^3 + K(s+2) = s^3 + 18s^2 + (K+108)s + 2(K+108)$$

or

$$p(s, K_c) = (s+6)^3 - 3K_c(s+2) = s^3 + 18s^2 + (108 - 3K_c)s + 2(108 - 3K_c)$$

From the first attempt of tracing the root locus, we note the presence of an extra singular point which can be computed with the simplified formula

$$s_2^* \quad \text{solution of} \quad \frac{3}{s+6} - \frac{1}{s+2} = 0 \quad \Leftrightarrow \quad 3(s+2) - (s+6) = 0 \quad \Leftrightarrow \quad s_2^* = 0$$

The corresponding value of K^* is such that the constant term of the closed loop polynomial is equal to 0, that is $K^* = -108$. Note that this corresponds to $K_c^* = 36$. We also note that for K^* the closed loop polynomial reduces to

$$p(s, K^*) = s^2(s+18)$$

From the root locus we may say that the closed loop system is asymptotically stable for $K > -108$ (crossing of the imaginary axis) or $K_c < 36$.

As a check we can write the Routh table (with simplifications).

$$\begin{array}{|cc} 1 & 108 - 3K_c \\ 18 & 2(108 - 3K_c) \\ 108 - 3K_c & \\ 108 - 3K_c & \end{array}$$

The root locus with respect to K is shown in Fig. 2. The red branches correspond to the negative root locus for K (and therefore the positive for K_c), the blue ones are the positive branches for K (negative for K_c).

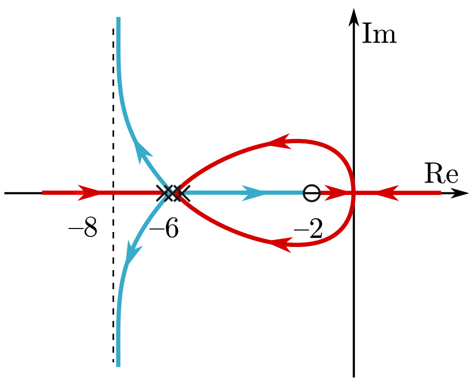

Figure 2: Exercise 1: Root locus w.r.t. K

Since the zeros are not modified by changes in the loop function gain, the closed loop system (complementary sensitivity function $T(s)$) will be

$$T(s) = \frac{K(s+2)}{p(s,K)}$$

For $K_c = 36$ (or $K = -108$) it becomes

$$T^*(s) = \frac{-108(s+2)}{s^2(s+18)}$$

The impulse response is the inverse Laplace transform of the transfer function, and therefore being

$$T^*(s) = \frac{-108(s+2)}{s^2(s+18)} = \frac{R_{11}}{s} + \frac{R_{12}}{s^2} + \frac{R_2}{s+18}$$

with

$$R_{12} = \left\{ \frac{-108(s+2)}{s^2(s+18)} s^2 \right\}_{s=0} = -12, \quad R_2 = \left\{ \frac{-108(s+2)}{s^2(s+18)} (s+18) \right\}_{s=-18} = \frac{16}{3}$$

and R_{11}computed either through the residue formula or by equating polynomial coefficiente. The result is

$$R_{11} = -\frac{16}{3}$$

and therefore the impulse response is

$$y_{imp}(t) = \left(R_{11} + t\, R_{12} + R_2\, e^{-18\,t} \right) \delta_{-1}(t)$$

Some recurring errors

- Many have directly drawn the root locus without the substitution $K = -3K_c$. The resulting root locus would have had a branch belonging to the negative locus going through the origin and so one would have stated that the closed loop system was asymptotically stable for $K_c > K_{c,crit}$ with $K_{c,crit}$ negative. On the other hand from the Routh analysis one finds that the closed loop system is asymptotically stable for $K_c < K_c^*$ with K_c^* positive. So one should have understood that there was a problem.
- Since $s^* = 0$ is a singular point of multiplicity 2 for the root locus corresponding to $K = -108$ (or $K_c = 36$), when one rewrites the closed loop polynomial $p(s, 36)$ the two roots in $s = 0$ should factor out making the factorization of $p(s, 36)$ trivial.
- You should always clearly distinguish the positive and negative root locus (use colours or whatever makes it clear).
- Do not confuse the complementary sensitivity function $T(s)$ with the control sensitivity function $S_u(s)$.
- Write and comment what you are doing!

Sol 2A) The system is the parallel of $-G(s)$ with the series of $H(s)$ and $F(s)$, that is

$$P(s) = H(s).F(s) - G(s) = \frac{s}{(s+10)^2} - \frac{1}{s+10} = \frac{-10}{(s+10)^2} = -0.1 \frac{1}{(1+0.1s)^2}$$

To obtain an asymptotic error in magnitude smaller equal to 0.01, we add a pole in $s = 0$ to make the system type1 and choose the controller gain such that

$$\frac{1}{|K_L|} \leq 0.01 \quad \Leftrightarrow \quad |K_L| \geq 100 \quad \Leftrightarrow \quad K_c \leq -1000.$$

We have chosen the negative inequality for K_c since the plant gain $K_p = -0.1$ is negative. We choose $K_c = -1000$. The extended plant is

$$\widehat{P}(s) = \frac{100}{s} \frac{1}{(1+0.1s)^2}$$

From the corresponding Bode plots, we see that in $\omega_c^* = 10$ rad/s we have

$$|\widehat{F}(j10)|_{dB} = 14\,\text{dB}, \quad \angle\widehat{F}(j10) = -180^\circ$$

We need a lead/lag combination, for example a lead with $m_a = 8$ and $\tau_a = 1/10$ we obtain a phase increase of 38° and an amplification ($\omega\tau = 1$) of 2.5 dB. A lag function with $m_i = 7$ and $\tau_i = 10$ ($\omega\tau = 100$), gives an attenuation of 16.5 dB and a lag of 4°. We obtain the desired crossover frequency with a phase margin of 34°. The Bode stability theorem guarantees closed loop stability.

Some recurring errors

- Some did not notice the signs in the block diagram.
- There are still problems in reducing at common denominator rational fractions ...
- The computation of the plant's gain was often wrong (either forgot to put the transfer function in the Bode canonical form or simply got it wrong).
- We have seen the general forms of the magnitude and phase for every term composing the Bode canonical form in order to explicitly avoid any computation. To understand the required action in the loop shaping it is useful (necessary in an exam) to show the Bode plots of the "modified plant".

- Do not forget to consider also the necessary part of the controller (and obtain the "modified plant") when checking the necessary action in terms of phase and magnitude to be taken through elementary functions.
- It is useless to study the closed loop stability after having determined the necessary part of the controller for the steady state requirements; the choice of the elementary functions will alter the result anyway.

Sol 3A) From the differential equations we have

$$A = \begin{pmatrix} 1 & 1 & -4 \\ 2 & 0 & -4 \\ 1 & 0 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & -1 \end{pmatrix}, \quad D = 0$$

The controllability matrix is

$$P = \begin{pmatrix} B & AB & A^2B \end{pmatrix} = \begin{pmatrix} 2 & 0 & 4 \\ 2 & 0 & 4 \\ 1 & -1 & 3 \end{pmatrix}, \quad \det(P) = 0, \quad \text{rank}(P) = 2 \quad Im(P) = \text{gen}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

If we proceed to the corresponding Kalman decomposition, choosing

$$T^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \Longrightarrow \qquad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

we obtain

$$\tilde{A} = \begin{pmatrix} 2 & -4 & 2 \\ 1 & -3 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad \tilde{C} = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix}$$

Being the uncontrollable subsystem asymptotically stable and its eigenvalue coincident with one of the desired eigenvalues, we can stabilize the full system through state feedback and assign the desired eigenvalues. To find the feedback in the new coordinates we can use the Ackermann formula applied to the controllable subsystem; $\tilde{\gamma}$ is the last row of the controllable matrix relative to $(\tilde{A}_{11}, \tilde{B}_1)$ and since -1 is already an eigenvalue of the closed loop system (as part of the uncontrollable subsystem), we need to assign $(-2, -3)$ and therefore $p^*(\lambda) = (\lambda + 2)(\lambda + 3) = \lambda^2 + 5\lambda + 6$. We have then

$$\tilde{P} = \begin{pmatrix} \tilde{B}_1 & \tilde{A}_{11}\tilde{B}_1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix} \quad P^{-1} = \begin{pmatrix} * & * \\ 1/2 & -1 \end{pmatrix} \quad P^{-1} = \begin{pmatrix} * \\ \tilde{\gamma}^T \end{pmatrix}$$

(here we have used a transpose in $\tilde{\gamma}^T$ to show it is a row vector), and

$$p^*(\tilde{A}_{11}) = \begin{pmatrix} 2 & -4 \\ 1 & -3 \end{pmatrix}^2 + 5\begin{pmatrix} 2 & -4 \\ 1 & -3 \end{pmatrix} + 6\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 16 & -16 \\ 4 & -4 \end{pmatrix}$$

Finally we have

$$\tilde{F}_1 = -\tilde{\gamma}^T p^*(\tilde{A}_{11}) = \begin{pmatrix} -4 & 4 \end{pmatrix}.$$

In the new coordinates we then have $\tilde{F} = \begin{pmatrix} \tilde{F}_1 & 0 \end{pmatrix}$ and in the original ones $F = \tilde{F}T$.

Note that now it is easier to compute the eigenvalues of the original system due to the upper block triangular form of the $\tilde{A}$ matrix, that is

$$\text{eig}(A) = \text{eig}(\tilde{A}) = \text{eig}\begin{pmatrix} 2 & -4 \\ 1 & -3 \end{pmatrix} \cup \{-1\} = \{1, -2\} \cup \{-1\}$$

To see if we can built an asymptotic observer, we have to verify if the system is detectable; being $\lambda = 1$ the only unstable eigenvalue, we can directly check its observability through the PBH test. This is done on the controllable part of the system to simplify computation: the uncontrollable part being asymptotically stable does not influence detectability. Being

$$\text{rank}\begin{pmatrix} \tilde{A}_{11} - I \\ \tilde{C}_1 \end{pmatrix} = \text{rank}\begin{pmatrix} 1 & -4 \\ 1 & -4 \\ 1 & -1 \end{pmatrix} = 2$$

the system is detectable. To see the order of the transfer function (number of poles) we can check if the eigenvalue -2 is observable or not using the PBH test

$$\text{rank}\begin{pmatrix} \tilde{A}_{11} + 2I \\ \tilde{C}_1 \end{pmatrix} = \text{rank}\begin{pmatrix} 4 & -4 \\ 1 & -1 \\ 1 & -1 \end{pmatrix} = 1$$

and therefore the only controllable and observable eigenvalue is $\lambda = 1$ so the transfer function will have only 1 pole. We can compute it from the controllable subsystem (2-dimensional)

$$P(s) = \tilde{C}_1(sI - \tilde{A}_{11})^{-1}\tilde{B}_1 = \frac{1}{s-1}$$

The hidden dynamics is asymptotically stable (and characterized by the eigenvalues -2 and -3) and therefore we can work with the transfer function directly which, having only one pole, is clearly stabilizable with a simple gain $C(s) = K_c$ since the closed loop pole polynomial is

$$p(s, K_c) = s - 1 + K_c.$$

Clearly choosing any $K_c > 1$ will stabilize the closed loop system.

Some recurring errors

- To simplify hand computation of the change of variables it is useful to find the simplest base for the $\mathrm{Im}(P)$ or $\mathrm{Ker}(O)$; for example here

$$\mathrm{span}\left\{\begin{pmatrix}2\\2\\1\end{pmatrix}, \begin{pmatrix}0\\0\\-1\end{pmatrix}\right\} = \mathrm{span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$$

- One could have computed the eigenvalues from the beginning on the original A matrix but this leads to a third order polynomial, so using the Kalman decomposition was useful.
- Similarly for the computation of the transfer function, computing $C(sI - A)^{-1}B$ would have been long and prone to errors; instead using the knowledge of the first decomposition we were able to reduce this computation to from a 3-dimensional to a 2-dimensional problem.

Sol 4A) The closed loop system has pole polynomial

$$p(s, K) = (s - 100)^2 + K(s - 10) = s^2 + (K - 200)\,s + 10000 - 10K$$

therefore the necessary and sufficient condition for the closed loop poles to have strictly negative real part is

$$K - 200 > 0 \qquad \text{and} \qquad 10000 - 10K > 0 \quad \Longrightarrow \quad 200 < K < 1000$$

The Bode canonical form is

$$L(s) = -\frac{K}{1000}\frac{(1 - s/10)}{(1 - s/100)^2}$$

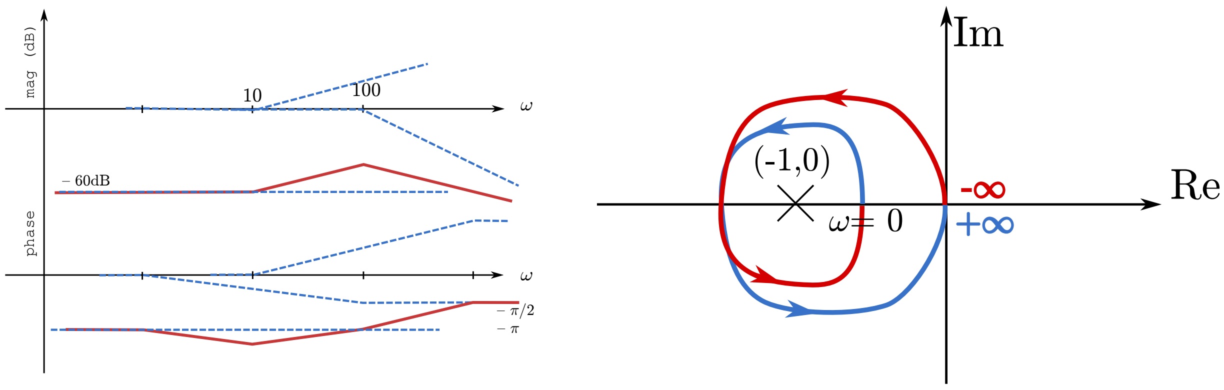

Figure 3: Exercise 4: Bode plots (left) and Nyquist plot (right)

Some recurring errors

- A recurring error was $(s - 100)^2 = -10^4(1 - s/10)^2$ or similar ...
- Bode plots are considered basic knowledge.
- In this system there were no poles on the imaginary axis but many (too many) have drawn closures at infinity in the Nyquist plot.
- The necessary condition for a second order polynomial is also a sufficient condition (for asymptotic stability)... No need to build the Routh table (and even get it wrong). It's a second order equation ...