Control Systems

System response

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Outline

• study a particular response, the step response, to a unit step as input
• characterize the asymptotic and transient response on the step response
• define the long term, asymptotic, behavior (steady-state) of a response and understand when it exists
• compute the steady-state response for different test inputs
The forced response to a **step input** is referred as **step response**

\[ u(t) = \delta_{-1}(t) \]

\[ x_0 = 0 \]

\[ U(s) = \frac{1}{s} \]

\[ X_s(s) = H(s)U(s) = H(s)\frac{1}{s} \]

\[ Y_s(s) = W(s)U(s) = W(s)\frac{1}{s} \]

integral property of the Laplace transform

\[ x_s(t) = \int_{0}^{t} h(\tau)d\tau \]

\[ y_s(t) = \int_{0}^{t} w(\tau)d\tau \]

the step response is equal to the integral of the impulse response
### Step response

Let $S$ be asymptotically stable

$$W(s) = \frac{N(s)}{\prod_{i=1}^{n}(s - p_i)}$$

distinct poles

$$Y_s(s) = W(s)U(s) = \frac{N(s)}{\prod_{i=1}^{n}(s - p_i)} \frac{1}{s}$$

$$Y_s(s) = \frac{R_0}{s} + \sum_{i=1}^{n} \frac{R_i}{s - p_i}$$

$$y_s(t) = \left( R_0 + \sum_{i=1}^{n} R_i e^{p_i t} \right) \delta_{-1}(t)$$

these terms decay since

$S$ be asymptotically stable $\Re[p_i] < 0$
**Step response**

stable system (A.S.)

\[ W(s) \]

\[ y_{\text{ZS}}(t) = y_{s}(t) \]

\[ y_{s}(t) = \left( R_{0} + \sum_{i=1}^{n} R_{i} e^{p_{i}t} \right) \delta_{-1}(t) \]

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**Diagram:***

- **Transient** region:
  - Initial response
  - Decreasing slope

- **Steady-state** region:
  - Constant value
  - "Asymptotic behavior"

**Labels:**
- \( R_{0} \)
- \( R_{i} \)
- \( p_{i} \)
- \( \delta_{-1}(t) \)
- \( y_{s}(t) \)
- \( y_{\text{ZS}}(t) \)
- \( \text{conceptual frontier} \)
- \( t \)
Step response

forced response (convolution integral + direct term)

\[ y_s(t) = \int_0^t Ce^{A(t-\tau)} Bu(\tau) d\tau + Du(t) \]

\[ = Ce^{A(t-\tau)} Bd\tau + D \]

\[ = C \int_0^t e^{A\sigma} Bd\sigma + D \]

\[ = C([A^{-1}e^{A\sigma} B]_{\sigma=t}^{\sigma=0} + D \]

\[ = CA^{-1}e^{At} B + D - CA^{-1}B \]

\underline{transient} \hspace{2cm} \underline{steady-state}

\[ u(t) = 1 \]

\[ \sigma = t - \tau \]
**Laplace transform**

**Final value theorem**

provided the limit on
the left exists

\[
\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)
\]

if \( sF(s) \) is analytic

(all the roots of the denominator of \( sF(s) \) in the open left half-plane)

example

\[
W(s) = \frac{1}{s + 1}
\]

\[
\begin{align*}
 u_1(t) &= \delta_{-1}(t) & U_1(s) &= \frac{1}{s} & sY_1(s) &= s \left( \frac{1}{s + 1} \right) \frac{1}{s} & \text{ok} \\
 u_2(t) &= t\delta_{-1}(t) & U_2(s) &= \frac{1}{s^2} & sY_2(s) &= s \left( \frac{1}{s + 1} \right) \frac{1}{s^2} & \text{no} \\
 u_3(t) &= \sin \omega t & U_3(s) &= \frac{\omega}{s^2 + \omega^2} & sY_3(s) &= s \left( \frac{1}{s + 1} \right) \left( \frac{\omega}{s^2 + \omega^2} \right) & \text{no}
\end{align*}
\]
**Step response: steady-state**

$S$ be asymptotically stable $\Re[p_i] < 0$  \[ W(s) = \frac{N(s)}{\prod_{i=1}^{n}(s - p_i)} \]  distinct poles

\[ Y_s(s) = W(s)U(s) = \frac{N(s)}{\prod_{i=1}^{n}(s - p_i)} \frac{1}{s} \]

by the definition of residue $R_0$

\[ R_0 = [sY_s(s)]_{s=0} = \left[sW(s) \frac{1}{s}\right]_{s=0} = W(0) \]

or the application of the final value theorem

\[ y_s(t) = R_0\delta_{-1}(t) + \sum_{i=1}^{n} R_i e^{p_it} \delta_{-1}(t) \]

these terms decay

\[ y_{ss}(t) = \lim_{s \to 0} sY_s(s) = \lim_{s \to 0} sW(s) \frac{1}{s} = W(0) = R_0 \]

**dc-gain**

note that \[ W(0) = D - CA^{-1}B \]
Step response: steady-state

Since the output (total) response to a step input is given by

\[ y(t) = y_{zi}(t) + y_s(t) \]

and the zero-input response tends to zero for an asymptotically stable system, we can state that:

The output of a linear asymptotically stable system to a unit step input tends to a constant value given by the system’s gain \( F(0) \)
Step response: steady-state

Even with non-zero initial condition, due to the asymptotic stability of $S$, all the responses tend to the same constant value. The final constant value coincides with the system gain which can also be zero (due to the presence of a zero in $s = 0$).

\[ P_1(s) = \frac{2}{(s+1)(s+2)} \]

\[ P_2(s) = \frac{2s}{(s+1)(s+2)} \]

\[ P_2(0) = 0 \]

Steady-state is independent from the initial conditions.
Step response: transient

**transient** is defined as the forced response minus the steady-state

\[ y_s(t) = R_0 \delta_{-1}(t) + \sum_{i=1}^{n} R_i e^{p_i t} \delta_{-1}(t) \]

\[ y_{ss}(t) = R_0 \]

\[ y_t(t) = \sum_{i=1}^{n} R_i e^{p_i t} \]

distinct poles case

alternative

\[ W(s) = \sum_{i=1}^{n} \frac{R'_i}{s - p_i} \]

\[ w(t) = \sum_{i=1}^{n} R'_i e^{p_i t} \]

\[ y_s(t) = \int_{0}^{t} w(\tau) d\tau = \sum_{i=1}^{n} \frac{R'_i}{p_i} \left[ e^{p_i t} - 1 \right] = \sum_{i=1}^{n} \frac{R'_i}{p_i} e^{p_i t} - \sum_{i=1}^{n} \frac{R'_i}{p_i} W(0) \]
Transient behavior of a system

Defined on a particular time response: step response

\[ y(t) \]

\[ y_{ss} \]

- **rise time**
- **peak time**
- **overshoot**
- **settling time**

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Transient

$t_r$ **rise time**: amount of time required for the signal to go from 10% to 90% of its final value

$y_{ss}$ **steady-state value**: asymptotic output value

$M_p$ **overshoot**: maximum excess of the output w.r.t. the final value (can be defined as a percentage of the final value). In a normalized $y(t)/y_{ss}$ plot the overshoot is given by the maximum of the normalized output minus one.

$t_p$ **peak time**: time required for the step response to reach the overshoot

$t_s$ **settling time**: amount of time required for the step response to stay within 2% of its final value for all future times
Transient

Quantities related to complex plane position of the poles

\[ \zeta = \sin \vartheta \]
Transient

Quantities related to complex plane position of the poles (real poles)

Comparison between two systems

\[ P_i(s) = \frac{-p_i}{s - p_i} \]

Single system

\[ P(s) = \frac{p_1 p_2}{(s - p_1)(s - p_2)} \]

we need to “wait” for the slower one

Lanari: CS - System response
Transient

Quantities related to complex plane position of the poles (complex poles)

\[ P(s) = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2} \]

Comparison between four systems with pair of complex poles and unit gain
Steady state

**Existence**

we require

1. the existence of the steady-state also for the state (not only for the output)
2. and the independence, of the asymptotic behavior, from the initial conditions \(x(0)\)

Being

\[
x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau
\]

\(x(t)\) will be independent for \(t \to \infty\) from the initial condition iff all the modes are converging (all the eigenvalues have negative real part)

i.e. the system is **asymptotically stable**
Steady state

Exists for asymptotically stable systems

Is independent from the initial state

Depends on the particular input applied to the system

- Polynomial inputs $\rightarrow$ Polynomial steady state
- Sinusoidal inputs $\rightarrow$ Sinusoidal steady state

(canonical test signals)
**Polynomial input**

Let the canonical input (signal of order $k$) be $u(t) = \frac{t^k}{k!} \delta_{-1}(t)$ with transform $U(s) = \frac{1}{s^{k+1}}$

For an asymptotically stable system with distinct poles, the output is

$$Y(s) = P(s)U(s) = \frac{N(s)}{\prod_{i=1}^{n}(s - p_i)} \frac{1}{s^{k+1}}$$

and expanding

$$Y(s) = \frac{R_{11}}{s} + \frac{R_{12}}{s^2} + \cdots + \frac{R_{1,k+1}}{s^{k+1}} + \sum_{i=1}^{n} \frac{R_i}{s - p_i}$$

these give in $t$ contributions that tend to 0 as $t$ increases

$$Y_{ss}(s) = \frac{R_{11}}{s} + \frac{R_{12}}{s^2} + \cdots + \frac{R_{1,k+1}}{s^{k+1}}$$

$$u(t) = \frac{t^k}{k!} \delta_{-1}(t)$$

steady-state

$$y_{ss}(t) = \left( R_{11} + R_{12}t + \cdots + R_{1,k+1} \frac{t^k}{k!} \right) \delta_{-1}(t)$$

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Steady state

Zero State response and Input

Response to a ramp input

Response to a ramp input: gain varies

$P(s) = \frac{10}{s + 10}$

$P_1(s) = \frac{2}{(s + 1)(s + 2)}$

$P_2(s) = \frac{2s}{(s + 1)(s + 2)}$

$P_3(s) = \frac{K_p}{(s + 1)(0.5s + 1)}$

$u(t) = t$ for $t > 0$

$K_p = 1$

$K_p = 2$

$K_p = -0.5$
**Sinusoidal input**

Input \( u(t) = \sin \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j} \)

\[ U(s) = \mathcal{L}\{\sin \omega t\} = \frac{\omega}{s^2 + \omega^2} = \frac{\omega}{(s + j\omega)(s - j\omega)} \]

System (asymptotically stable)

\[ P(s) = \frac{N(s)}{\prod (s - p_i)} \]

Output

\[ Y(s) = P(s)U(s) = \frac{N'(s)}{\prod (s - p_i)} + \frac{R_1}{s - j\omega} + \frac{R_2}{s + j\omega} \]

with

\[ R_1 = [(s - j\omega)Y(s)]_{s=j\omega} = \left[ P(s)\frac{\omega}{s + j\omega} \right]_{s=j\omega} = \frac{1}{2j}P(j\omega) \]

\[ R_2 = [(s + j\omega)Y(s)]_{s=-j\omega} = \left[ P(s)\frac{\omega}{s - j\omega} \right]_{s=-j\omega} = -\frac{1}{2j}P(-j\omega) = R_1^* \]
Sinusoidal input

Asymptotic stability

\[
L^{-1}\left\{ \frac{N'(s)}{\prod(s-p_i)} \right\} = L^{-1}\left\{ \sum \sum \frac{R_{ik}}{(s-p_i)^{m_i-k}} \right\} \rightarrow 0 \quad \text{when} \quad t \to \infty
\]

Asymptotic behavior (steady-state) is

\[
y_{ss}(t) = L^{-1}\left\{ \frac{R_1}{s-j\bar{\omega}} + \frac{R_2}{s+j\bar{\omega}} \right\}
\]

but being \( P(j\bar{\omega}) = |P(j\bar{\omega})|e^{j\angle P(j\bar{\omega})} \) with \( |P(-j\bar{\omega})| = |P(j\bar{\omega})| \) and \( \angle P(-j\bar{\omega}) = -\angle P(j\bar{\omega}) \)

we have

\[
y_{ss}(t) = R_1e^{j\bar{\omega}t} + R_2e^{-j\bar{\omega}t}
\]

\[
= \frac{1}{2j} \left( P(j\bar{\omega})e^{j\bar{\omega}t} - P(-j\bar{\omega})e^{-j\bar{\omega}t} \right)
\]

\[
= \frac{1}{2j} \left( |P(j\bar{\omega})|e^{j\angle P(j\bar{\omega})} e^{j\bar{\omega}t} - |P(-j\bar{\omega})|e^{-j\angle P(j\bar{\omega})} e^{-j\bar{\omega}t} \right)
\]

\[
= \frac{|P(j\bar{\omega})|}{2j} \left( e^{j(\bar{\omega}t+\angle P(j\bar{\omega}))} - e^{-j(\bar{\omega}t+\angle P(j\bar{\omega}))} \right)
\]

\[
= |P(j\bar{\omega})| \sin(\bar{\omega}t + \angle P(j\bar{\omega}))
\]
Sinusoidal input

The steady-state response of an asymptotically stable system $P(s)$ to a sinusoidal input $u(t) = \sin \omega t$ is given by

$$y_{ss}(t) = |P(j\omega)| \sin(\omega t + \angle P(j\omega))$$

- steady-state has same frequency than input
- can be
  - amplified $|P(j\omega)| > 1$
  - attenuated $|P(j\omega)| < 1$
- also phase variation
- depends only on the frequency of the input and the system characteristics

$|P(j\omega)|$ gain curve
$\angle P(j\omega)$ phase curve
Sinusoidal input

![Graph showing the frequency response of a system to a sinusoidal input. The graph plots gain against frequency in radians per second on a logarithmic scale.]