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Nonhomogeneous Nilpotent Approximations for Nonholonomic Systems With Singularities

Marilena Vendittelli, Giuseppe Oriolo, Frédéric Jean, and Jean-Paul Laumond

***Abstract*—Nilpotent approximations are a useful tool for analyzing and controlling systems whose tangent linearization does not preserve controllability, such as nonholonomic mechanisms. However, conventional homogeneous approximations exhibit a drawback: in the neighborhood of singular points (where the system growth vector is not constant) the vector fields of the approximate dynamics do not vary continuously with the approximation point. The geometric counterpart of this situation is that the sub-Riemannian distance estimate provided by the classical Ball-Box Theorem is not uniform at singular points. With reference to a specific family of driftless systems, we show how to build a nonhomogeneous nilpotent approximation whose vector fields vary continuously around singular points. It is also proven that the privileged coordinates associated to such an approximation provide a uniform estimate of the distance.**

***Index Terms*—Nilpotent approximations, nonholonomic systems, singularities, sub-Riemannian distance.**

I. Introduction

Studying local properties of nonlinear systems through some approximation of the original dynamics is often the only viable approach to the solution of difficult synthesis problems. Tangent linearization, the most common technique, may not preserve the structural properties of the system; a well-known example of this situation are nonholonomic mechanisms, that cannot be stabilized by smooth feedback. To deal with these cases, it is convenient to resort to a *nilpotent approximation* (NA), a higher order approximation with increased adherence to the original dynamics. Among the classical applications of NAs, we mention the study of sufficient controllability conditions for systems with drift [2], [14] and of stabilizability properties for nonsmoothly stabilizable systems [6], [9].

Various techniques are available for computing NAs (e.g., [1] and [4]). These require first to express the original dynamics in a privileged coordinate system centered at the approximation point, defined on the basis of the control Lie algebra. Then, the transformed vector fields are expanded in Taylor series; by truncating the expansion at a proper order, one obtains a nilpotent system which is polynomial, homogeneous and triangular. As shown in [5], homogeneity may be essential for preserving controllability and stabilizability; for this reason, homogeneous NAs have been used in the above applications.

There is, however, a situation in which homogeneous NAs exhibit a drawback: in the neighborhood of singular points (where the growth vector changes), both the privileged coordinate system and the truncation order change. Hence, in the presence of singularities, homogeneous NAs vary discontinuously with the approximation point. Another consequence is that the sub-Riemannian distance estimate provided by the Ball-Box Theorem [1] is not uniform around singularities: when approaching a singular point through a sequence of regular points, the region of validity for such estimate tends to zero [8].

The above problem is particularly critical when nilpotent approximations are used to design approximate steering laws to be applied iteratively in a feedback fashion. This approach, based on the general framework of [11], was proposed in [15] for achieving stabilization of a particular nonsmoothly stabilizable, nonnilpotentizable system. Within this scheme, continuity is also essential for estimating the steering error due to the approximation in order to prove stability. Another use of NAs that requires continuity is the evaluation of the complexity of nonholonomic motion planning problems [7].

In this note, we show that continuity in the presence of singularities may be achieved by giving up the homogeneity property. For five-dimensional two-input driftless systems with generic singularities, we build *nonhomogeneous* NAs that preserve structural properties and vary continuously with the approximation point over a finite number of subsets that cover the singular locus. By doing so, we associate a continuous approximation procedure to each point. As a byproduct, a uniform estimate of the sub-Riemannian distance is also obtained.

II. Background Material

We recall some basic tools used in sub-Riemannian geometry following [1]. While the general setting is that of differentiable manifolds, the local nature of our study allows the restriction to $\mathbb{R}^n$. Consider a driftless control system

$$\dot{x} = \sum_{i=1}^{m} g_i(x)\, u_i, \qquad x \in \mathbb{R}^n \tag{1}$$

where $g_1, \dots, g_m$ are C^∞ vector fields on $\mathbb{R}^n$ and the input vector $u(t) = (u_1(t), \dots, u_m(t))$ takes values on $\mathbb{R}^m$. Given $x_0 \in \mathbb{R}^n$, let η be a trajectory of (1) originating from x_0 under an input function $u(t)$, $t \in [0, T]$. We define its *length* as

$$\text{length}(\eta) = \int_0^T \sqrt{u_1^2(t) + \cdots + u_m^2(t)}\, dt.$$

A point $x = \eta(t)$, for $t \in [0, T]$, is said to be *accessible* from x_0.

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System (1) induces a *sub-Riemannian distance* d on $\mathbb{R}^n$, defined as

$$d(x_1, x_2) = \inf_{\eta} \text{ length}(\eta) \tag{2}$$

with the infimum taken over all trajectories η joining x_1 to x_2. Chow's Theorem states that any two points in $\mathbb{R}^n$ are accessible from each other (i.e., $d(x_1, x_2) < \infty$) if the elements of the Lie Algebra $\mathcal{L}_g$ generated by the g_i's span the tangent space $T_{x_0}\mathcal{M}$ at each point x_0 (Lie algebra rank condition, or LARC). As (1) is driftless, the LARC implies controllability in any usual sense [14]. Throughout this note, we assume that the LARC is satisfied.

Take $x_0 \in \mathbb{R}^n$ and let $L^s(x_0)$ be the vector space generated by the values at x_0 of the brackets of $g_1, \ldots, g_m$ of length $\leq s$, $s = 1, 2, \ldots$ (the g_i's are brackets of length 1). The LARC guarantees that there exists a smallest integer $r = r(x_0)$ such that $\dim L^r(x_0) = n$, called the *degree of nonholonomy* at x_0.

Let $n_s(x) = \dim L^s(x)$, $s = 1, \ldots, r$. A point x_0 is *regular* if the *growth vector* $(n_1(x), \ldots, n_r(x))$ is constant around x_0; otherwise x_0 is *singular*. In particular, points where r changes are singular. Regular points are an open and dense set in $\mathbb{R}^n$.

A. Nilpotent Approximations and Privileged Coordinates

Consider a smooth real-valued function f. Call *first-order nonholonomic derivatives* of f the Lie derivatives $g_i f$ of f along g_i, $i = 1, \ldots, m$. Call $g_i(g_j f)$, $i, j = 1, \ldots, m$, the *second-order nonholonomic derivatives* of f, and so on.

Definition 1: A function f is *of order* $\geq s$ at x_0 if its nonholonomic derivatives of order $\leq s-1$ vanish at x_0. If f is of order $\geq s$ and not of order $\geq s+1$ at x_0, it is *of order* s at x_0.

Equivalently, if f is of order s at x_0, then $f(x) = O(d^s(x_0, x))$.

Definition 2: A vector field h is *of order* $\geq q$ at x_0 if, for every s and every f of order s at x_0, hf has order $\geq q+s$ at x_0. If h is of order $\geq q$ but not $\geq q+1$, it is *of order* q at x_0.

It is easy to show that g_i, $i = 1, \ldots, m$, has order ≥ -1, bracket $[g_i, g_j]$, $i, j = 1, \ldots, m$, has order ≥ -2, and so on.

Definition 3: A system

$$\dot{x} = \sum_{i=1}^{m} \hat{g}_i(x)\, u_i$$

defined on a neighborhood of x_0, is a *nilpotent approximation* (NA) of system (1) at x_0 if

a) the vector fields $g_i - \hat{g}_i$ are of order ≥ 0 at x_0;
b) its Lie algebra is *nilpotent* of step $s > r(x_0)$.

This definition is equivalent to those in [1] and [5]. Property a) implies the preservation of growth vector and LARC.

Algorithms for computing nilpotent approximations are based on the fact that at each point one can define a set of locally defined privileged coordinates.

Definition 4: Define the integer w_j, $j = 1, \ldots, n$ by setting $w_j = s$ if $n_{s-1} < j \leq n_s$, with $n_s = n_s(x_0)$ and $n_0 = 0$. The local coordinates $z = (z_1, \ldots, z_n)$ centered at x_0 are *privileged* if the order of z_j at x_0 equals w_j, for $j = 1, \ldots, n$. In this case, w_j is called the *weight* of coordinate z_j.

The order of functions and vector fields expressed in privileged coordinates can be computed in an algebraic way.

- The order of monomial $z_1^{\alpha_1} \ldots z_n^{\alpha_n}$ equals its weighted degree $w(\alpha) = w_1\alpha_1 + \ldots + w_n\alpha_n$.
- The order of a function $f(z)$ at $z = 0$ (the image of x_0) is the least weighted degree of the monomials actually appearing in the Taylor expansion of f at 0.
- The order of a vector field $h(z) = \sum_{j=1}^{n} h_j(z)\partial_{z_j}$ at $z = 0$ is the least weighted degree of the monomials actually appearing in the Taylor expansion of h at 0:
$$h(z) \sim \sum_{\alpha, j} a_{\alpha,j} z_1^{\alpha_1} \ldots z_n^{\alpha_n} \partial_{z_j}$$
considering $a_{\alpha,j} z_1^{\alpha_1} \ldots z_n^{\alpha_n} \partial_{z_j}$ as a monomial and assigning to ∂_{z_j} the weight $-w_j$.

For our developments it is convenient to define the notion of approximation procedure.

Definition 5: An *approximation procedure* of system (1) on a given open domain $\mathcal{V} \subset \mathbb{R}^n$ is a function AP which associates to each point $x_0 \in \mathcal{V}$ a smooth mapping $z : \mathbb{R}^n \to \mathbb{R}^n$ and a driftless control system Ψ on $\mathbb{R}^n$, given by the vector fields $\hat{g}_1, \ldots, \hat{g}_m$, such that

- $z = (z_1, \ldots, z_n)$ restricted to a neighborhood Ω of x_0 are privileged coordinates at x_0;
- the pull-backs $z^*\hat{g}_i$ of the vector fields $\hat{g}_i$ by z define on Ω an NA of (1) at x_0.

In other words, Ψ is an NA at 0 of (1) expressed in the z coordinates. One example of such procedure is recalled here.

B. Homogeneous Approximation Procedure

Consider (1) and an approximation point $x_0 \in \mathbb{R}^n$. An algorithm for computing a set of privileged coordinates and a nilpotent approximation at x_0 is the following [1].

1) Compute at x_0 the growth vector $(n_1, \ldots, n_r)$ and the weights $w_1, \ldots, w_n$ as described previously.
2) Choose vector fields $\gamma_1, \ldots, \gamma_n$ such that their values at x_0 form a basis of $L^r(x_0) = T_{x_0}\mathbb{R}^n$ and such that

$$\gamma_{n_{s-1}+1}(x), \ldots, \gamma_{n_s}(x) \in L^s(x), \qquad s = 1, \ldots, r$$

for any x in a neighborhood of x_0, with $n_0 = 0$.
3) From the original coordinates x, compute local coordinates y as

$$y = \Gamma^{-1}(x - x_0)$$

where Γ is the $n \times n$ matrix whose elements Γ_{ij} are defined by $\gamma_j(x_0) = \sum_{i=1}^{n} \Gamma_{ij}\partial_{x_i}|_{x_0}$.
4) Build privileged coordinates $z = (z_1, \ldots, z_n)$ around x_0 via the recursive formula

$$z_j = y_j + \sum_{k=2}^{w_j - 1} h_k(y_1, \ldots, y_{j-1}), \qquad j = 1, \ldots, n \tag{3}$$

where

$$h_k(y_1, \ldots, y_{j-1}) = -\sum_{\substack{|\alpha|=k \\ w(\alpha)<w_j}} m_j\, \gamma_1^{\alpha_1} \ldots \gamma_{j-1}^{\alpha_{j-1}} \left(y_j + \sum_{q=2}^{k-1} h_q \right)(x_0)$$

with $m_j = \prod_{i=1}^{j-1} y_i^{\alpha_i}/\alpha_i!$ and $|\alpha| = \sum_{i=1}^{n} \alpha_i$.
5) Express the dynamics of the original system in privileged coordinates

$$\dot{z} = \sum_{i=1}^{m} g_i(z) u_i.$$

6) Expand the vector fields $g_i(z)$ in Taylor series at 0 and express them in terms of vector fields that are homogeneous w.r.t. the weighted degree

$$g_i(z) = g_i^{(-1)}(z) + g_i^{(0)}(z) + g_i^{(1)}(z) + \cdots.$$

7) Let $\hat{g}_i(z) = g_i^{(-1)}(z)$, and define the approximate system as

$$\dot{z}_j = \sum_{i=1}^{m} \hat{g}_{ij}(z_1, \ldots, z_{j-1}) u_i, \qquad j = 1, \ldots, n \tag{4}$$

where the $\hat{g}_{ij}$'s are homogeneous polynomials of weighted degree $w_j - 1$.

System (4) is an NA (triangular by construction) of the original dynamics (1) in the z coordinates, hereafter referred to as a *homogeneous* NA.

Strictly speaking, the above algorithm does not represent an approximation procedure because Step 2 contains a choice. Assume, however, that there exists an open domain $\mathcal{V}$ containing x_0 where a unique way can be specified for choosing the vector fields γ_j, $j = 1, \ldots, n$. By doing so, one obtains an approximation procedure AP on $\mathcal{V}$. An example of this construction will be given in Section IV.

C. Distance Estimation

Privileged coordinates provide the following estimate of the sub-Riemannian distance d.

Ball-Box Theorem: Consider system (1) and a set of privileged coordinates $z = (z_1, \ldots, z_n)$ at x_0. There exist positive constants c_0, C_0 and ϵ_0 such that, for all x with $d(x_0, x) < \epsilon_0$

$$c_0 f(z) \leq d(x_0, x) \leq C_0 f(z) \tag{5}$$

where $f(z) = |z_1|^{1/w_1} + \cdots + |z_n|^{1/w_n}$.

III. Objective

Assume we wish to control system (1) around a point $\bar{x}$ by means of nilpotent approximations computed in the vicinity of $\bar{x}$ (for example, using the approach in [15]). To this end, we use an approximation procedure AP defined on an open domain $\mathcal{V}$ including $\bar{x}$ to compute a nilpotent approximation Ψ and the associated privileged coordinates z at $x_0 \in \mathcal{V}$.

To guarantee that the structure of the NA does not change in $\mathcal{V}$—which would hinder its use for control synthesis—it is essential that AP is a *continuous*[1] function in $\mathcal{V}$; i.e., both the privileged coordinates and the NA must vary continuously w.r.t. the approximation point $x_0 \in \mathcal{V}$. If $\bar{x}$ is regular, the homogeneous approximation procedure satisfies this requirement; however, if $\bar{x}$ is singular, the growth vector and the associated privileged coordinates weights change around the point, implying that the procedure is discontinuous at $\bar{x}$.

A similar difficulty arises when considering distance estimation. Around a regular point $\bar{x}$, coordinates z and constants c_0, C_0 and ϵ_0 depend continuously on the approximation point x_0. This is not true at a singular point. In particular, if $\{x_i\}$ is a sequence of regular points converging to a singular point x_∞, then ϵ_i tends to 0 although ϵ_∞ is nonzero. Hence, if $\bar{x}$ is singular, the estimate (5) does not hold *uniformly* in $\mathcal{V}$; that is, there is no $\epsilon > 0$ such that the estimate holds for any x_0 and x in $\mathcal{V}$ that satisfy $d(x_0, x) < \epsilon$.

The objective of this note may now be more clearly stated. With reference to a family of five-dimensional driftless systems with singularities, it will be shown that around each point $\bar{x} \in \mathbb{R}^5$ it is possible to define an approximation procedure which is continuous at $\bar{x}$. In particular, we prove that there exists a finite set of continuous approximation procedures with open domains of definition covering $\mathbb{R}^5$. As a consequence, we also obtain a modified version of the Ball-Box Theorem yielding an estimate of the sub-Riemannian distance which is uniform w.r.t. the approximation point x_0. Apart from its intrinsic significance, the latter development will be essential in deriving (along the lines of [1, Prop. 7.29]) a uniform estimate of the steering error arising from the use of NAs.

[1]The function AP takes values in the product of the set of smooth mappings from $\mathbb{R}^n$ to itself with the set of m-tuples of smooth vector fields on $\mathbb{R}^n$, which can be equipped with the product topology induced by the C^0 topology on $C^\infty(\mathbb{R}^n)$. The continuity of AP is relative to this topology.

IV. Family of Systems With Singularities

Consider the family of driftless controllable systems

$$\dot{x} = g_1(x)u_1 + g_2(x)u_2, \qquad x \in \mathbb{R}^5, \quad g_1, g_2 \in C^\infty \tag{6}$$

having growth vector (2,3,5) at regular points and (2,3,4,5) at singular points. A generic (for the C^∞ Whitney topology) pair of vector fields in $\mathbb{R}^5$ satisfies this assumption (except possibly for a set of codimension ≥ 4, where the growth vector may be (2, 3, 4, 4, 5). For example, the so-called *general two-trailer system*, consisting of a unicycle towing two off-hooked trailers, belongs to this family [15].

Under the above assumption, system (6) cannot be transformed in *chained form* at regular points [12]. This also implies that the system is not *flat*; alternatively, one may check that the conditions [13] for flatness are violated. If the particular instance of system (6) under consideration is exactly nilpotentizable, one may use the algorithm of [10] to achieve exact steering between arbitrary points; otherwise, no exact steering methods are available. Therefore, it is in general of interest to define an approximation procedure of (6), either for approximate steering or distance estimation.

Let us apply the homogeneous approximation procedure. First, we recall some algebraic machinery introduced in [14]. Denote by $L(X_1, X_2)$ the free Lie algebra in the *indeterminates* $\{X_1, X_2\}$. The following brackets are the first eight elements of a P. Hall basis of $L(X_1, X_2)$

$$\begin{aligned} X_1,\ X_2,\ X_3 &= [X_1, X_2],\ X_4 = [X_1, [X_1, X_2]] \\ X_5 &= [X_2, [X_1, X_2]],\ X_6 = [X_1, [X_1, [X_1, X_2]]] \\ X_7 &= [X_2, [X_1, [X_1, X_2]]], X_8 = [X_2, [X_2, [X_1, X_2]]]. \end{aligned}$$

Consider (6) and let E_g be the *evaluation map* which assigns to each $P \in L(X_1, X_2)$ the vector field obtained by plugging in g_i for the corresponding indeterminate X_i ($i = 1, 2$). The vector fields $g_3, \ldots, g_8$ are given by $g_j = E_g(X_j)$, $j = 3, \ldots, 8$. Denote by $\mathcal{V}_r$ the open set of regular points, where the growth vector is (2,3,5). In each point of $\mathcal{V}_r$, a basis of the $T_{x_0}\mathbb{R}^5$ is given by the value of $B_r = \{g_1, \ldots, g_4, g_5\}$.

At a singular point, where the growth vector is (2,3,4,5), we need one bracket of length 3 and one of length 4 to span the tangent space. Candidate bases are given by the value of the sets $B_{ij} = \{g_1, g_2, g_3, g_i, g_j\}$, $i = 4, 5$, $j = 6, 7, 8$. Each B_{ij} has rank 5 on an open set $\mathcal{V}_{ij} \subseteq \mathbb{R}^5$. The union of the six $\mathcal{V}_{ij}$ contains the singular locus $\mathcal{V}_s$ and some regular points.

Consider now a point x_0 in $\mathbb{R}^5$. To define a homogeneous approximation procedure on the basis of the algorithm of Section II-B, we must instantiate Step 2 depending on the nature of x_0. If, to perform Step 2, we choose B_r, we obtain a procedure AP_r defined on $\mathcal{V}_r$; if we choose a B_{ij}, we obtain a procedure AP_{ij} defined on the corresponding $\mathcal{V}_{ij}$. In formulas

$$\begin{aligned} AP_r(x_0) &= (z_r, \Psi_r), \quad \text{for } x_0 \in \mathcal{V}_r \\ AP_{ij}(x_0) &= \begin{cases} (z_{ij,r}, \Psi_{ij,r}), & \text{for } x_0 \in \mathcal{V}_{ij} \cap \mathcal{V}_r \\ (z_{ij,s}, \Psi_{ij,s}), & \text{for } x_0 \in \mathcal{V}_{ij} \cap \mathcal{V}_s \end{cases}. \end{aligned}$$

At $x_0 \in \mathcal{V}_{ij} \cap \mathcal{V}_r$, both AP_r and AP_{ij} are defined and continuous. Instead, at $x_0 \in \mathcal{V}_{ij} \cap \mathcal{V}_s$, AP_{ij} is not continuous near x_0, while AP_r is not defined. Therefore, no homogeneous approximation procedure is continuous near a singular point—correspondingly, no such procedure gives a uniform distance estimation on the corresponding $\mathcal{V}_{ij}$.

In the following sections, we show that nonhomogeneous NAs solve the aforementioned difficulties.

V. Nonhomogeneous Approximation Procedure

With reference to (6), we intend to show that, given a domain $\mathcal{V}_{ij}$, it is possible to devise a nonhomogeneous approximation procedure which is continuous at each point—whether regular or singular—of

$\mathcal{V}_{ij}$. For illustration, consider first the domain $\mathcal{V}_{46}$ (equal to the whole state space for the aforementioned general two-trailer system [15]).

The key point is to modify the homogeneous approximation procedure given in Section II-B by *assigning* to the coordinate z_5 its maximum weight, i.e., $w_5 = 4$. The modified procedure, denoted by AP_{46}^{nh}, is detailed here (compare with Section II-B).

1) Set the weights to 1, 1, 2, 3, 4.
2) Choose B_{46} as a set of vector fields.
3)–6) As in **Section II-B**. We get

$$g_i(z) = g_i^{(\leq -1)}(z) + g_i^{(0)}(z) + g_i^{(1)}(z) + \cdots .$$

$g_i^{(\leq -1)}(z)$ is the sum of all terms of weighted degree ≤ -1.
7) Let $\bar{g}_i(z) = g_i^{(\leq -1)}(z)$, and define the approximate system Ψ_{46}^{nh} as

$$\dot{z}_j = \sum_{i=1}^{2} \bar{g}_{ij}(z_1, \ldots, z_{j-1}) u_i, \qquad j = 1, \ldots, 5. \tag{7}$$

The inclusion of terms of weighted degree ≤ -1 in $\bar{g}_i(z)$ is due to the new assignment of weights. In particular, having now set $w_5 = 4$, ∂_{z_5} is of weighted degree -4. As a consequence, the weighted degree of a monomial $a_\alpha z_1^{\alpha_1} \ldots z_n^{\alpha_n} \partial_{z_j}$, computed with the new weights, is not equal to its order. Thus, at regular points the first monomials actually appearing in the Taylor expansion of the fifth component of $g_i(z)$ are of weighted degree $< w_5 - 1$. These monomials are automatically zero at singular points, for z_5 becomes there of order 4.

Theorem 1: The approximation procedure AP_{46}^{nh} depends continuously on x_0 in $\mathcal{V}_{46}$.

Proof: First, observe that the system of coordinates provided by AP_{46}^{nh} is privileged. In fact, setting $w_5 = 4$ affects only the expression [computed by (3)] of z_5 at regular points of $\mathcal{V}_{46}$, where additional, higher-degree terms appear w.r.t. the expression provided by AP_{46}. This does not affect the order of z_5, which will still be 3 at regular points. At singular points, the coordinates provided by AP_{46}^{nh} and by AP_{46} coincide. Hence, z are privileged in $\mathcal{V}_{46}$ since they have order (1, 1, 2, 3, 3) at regular points and (1, 1, 2, 3, 4) at singular points.

We now show that Ψ_{46}^{nh} is a nilpotent approximation of (6) in $\mathcal{V}_{46}$, expressed in the z coordinates. At singular points, Ψ_{46}^{nh} coincides with the homogeneous NA $\Psi_{46,s}$ obtained by applying AP_{46}. At regular points of $\mathcal{V}_{46}$, the order of privileged coordinates is (1, 1, 2, 3, 3) and, therefore, the *homogeneous* approximation of Ψ_{46}^{nh} at $z = 0$, obtained by applying AP_{46} to (7), coincides with $\Psi_{46,r}$. Hence, the homogeneous NA of Ψ_{46}^{nh} at $z = 0$ is also the homogeneous NA of (6) at x_0, expressed in z. This proves condition a) of Definition 3 . To prove b), consider that $\bar{g}_1, \bar{g}_2$ of system (7) are, by construction, of weighted degree ≤ -1. Thus, their brackets of length ≥ 5 are of weighted degree ≤ -5. However, no monomial can be of weighted degree < -4, so that all brackets of length > 4 must be zero, i.e., (7) is nilpotent of step 5.

Finally, coordinates z and Ψ_{46}^{nh} are continuous in $\mathcal{V}_{46}$ by construction, and so is AP_{46}^{nh}. ■

Ψ_{46}^{nh} has the same polynomial, triangular structure of the homogeneous NA (4). The distinctive feature of Ψ_{46}^{nh} is its nonhomogeneity: function $\bar{g}_{i5}(z_1, \ldots, z_4)$, $i = 1, 2$, is the sum of two polynomials of homogeneous degree 2 and 3, respectively. At singular points the coefficients of the monomials of homogeneous degree 2 vanish, so that only a polynomial of homogeneous degree 3 is left. We call AP_{46}^{nh} a *nonhomogeneous* approximation procedure and Ψ_{46}^{nh} a *nonhomogeneous* NA.

Since Ψ_{46}^{nh} satisfies Definition 3, which implies the LARC, we conclude that Ψ_{46}^{nh} preserves the controllability of the original system. In view of the absence of drift, this also guarantees stabilizability of the approximate system via continuous time-varying feedback, by the result of [3].

In a generic domain $\mathcal{V}_{ij}$, a nonhomogeneous approximation procedure AP_{ij}^{nh} is obtained by choosing B_{ij} in Step 2. The associated NA is denoted by Ψ_{ij}^{nh}.

The state space $I\!R^5$ of system (6) is given by the union of $\mathcal{V}_r$ and the six $\mathcal{V}_{ij}$'s defined in Section IV. If one of the $\mathcal{V}_{ij}$'s covers the whole state space (i.e., if one of the B_{ij}'s gives a basis at every point), then AP_{ij}^{nh} provides at each point a system of privileged coordinates and a nilpotent approximation which depend continuously on the approximation point. In general, however, a globally valid basis may not exist; if so, there exists no approximation procedure (homogeneous or nonhomogeneous) that is defined and continuous everywhere. Still, around each point there exists at least one continuous approximation procedure: either AP_r or one of the AP_{ij}^{nh}.

For practical purposes one may also wish to associate a single approximation procedure to each point of the state space. To this end, one may partition the state space into seven subsets with nonempty interior:

$$\mathcal{D}_r = \{x \in I\!R^5 : |\det \Gamma_r| \geq |\det \Gamma_{hl}|,\ h = 4, 5,\ l = 6, 7, 8\}$$
$$\mathcal{D}_{ij} = \left\{ x \in I\!R^5, x \notin \mathcal{D}_r : \begin{array}{l} |\det \Gamma_{ij}| > |\det \Gamma_{hl}|, \{hl\} < \{ij\} \\ |\det \Gamma_{ij}| \geq |\det \Gamma_{hl}|, \{hl\} \geq \{ij\} \end{array} \right\}$$

for $i, h = 4, 5,\ j, l = 6, 7, 8$. Here, Γ_r and Γ_{hl} are the 5×5 matrices whose columns are, respectively, the vectors of coordinates of the vector fields $\{g_1, \ldots, g_5\}$ and $\{g_1, g_2, g_3, g_h, g_l\}$ at x, and couples of indices have been ordered lexicographically. Each $\mathcal{D}_{ij}$ (respectively, $\mathcal{D}_r$) is included in $\mathcal{V}_{ij}$ (respectively, $\mathcal{V}_r$); therefore, by taking AP_{ij}^{nh} on $\mathcal{D}_{ij}$ and AP_r on $\mathcal{D}_r$ we define on $I\!R^5$ a unique approximation procedure whose restriction to each of the seven subsets is continuous.

VI. Uniform Estimation of Sub-Riemannian Distance

We now address the problem of obtaining a uniform estimate of the sub-Riemannian distance as a function of privileged coordinates. To this end, we first sketch the procedure for estimating uniformly the sub-Riemannian distance through the lifting method, and then show that an estimate based on privileged coordinates can be obtained by computing the relationship between the latter and the lifted privileged coordinates [such as (11)].

A. Lifting of the Control System

We first desingularize the system using the *lifting method*, based on the following result.

Lemma [8]: Consider (1) and $x_0 \in I\!R^n$. There exist an integer $\tilde{n} \geq n$; a neighborhood $\tilde{U} \subset I\!R^{\tilde{n}}$ of $(x_0, 0)$; coordinates (x, ξ) on $\tilde{U}$, where $\xi = (\xi_1, \ldots, \xi_{\tilde{n}-n})$; and smooth vector fields $\tilde{g}_i$ on $\tilde{U}$ in the form[2]

$$\tilde{g}_i(x, \xi) = g_i(x) + \sum_{j=1}^{\tilde{n}-n} b_{ij}(x, \xi) \partial_{\xi_j}$$

with the b_{ij}'s smooth functions on $I\!R^{\tilde{n}}$, such that the system defined by the *lifted* vector fields $\tilde{g}_1, \ldots, \tilde{g}_m$ satisfies the LARC and has no singular point in $\tilde{U}$.

[2]With a little abuse of notation, we denote by g_i also the vector fields obtained by extending the input vector fields of system (1) with $\tilde{n} - n$ coordinates equal to zero.

Let $(x_1, 0)$ be a point in $\tilde{U}$ and $u(t), t \in [0, T]$, be an input function. They define a trajectory in $\mathbb{R}^{\tilde{n}}$ steering the lifted system from $(x_1, 0)$ to (x_2, ξ), the solution at $t = T$ of the differential equation

$$(\dot{x}(t), \dot{\xi}(t)) = \sum_{i=1}^{m} \tilde{g}_i(x(t), \xi(t))\, u_i(t)$$

with initial condition $(x(0), \xi(0)) = (x_1, 0)$. Using the definition of the lifted vector fields, we write these equations as

$$\dot{x}(t) = \sum_{i=1}^{m} g_i(x(t))\, u_i(t)$$
$$\dot{\xi}_j(t) = \sum_{i=1}^{m} b_{ij}(x(t), \xi(t))\, u_i(t), \qquad j = 1, \dots, \tilde{n} - n$$

with $x(0) = x_1$, $\xi_j(0) = 0$. The first equation represents the original system in $\mathbb{R}^n$. Therefore, the canonical projection of the trajectory in $\mathbb{R}^{\tilde{n}}$ associated to $u(t)$ and steering the lifted system from $(x_1, 0)$ to (x_2, ξ) is the trajectory in $\mathbb{R}^n$ associated to the same $u(t)$ and steering the original system from x_1 to x_2. In particular, the two trajectories have the same length.

The sub-Riemannian distance between x_1 and x_2 in a neighborhood of x_0 is

$$d(x_1, x_2) = \inf_{\xi \in \mathbb{R}^{\tilde{n}-n}} \tilde{d}((x_1, 0), (x_2, \xi)) \tag{8}$$

where $\tilde{d}$ is the sub-Riemannian distance for the lifted system.

B. Distance Estimation

As in Section V, consider for illustration the case $x_0 \in \mathcal{V}_{46}$. Our objective is to build a regular system in some space $\mathbb{R}^{\tilde{n}}$ such that its canonical projection on $\mathbb{R}^5$ near x_0 coincides with the original system. Let $b_1(x, \xi)$ and $b_2(x, \xi)$ be C^∞ functions on $\mathbb{R}^5 \times \mathbb{R}$ and set

$$\tilde{g}_i(x, \xi) = g_i(x) + b_i(x, \xi)\partial_\xi, \qquad i = 1, 2. \tag{9}$$

For a generic (for the C^3 topology) choice of the b_i's, the lifted system defined by $\tilde{g}_1, \tilde{g}_2$ on $\mathbb{R}^{\tilde{n}} = \mathbb{R}^6$ will have growth vector (2,3,5,6) at $(x_0, 0)$. Hence, this system satisfies the LARC and has no singular point in a neighborhood $\tilde{U}_{x_0}$ of $(x_0, 0)$.

Consider the first eight elements of a P. Hall basis as given in Section IV and the evaluation map $E_{\tilde{g}}$ assigning to each element of the Lie Algebra in the indeterminates $\{X_1, X_2\}$ the vector field obtained by plugging in the $\tilde{g}_i$, $i = 1, 2$, for the corresponding X_i. Denoting by $\tilde{g}_3, \dots, \tilde{g}_8$ the vector fields given by $\tilde{g}_j = E_{\tilde{g}}(X_j)$, $j = 3, \dots, 8$, we can also write

$$\tilde{g}_i(x, \xi) = g_i(x) + b_i(x, \xi)\partial_\xi, \qquad i = 3, \dots, 6.$$

Reducing (if needed) $\tilde{U}_{x_0}$ so that $\tilde{U}_{x_0} \subset \mathcal{V}_{46} \times \mathbb{R}$, and using the genericity of b_1 and b_2, we can assume that $\{\tilde{g}_1(x, \xi), \dots, \tilde{g}_6(x, \xi)\}$ has rank 6 at any point $(x, \xi) \in \tilde{U}_{x_0}$.

Let $(x_1, 0) \in \tilde{U}_{x_0}$. We want to compute privileged coordinates in $\mathbb{R}^6$ around $(x_1, 0)$ for the lifted control system, and compare them with $z_1, \dots, z_5, \xi$, where the z_i's are the coordinates constructed in Section V. To this end, we follow Steps 1–4 of the procedure given in Section V.

1) Set the weights to 1, 1, 2, 3, 4, 4.

2) For the choice of the vector fields, note first that, being $x_1 \in \mathcal{V}_{46}$, we have

$$g_5(x_1) = \rho_1 g_1(x_1) + \dots + \rho_4 g_4(x_1) + \rho g_6(x_1) \tag{10}$$

where $\rho = 0$ if x_1 is singular. Set $\tilde{g}_5' = \tilde{g}_5 - \rho_1\tilde{g}_1 - \dots - \rho_4\tilde{g}_4$ and choose vector fields $\tilde{g}_1, \dots, \tilde{g}_5', \tilde{g}_6$. At $(x_1, 0)$, we have

$$\tilde{g}_i(x_1, 0) = g_i(x_1, 0) + b_i^0\partial_\xi, \qquad i = 1, \dots, 4, 6$$
$$\tilde{g}_5'(x_1, 0) = \rho g_6(x_1, 0) + b_5^0\partial_\xi$$

where $b_i^0 = b_i(x_1, 0)$, $i = 1, \dots, 6$. Note that $\beta = b_5^0 - \rho b_6^0$ is nonzero.

3) Compute local coordinates $\tilde{y} = (\tilde{y}_1, \dots, \tilde{y}_6)$ as

$$\tilde{y} = \tilde{\Gamma}^{-1}\begin{pmatrix} x - x_1 \\ \xi \end{pmatrix}$$

where $\tilde{\Gamma}$ is the matrix whose columns are the values of the vector fields $\tilde{g}_1, \dots, \tilde{g}_6$ at $(x_1, 0)$. Being $x - x_1 = \Gamma_{46}\, y$, where $y = (y_1, \dots, y_5)$ and Γ_{46} is the 5×5 matrix whose columns are the values of $g_1, \dots, g_4, g_6$ at x_1, we obtain

$$\tilde{y}_i = y_j, \qquad j = 1, \dots, 4$$
$$\tilde{y}_5 = \frac{1}{\beta}(\xi - b_6^0 y_5 - b_4^0 y_4 - \dots - b_1^0 y_1)$$
$$\tilde{y}_6 = y_5 - \frac{\rho}{\beta}(\xi - b_6^0 y_5 - b_4^0 y_4 - \dots - b_1^0 y_1) = y_5 - \rho\tilde{y}_5.$$

4) Define privileged coordinates $(\tilde{z}_1, \dots, \tilde{z}_6)$ around $(x_1, 0)$ using **(3)**. Since $\tilde{g}_i y_j = g_i y_j$ for $i \le 4$, we have $\tilde{z}_j = z_j$ for $j \le 4$. The last two coordinates have the form

$$\tilde{z}_5 = \frac{\xi}{\beta} + \psi_5(z_1, \dots, z_5)$$
$$\tilde{z}_6 = z_5 - \rho\tilde{z}_5. \tag{11}$$

The privileged coordinates provided by AP_{46}^{nh} led to an expression of $\tilde{z}_6$ depending only on z_5 and $\tilde{z}_5$. The same coordinates are now used to derive an estimate of d.

Theorem 2: Let $S \subset \mathcal{V}_{46}$ be a compact set. There exist c, C and $\epsilon > 0$ such that, for all $x_1 \in S$ and all x with $d(x_1, x) < \epsilon$

$$cf'(z) \le d(x_1, x) \le Cf'(z) \tag{12}$$

where

$$f'(z) = |z_1| + |z_2| + |z_3|^{1/2} + |z_4|^{1/3} + \min\left(\left|\frac{z_5}{\rho}\right|^{1/3}, |z_5|^{1/4}\right) \tag{13}$$

with $\rho = \det(\Gamma_r)/\det(\Gamma_{46})$ and Γ_r the 5×5 matrix whose columns are the values of $g_1, \dots, g_5$ at x_1.

Proof: Consider $x_0 \in \mathcal{V}_{46}$. We first prove the result for a compact neighborhood N_{x_0} of x_0 such that $N_{x_0} \times \{0\} \subset \tilde{U}_{x_0}$. At any point $\tilde{x}_1 = (x_1, 0) \in N_{x_0} \times \{0\}$, the Ball-Box Theorem guarantees the existence of $\tilde{c}_1, \tilde{C}_1$ and $\tilde{\epsilon}_1 > 0$ such that an inequality like (5) holds for $\tilde{d}$ if $\tilde{d}(\tilde{x}_1, \tilde{x}) < \tilde{\epsilon}_1$. Moreover, $(x_1, 0)$ is a regular point and, by construction, $\tilde{z}_1, \dots, \tilde{z}_6$ around $(x_1, 0)$ vary continuously with x_1. Then, $\tilde{c}_1, \tilde{C}_1$ and $\tilde{\epsilon}_1$ are continuous functions of x_1 and have finite, nonzero extrema on the compact set N_{x_0}. Hence, there exist $\tilde{c}$, $\tilde{C}$ and $\tilde{\epsilon} > 0$ such that, for any $x_1 \in N_{x_0}$ and any $\tilde{x} = (x, \xi)$ such that $\tilde{d}(\tilde{x}_1, \tilde{x}) < \tilde{\epsilon}$, it is

$$\tilde{c}f(\tilde{z}) \le \tilde{d}(\tilde{x}_1, \tilde{x}) \le \tilde{C}f(\tilde{z}) \tag{14}$$

where $f(\tilde{z}) = |\tilde{z}_1| + |\tilde{z}_2| + |\tilde{z}_3|^{1/2} + |\tilde{z}_4|^{1/3} + |\tilde{z}_5|^{1/3} + |\tilde{z}_6|^{1/4}$. According to (8), it is

$$d(x_1, x) = \inf_{\xi \in \mathbb{R}} \tilde{d}((x_1, 0), (x, \xi)).$$

Being $\partial\tilde{z}_5/\partial\xi = 1/\beta$ nonzero and using (11), we may write

$$\inf_{\xi \in \mathbb{R}} f(\tilde{z}) = \inf_{\tilde{z}_5 \in \mathbb{R}} \left(|z_1| + \dots + |z_4|^{1/3} + |\tilde{z}_5|^{1/3} + |z_5 - \rho\tilde{z}_5|^{1/4}\right).$$

The infimum is attained at $\tilde{z}_5 = z_5/\rho$ if $|z_5| \leq \rho^4$ and at $\tilde{z}_5 = 0$ if $|z_5| \geq \rho^4$. This, together with the estimate (14) of $\tilde{d}$, gives the estimate of $d(x_1, x)$, with $c = \tilde{c}$ and $C = \tilde{C}$. The expression of ρ is easily derived from (10).

Having proven the result on a compact neighborhood N_{x_0} of each $x_0 \in \mathcal{V}_{46}$, let now S be a compact subset of $\mathcal{V}_{46}$. The union of the interiors V_{x_0} of N_{x_0}, $x_0 \in S$, is a covering of S by open sets, from which we can extract a finite covering $\cup V_i$; equation (12) holds on each V_i with constants c_i, C_i and ϵ_i. Setting $\epsilon = \min_i \epsilon_i$, $c = \min_i c_i$, and $C = \max_i C_i$, the thesis follows. ■

Note the following points.

- The estimate does not depend on the choice of the lifting.
- When x_1 is a singular point, the continuous function ρ equals zero and Theorem 2 is simply the Ball-Box Theorem at a singular point. On the other hand, when x_1 is regular and far enough from the singular locus, it may be certainly assumed that $\rho > \epsilon$ (reducing ϵ if needed). In this case, condition $d(x_1, x) < \epsilon$ implies $|z_5| \leq \rho^4$, and Theorem 2 turns out to be the Ball-Box Theorem at a regular point.
- A uniform estimate of the form (12)–(13) holds for compact subsets of the generic $\mathcal{V}_{ij}$, with the privileged coordinates defined by AP_{ij}^{nh} and $\rho_{ij} = \det \Gamma_r / \det \Gamma_{ij}$ in place of ρ. The same is true on compact subsets of $\mathcal{V}_r$, with the privileged coordinates defined by AP_r and $\rho = \rho_r = 1$ in place of ρ; in this case, the estimate (12)–(13) coincides with that of the classical Ball-Box theorem.

If $\mathcal{V}_{46}$ covers the whole state space, Theorem 2 directly provides a uniform estimation of d on $\mathbb{R}^5$. Even in the general case, however, it is possible to obtain the same result; in fact, given any compact subset $K \subset \mathbb{R}^5$, we can write $K = \left(\bigcup_{i,j} K_{ij}\right) \cup K_r$, having set $K_{ij} = K \cap \mathcal{D}_{ij}$ and $K_r = K \cap \mathcal{D}_r$. Estimate (12)–(13) holds on K_r as well as each K_{ij}; a uniform distance estimation over K is then obtained by computing the appropriate extremal values of c, C and ϵ over the subset.

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Control With Disturbance Preview and Online Optimization

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***Abstract*—We present a intuitive and self-contained formulation of a stability preserving receding horizon control strategy for a system where limited preview information is available for the disturbances. The simplicity of the derivation is due to (and its benefits somewhat offset by) a set of stringent and highly structured assumptions. The formulation uses a suboptimal value function for terminal cost, and relies on optimization strategies that only require a trivial improvement property, allowing implementation as an "anytime" algorithm. The nature of this strategy's performance is clarified with linear examples.**

***Index Terms*—Anytime, disturbance preview, model predictive control, receding horizon control.**

I. INTRODUCTION

Performance advances in microprocessors have spurred the interest in receding horizon, also termed model predictive, control strategies. An excellent review of the growth of the field is given in [1]. Of particular interest to this note are [2], [3], especially [4], [5], and the suboptimality results of [6].

We extend the methods of receding horizon control to the case where a discrete nonlinear dynamic system is driven by disturbances, and where consistent finite length previews of these disturbances are available. We consider the problem as a dynamic game between control and disturbance. From this perspective, it is generally the case that advanced knowledge of the disturbance is both desirable *and* expensive. Hence, in some cases a limited preview will be available through additional sensors, intelligence, or short term predictive models (e.g., the

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