AN APPLICATION OF NONLINEAR MODEL MATCHING TO THE DYNAMIC CONTROL OF ROBOT ARM WITH ELASTIC JOINTS

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Abstract. The paper is concerned with dynamic control of robot arm with non negligible joint elasticity. In this case nonlinear control techniques based on decoupling and nonlinearity compensation via *static* state-feedback cannot be applied for the most common robot structures. This is because the mathematical model associated with these structures is such that the necessary and sufficient conditions for the existence of the decoupling control laws fail to hold. Motivated by these facts, the problem of using a *dynamic* state-feedback is considered. The main purpose of the paper is to show that one can design a dynamic state-feedback compensator, which makes the external behavior of the controlled robot identical to the one of a prescribed decoupled linear model. The particular example considered is a planar robot arm with two elastic joints.

Keywords. Nonlinear control systems; decoupling; model matching; robots; system graphs.

INTRODUCTION

The problem of decoupling the dynamics of robot arms via static state-feedback has received large attention in the last years and is an appealing approach for achieving a complete control of the external behavior of an anthropomorphic manipulator.

For robots with rigid joints this control technique has been labeled in a number of different ways in the past (see e.g. Brady and others, 1982) and it has been shown that the resulting control is robust (Nicosia, Nicolò and Lentini, 1981); moreover the nonlinear decoupling has the nice property that it completely linearizes the state dynamics (Tarn and others, 1984).

For robots with elastic transmission between actuators and arms, as belts or harmonic drives, the effect of joint elasticities is such that the decoupling technique cannot be applied for the most common robot mechanical structures. This is because the mathematical model associated with these structures is such that the necessary and sufficient conditions for the existence of a decoupling law fail to hold. There are special but rather unusual mechanical structures for which such conditions hold (De Simone and Nicolò, 1985); however, in these cases, the computation of the relevant control laws is quite cumbersome. In alternative, an approximate decoupling may be obtained using a controller based on singular perturbation theory as done by Ficola, Marino and Nicosia (1983).

In this paper we show that exact decoupling can be achieved for robot with elastic joints via the nonlinear model matching theory developed by Di Benedetto and Isidori (1984). In this case the resulting control law is a dynamic state-feedback which makes the external behavior of the controlled robot identical to the one of a prescribed decoupled linear model. Moreover, we show that the system graph representation of Kasinski and Levine (1983) gives a fruitful insight into the model structure, thus allowing reduction of the dynamic order of the controller and complete linearization of the state dynamics.

MATHEMATICAL MODEL OF ELASTIC ROBOTS AND STATIC DECOUPLING TECHNIQUE

The mechanical structure of a robot is constituted by N+1 bodies interconnected through N joints. The body between two joints is called a link. The joints are activated by motors with transmission gears or belts; when the links and the transmissions are assumed to be rigid the dynamical behavior is that of a chain of N rigid bodies. In this case the equation of motion in matrix form is

$$\ddot{q} = B(q)^{-1}[m(t)-e(q)-c(q,\dot{q})] \tag{1}$$

where q is the N-vector of joint variables giving the relative displacement between two adjacent links, B(q) is the N × N inertial matrix, m(t) is the N-vector of generalized forces delivered by the motors, e(q) is the N-vector of conservative forces and $c(q,\dot{q})$ is the N-vector collecting centrifugal and Coriolis forces.

When the transmissions are not rigid the N actuating bodies of the motors are elastically coupled to the driven links; therefore the dynamical behavior is that of 2N rigid bodies with only N actuated bodies and the others transmitting elasticity; this is the case of interest here. The equation of motion in matrix form is still eq. (1), but with the following peculiarities:

- the number of second order equations is 2N;
- q is a 2N-vector in which q_{2i} denotes the displacement of link i w.r.t. link i-1 and q_{2i-1} denotes the displacement of the driving body of joint i w.r.t. link i-1, for i = 1,...,N;
- B(q) is the 2N × 2N inertial matrix of the 2N rigid bodies;
- e(q) and $c(q,\dot{q})$ are 2N-vectors and e(q) includes the effects of elasticity;
- m(t) is a 2N-vector with the even components equal to zero.

Starting from mechanical parameters, the model (1) is given automatically by the DYMIR code both for rigid and elastic robots (Cesareo, Nicolò and Nicosia, 1984). Eq. (1) may be rewritten in the standard form

$$\begin{aligned}\dot{x} &= f(x) + g(x)u \\ y &= h(x)\end{aligned} \tag{2}$$

with state $x = \begin{bmatrix} x_p^T & x_v^T \end{bmatrix}^T = \begin{bmatrix} q^T & \dot{q}^T \end{bmatrix}^T \in X = R^n$, input $u \in R^m$ and output $y \in R^{\ell}$. In the elastic case n = 4N, m = ℓ = N. The vector f and the m columns $g_1,\ldots,g_m$

of the matrix g are smooth vector fields defined on an open subset of R^n; the expressions for f and g are given by:

$$f(x) = \begin{bmatrix} x_v \\ -B(x_p)^{-1}[c(x_p,x_v)+e(x_p)] \end{bmatrix},$$

$$g(x) = \begin{bmatrix} 0 \\ B(x_p)^{-1}\mathrm{diag}\{\begin{bmatrix}1\\0\end{bmatrix}\} \end{bmatrix}$$

Moreover, in (2) h is a smooth vector-valued function; in our case the output y may be defined as the vector of link displacements $x_{2i} = q_{2i}$ $(i = 1,\dots,N)$. The input u collects only the non-zero components of m(t).

An attractive approach to the control of nonlinear systems is the one based on decoupling state-feedback laws. The *static* state-feedback decoupling problem is defined as follows; given a system in the form (2), find a feedback $\alpha(x)$ and a state dependent change of coordinates $\beta(x)$ in the input space such that the closed-loop system formed by the composition of (2) with

$$u = \alpha(x) + \beta(x)v \tag{3}$$

has the i-th output dependent only on the i-th component of the new input.

The problem is now well understood (Isidori and co-workers, 1981; Sinha, 1977); the basic concepts of decoupling theory are summarized in what follows. Given a vector field f on X and a smooth function h, recall that the Lie derivative of h in the direction of the field f is the function

$$L_f h = (\frac{\partial h}{\partial x_1} \dots \frac{\partial h}{\partial x_n})f.$$

The k-th order Lie derivative is $L_f^k h = L_f(L_f^{k-1}h)$ while $L_f^o h = h$.

Definition 1. The characteristic number ρ_j associated with the output y_j is the largest integer such that for all $k < \rho_j$

$$L_{g_i} L_f^k h_j = 0, \quad \forall i \in \{1,\dots,m\}.$$

In the nondegenerate case ($\rho_j \neq \infty$, $\forall j$) we have:

Definition 2. The decoupling matrix A(x) associated with the system (2), with $m = \ell$, is the $\ell \times \ell$ matrix with smooth entries

$$a_{ji}(x) = L_{g_i} L_f^{\rho_j} h_j.$$

The main result of static state-feedback decoupling theory given by Isidori and co-workers (1981) is:

Theorem 1. A necessary and sufficient condition for the existence of (α,β) which solves the decoupling problem is that the decoupling matrix A(x) is nonsingular.

When Theorem 1 applies, a possible decoupling control is given by (3) with:

$$\begin{aligned}\alpha(x) &= -A(x)^{-1}\mathrm{col}\{L_f^{(\rho_1+1)}h_1,\dots,L_f^{(\rho_\ell+1)}h_\ell\} \\ \beta(x) &= A(x)^{-1}\end{aligned} \tag{4}$$

It has been shown (Freund, 1982; Nicosia, Nicolò and Lentini, 1981; Tarn and others, 1984) that for the rigid robot the corresponding matrix A(x) is nonsingular so that (4) provides a solution to the decoupling problem. On the contrary, in the elastic case considered here the matrix A(x) is in general singular (De Simone and Nicolò, 1985), except for particular mechanical structures. We will see later on that the condition of Theorem 1 fails to hold for a configuration present in most robots. As a consequence, static state-feedback is no longer sufficient in order to solve the decoupling problem when joint elasticity is taken into account.

MODEL MATCHING VIA DYNAMIC STATE FEEDBACK

In this section we summarize some recent results concerned with the use of dynamic feedback compensation in order to match a prescribed input-output behavior and we find, as a byproduct, a sufficient condition for decoupling via dynamic feedback.

A *dynamic* state-feedback is a control mode in which the inputs $u_1,\dots,u_m$ are related to the state x of the process (2) and to other input variables $v_1,\dots,v_\mu$ by means of equations of the form

$$\begin{aligned}\dot\xi &= a(x,\xi) + b(x,\xi)v \\ u &= c(x,\xi) + d(x,\xi)v.\end{aligned} \tag{5}$$

These equations characterize a dynamical system - the state-feedback compensator - whose state ξ evolves on an open subset of R^ν. The $\nu \times 1$ vector $a(x,\xi)$, the $\nu \times \mu$ matrix $b(x,\xi)$, the $m \times 1$ vector $c(x,\xi)$ and the $m \times \mu$ matrix $d(x,\xi)$ have entries which are smooth functions defined on an open subset of $R^n \times R^\nu$.

The composition of (2) with (5) defines a new dynamical system with inputs $v_1,\dots,v_\mu$, outputs $y_1,\dots,y_\ell$ and state $\hat x = (x,\xi)$ given by equations of the form

$$\begin{aligned}\dot{\hat x} &= \hat f(\hat x) + \hat g(\hat x)v \\ y &= \hat h(\hat x)\end{aligned} \tag{6}$$

with

$$\hat f(x,\xi) = \begin{bmatrix} f(x)+g(x)c(x,\xi) \\ a(x,\xi)\end{bmatrix}, \hat g(x,\xi) = \begin{bmatrix} g(x)d(x,\xi) \\ b(x,\xi)\end{bmatrix},$$

$$\hat h(x,\xi) = h(x)$$

We shall see that dynamic compensation enables us to match, if some conditions are verified, a prescribed behavior between inputs and outputs in the composed (closed-loop) system (6). In particular, we will be able to obtain a response of the form

$$y(t) = Q(t,(x^o,\xi^o)) + \int_0^t W_M(t-\tau)v(\tau)d\tau \tag{7}$$

where $W_M(t)$ is the impulse-response of a fixed linear model

$$\begin{aligned}\dot z &= Az + Bv \\ y &= Cz\end{aligned} \tag{8}$$

On the first term $Q(t,(x^o,\xi^o))$ of the right-hand-side of (7), which corresponds to the zero-input response (and clearly depends - possibly in a nonlinear manner - on the initial state), we do not impose at this point any particular constraint.

If the process (2) and the compensator (5) were linear, then a well known necessary and sufficient condition for the existence of solutions to this problem would be the one based on the comparison of the behavior of the transfer functions of the process and of the model as $s \to \infty$ (Malabre, 1982). We recall that the behavior of a strictly proper transfer function W(s) for $s \to \infty$ is fully described by the so-called Smith-McMillan factorization at the infinity (Dion and Commault, 1982).

$$W(s) = Q(s)\Lambda(s)P(s)$$

in which Q(s) and P(s) are biproper rational matrices (i.e. proper rational matrices with an inverse which is also proper) and

$$\Lambda(s) = \mathrm{diag}\{I_{\delta_1}\frac{1}{s},\ I_{\delta_2}\frac{1}{s^2}, \ldots, I_{\delta_q}\frac{1}{s^q},\ 0\}$$

The sequence $\{\delta_1, \delta_2, \ldots\}$ is said to characterize the structure at infinity of the given linear system. Sometimes, one considers a sequence $\{r_0, r_1, \ldots\}$ related to the former in this way

$$r_0 = \delta_1\ ,\quad r_i = \delta_{i+1} + r_{i-1}\ ,\quad i \geq 1 \tag{9}$$

and whose computation is rather easy on a realization (A,B,C) of the transfer function W(s).

If the process to be compensated is nonlinear, one may still establish a sufficient condition for the existence of solutions to a model matching problem by means of a suitable extension of the notion of structure at infinity, see Nijmeijer and Schumacher (1985). This extension, which is based upon the consideration of the so-called maximal controlled invariant subspace Algorithm of Isidori and coworkers (1981) and is described in full in Appendix 1, associates with any nonlinear system of the form (2), i.e. with any triplet (f,g,h), a string of integers $\{r_0, r_1, \ldots\}$ which shares much of the properties of the sequence defined by means of (9).

A (sufficient) condition for the solvability of the problem of matching the external behavior of a prescribed linear model can be found in the following way. With the given process (2), i.e. with the triplet (f,g,h), we associate its structure at the infinity $\{r_0, r_1, \ldots\}$. Moreover, from the triplet (f,g,h) and from the triplet (A,B,C) which characterizes the model to be matched, we define an *enlarged* system as follows

$$\dot{x} = f(x) + g(x)u$$
$$\dot{z} = Az + Bv$$
$$e = h(x) - Cz.$$

These equations may be written in more condensed form as

$$\dot{x}^E = f^E(x^E) + g^E(x^E)u^E \qquad e = h^E(x^E) \tag{10}$$

letting $x^E = (x,z)$, $u^E = (u,v)$ and

$$f^E(x,z) = \begin{bmatrix} f(x) \\ Az \end{bmatrix},\quad g^E(x,z) = \begin{bmatrix} g(x) & 0 \\ 0 & B \end{bmatrix},$$

$$h^E(x,z) = h(x) - Cz.$$

With the triplet (f^E, g^E, h^E) we associate the structure at infinity, denoted $\{r_0^E, r_1^E, \ldots\}$.

The coincidence between the structure at infinity of the triplet (f,g,h) and that of the triplet (f^E, g^E, h^E) is exactly the condition we were looking for. As a matter of fact, the following result has been shown to hold.

Theorem 2. Let (f,g,h) and (A,B,C) be given. If

$$r_k^E = r_k \tag{11}$$

for all $k \geq 0$, then there exists a dynamic feedback compensator under which the input-output behavior of (2) becomes of the form (7) with $W_M(t) = C\exp(At)B$.

The proof of this Theorem, which is constructive, may be found in Di Benedetto and Isidori (1984). It may be worth observing that the compensator (5) thus determined incorporates the dynamics of the model to be followed (i.e. $\dot{\xi} = A\xi + Bv$). In particular, its dimension ν is equal to that of the linear model.

The conditions provided by this Theorem may be used, in particular, in order to check the possibility of matching the external behavior of a linear model with transfer function

$$W_M(s) = \left[\ \mathrm{diag}\{I_\ell \frac{1}{s^\delta}\}\ \vdots\ 0\ \right] \tag{12}$$

where δ is a suitable integer.

The external behavior of a linear system with transfer function (12) is clearly *decoupled*. Thus, the dynamic state-feedback compensator which enables us to match this model is such as to impose a decoupled behavior between inputs and outputs. In the closed-loop system, each component of the output is influenced only by the corresponding component of the input, independently from the specific form of the zero-input response $Q(t,(x^o,\xi^o))$.

APPLICATION OF THE MODEL MATCHING THEORY TO THE ROBOT ARM

In this section we describe the application of the previous theoretical results to the problem of controlling a robot arm with elastic joints. In particular we will consider as an example the two-link planar robot arm whose complete mathematical model is described in Appendix 2.

By means of the Algorithm of Appendix 1 we will first compute the sequence $\{r_0, r_1, \ldots\}$ and then see that it is possible to match a linear model with transfer function of the form (12), with $\delta = k^* + 1$. This is done by imposing the same structure at infinity of the original system on the system (10) which includes the model to be matched.

We begin with the computation of the sequence $\{r_0, r_1, \ldots\}$ for the triplet (f,g,h) which describes the robot arm system. In the initial step we make use of the output functions $h_1 = x_2$, $h_2 = x_4$; we have

$$\Omega_o = \mathrm{sp}\{dh_1, dh_2\} = \mathrm{sp}\begin{Bmatrix} 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0 \end{Bmatrix}$$

so that $s_o = \dim \Omega_o = 2$, and $\lambda_o = [h_1, h_2]^T$.

To start the 0-th iteration we need to find r_o i.e. we just have to compute the rank of the matrix

$$A_o = d\lambda_o \cdot g = \begin{bmatrix} dh_1 \\ dh_2 \end{bmatrix}\begin{bmatrix} g_1 & g_2 \end{bmatrix} = 0 \rightarrow r_o = 0.$$

Since $r_o = 0$, the equations (A1) in Appendix 1 are trivial. To construct Ω_1 we have to take functions from the set $\Lambda_o = \{L_f h_j, L_{g_i} h_j;\ i = 1,2;\ j = 1,2\}$ such that their differentials are linear independent with respect to each other and to the set of differentials which span Ω_o. Since we have $L_f h_1 = x_6$, $L_f h_2 = x_8$, $L_g h = 0$, we obtain directly

$$\Omega_1 = \Omega_o \oplus \mathrm{sp}\{dL_f h_1, dL_f h_2\} = \Omega_o \oplus \mathrm{sp}\begin{Bmatrix} 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1 \end{Bmatrix}$$

and hence $s_1 = \dim \Omega_1 = 4$. Thus, we set $\lambda_1 = [h_1, h_2, L_f h_1, L_f h_2]^T$.

In the 1-st iteration, r_1 is given by the rank of the matrix

$$A_1 = d\lambda_1 \cdot g = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & g_{62} & g_{82} \end{bmatrix}^T \rightarrow r_1 = 1.$$

At this point consider the 4 by 4 permutation matrix P_1 with $P_{11} = [0\ 0\ 1\ 0]$ which is such that $P_{11}A_1$ selects r_1 linearly independent rows from A_1. In this case $r_1 = 1$ and the third row of A_1 is chosen. Define further $B_1 = d\lambda_1 \cdot f = [x_6 x_8 f_6 f_8]^T$ and solve for an m-vector α and an $m \times m$ invertible matrix β (m = 2) the equations

$$P_{11}A_1\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = -P_{11}B_1,\quad P_{11}A_1\begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix} = K = [0\quad 1]$$

where for K we can choose any matrix of real numbers of rank equal to r_1. We obtain:

$\alpha_1 = 0,\ \alpha_2 = -f_6/g_{62},\ \beta_{11} = 1,\ \beta_{22} = 1/g_{62},\ \beta_{12} = \beta_{21} = 0$ from which we can construct a static feedback law

$$u = \alpha(x) + \beta(x)w \qquad (13)$$

that gives the new vector fields $\tilde{f} = f + g\alpha$, $\tilde{g}_i = (g\beta)_i$ (see Appendix 2). Next, select a maximal number of functions from the set

$$\Lambda_1 = \{L_{\tilde{f}}(L_f h_j),\ L_{\tilde{g}_i}(L_f h_j);\ i = 1,2;\ j = 1,2\}$$

with linearly independent differentials. Noting that $L_f h = L_{\tilde{f}} h$, we have

$$L_{\tilde{f}}^2 h_1 = 0,\ L_{\tilde{f}}^2 h_2 = \tilde{f}_8,\ L_{\tilde{g}_1} L_{\tilde{f}} h_1 = L_{\tilde{g}_1} L_{\tilde{f}} h_2 = 0,$$

$$L_{\tilde{g}_2} L_{\tilde{f}} h_1 = 1,\ L_{\tilde{g}_2} L_{\tilde{f}} h_2 = \tilde{g}_{82}$$

so that the only candidate is $L_{\tilde{f}}^2 h_2$ since $\tilde{g}_{82}$ depends on x_4 only. We obtain in fact

$$\Omega_2 = \Omega_1 \oplus \mathrm{sp}\{dL_{\tilde{f}}^2 h_2\} = \Omega_1 \oplus \mathrm{sp}\{0 * \frac{\partial \tilde{f}_8}{\partial x_3} * 0 * 0\ 0\}$$

where $(\partial \tilde{f}_8/\partial x_3) = K_2/N_2 A_2 \neq 0$ assures the linear independency of the new row w.r.t. the previous ones (* denotes non relevant terms). So $s_2 = \dim \Omega_2 = 5$ and $\lambda_2 = [h_1, h_2, L_{\tilde{f}} h_1, L_{\tilde{f}} h_2, L_{\tilde{f}}^2 h_2]^T$.

For the 2-nd iteration

$$A_2 = d\lambda_2 \cdot g = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & g_{62} & g_{82} & * \end{bmatrix}^T \rightarrow r_2 = 1.$$

Again we need to find (α,β) as a solution of a matrix equation similar to the one considered in the previous step. However, as long as $r_k = 1$ we obtain the same (α,β) i.e. the same $\tilde{f}, \tilde{g}$ and so we will bypass this part of the computations; thus, the new searching set is:

$$\Lambda_2 = \{L_{\tilde{f}}(L_{\tilde{f}}^2 h_2), L_{\tilde{g}_1}(L_{\tilde{f}}^2 h_2), L_{\tilde{g}_2}(L_{\tilde{f}}^2 h_2)\}.$$

We have

$$L_{\tilde{g}_1} L_{\tilde{f}}^2 h_2 = 0, L_{\tilde{g}_2} L_{\tilde{f}}^2 h_2 = \partial \tilde{f}_8/\partial x_6 = -2\,(A_3/A_2) x_6 \sin x_4,$$

$$L_{\tilde{f}}^3 h_2 = (\partial \tilde{f}_8/\partial x_2) x_6 + (\partial \tilde{f}_8/\partial x_3) x_7 + (\partial \tilde{f}_8/\partial x_4) x_8$$

and then

$$\Omega_3 = \Omega_2 \oplus \mathrm{sp}\{dL_{\tilde{f}}^3 h_2\} = \Omega_2 \oplus \mathrm{sp}\{0 * 0 * 0 * \frac{\partial \tilde{f}_8}{\partial x_3} *\}$$

so that $s_3 = \dim \Omega_3 = 6$, $\lambda_3 = [h_1, h_2, L_{\tilde{f}} h_1, L_{\tilde{f}} h_2, \ldots L_{\tilde{f}}^3 h_2]^T$.

The 3-rd step of the algorithm starts computing

$$A_3 = d\lambda_3 \cdot g = \begin{bmatrix} 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0 * * * * \end{bmatrix}^T \rightarrow r_3 = 1.$$

Furthermore

$$\Lambda_3 = \{L_{\tilde{f}}(L_{\tilde{f}}^3 h_2), L_{\tilde{g}_1}(L_{\tilde{f}}^3 h_2), L_{\tilde{g}_2}(L_{\tilde{f}}^3 h_2)\}$$

where $L_{\tilde{g}_1} L_{\tilde{f}}^3 h_2 = 0$, $L_{\tilde{g}_2} L_{\tilde{f}}^3 h_2$ is a function of x_2, x_4, x_6, x_8 only, while

$$L_{\tilde{f}}^4 h_2 = (\partial \tilde{f}_8/\partial x_3) \tilde{f}_7 + \phi(x_2, x_3, x_4, x_6, x_7, x_8).$$

Notice that $(\partial \tilde{f}_7/\partial x_1) = K_1/N_1 JRZ_2 \neq 0$; thus, we obtain

$$\Omega_4 = \Omega_3 \oplus \mathrm{sp}\{dL_{\tilde{f}}^4 h_2\} = \Omega_3 \oplus \mathrm{sp}\{(\frac{\partial \tilde{f}_8}{\partial x_3} \frac{\partial \tilde{f}_7}{\partial x_1}) * * * 0 * * * \}$$

and $s_4 = \dim \Omega_4 = 7$, $\lambda_4 = [h_1 h_2 L_{\tilde{f}} h_1 L_{\tilde{f}} h_2 \ldots L_{\tilde{f}}^4 h_2]^T$.

In the 4-th step,

$$A_4 = d\lambda_4 \cdot g = \begin{bmatrix} 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0 * * * * * \end{bmatrix}^T \rightarrow r_4 = 1.$$

In Λ_4, constructed similarly to the previous steps, the only function which depends on x_5 is $L_{\tilde{f}}^5 h_2$; this has the form

$$L_{\tilde{f}}^5 h_2 = (\partial \tilde{f}_8/\partial x_3) \cdot (\partial \tilde{f}_7/\partial x_1) x_5 + \psi(x_1, x_2, x_3, x_4, x_6, x_7, x_8)$$

and we have that

$$\Omega_5 = \Omega_4 \oplus \mathrm{sp}\{dL_{\tilde{f}}^5 h_2\} = \Omega_4 \oplus \mathrm{sp}\{* * * * (\frac{\partial \tilde{f}_8}{\partial x_3} \frac{\partial \tilde{f}_7}{\partial x_1}) * * *\}.$$

Thus, $s_5 = \dim \Omega_5 = 8 = \dim x$ and hence the Algorithm stops at $k^* = 5$. Last we have to compute the rank of the matrix

$$A_5 = d\lambda_5 \cdot g = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & (g_{51} \frac{\partial \tilde{f}_8}{\partial x_3} \frac{\partial \tilde{f}_7}{\partial x_1}) \\ 0 & 0 & g_{62} g_{82} & * & * & * & & * \end{bmatrix}^T$$

which gives $r_{k^*} = 2$.

Notice that, for the particular example considered, we have found that the set of functions whose differentials span Ω_{k^*}, is obtained by taking the Lie derivatives of the system outputs only with respect to the vector field $\tilde{f}$; these functions may be used to define a new set of local coordinates in terms of which the system is described by much simpler expressions.

To conclude this section, we consider the problem of finding what kind of linear and decoupled model it is possible to match. Let the linear model be expressed in the parametrized form (8), with the condition that $c_j A^k b_i = 0$, $\forall k$ if $i \neq j$, i.e. having a decoupled structure of input-output channels. The model matching problem is solvable if the enlarged system (robot arm plus model, see (10)) has the same structure at the infinity as the original system (the robot arm alone).

The procedure is then the following: apply the Algorithm to (f^E, g^E, h^E) and compute the matrices A_k^E of the enlarged system; then, imposing the equality between r_k^E (rank of A_k^E) and r_k for $k = 1,\ldots,k^*$, derive conditions on the triplet (A,B,C). The computation of Lie derivatives is greatly simplified being the model to be matched a linear one and will be omitted here. After $k^* = 5$ steps of the Algorithm, we find that the constraint

$$CA^k B = 0 \quad \text{for} \quad k = 0,1,\ldots,4$$

is such as to make the condition (11) satisfied for all $k \geq 0$.

This means that the input-output behavior of the considered robot arm can be made equal via dynamic feedback to that of a decoupled linear system constituted by two chains of six integrators each. In this case (12) becomes $W_M(s) = \mathrm{diag}\{1/s^6, 1/s^6\}$. Then, a dynamic controller can be constructed just following the arguments of the proof of Theorem 2. The obtained controller will be of order $\ell \cdot (k^*+1) = 12$, with a dynamics inherited from the one of the matched model. However, the explicit derivation of this controller will be avoided here.

ANALYSIS OF THE MODEL STRUCTURE VIA SYSTEM GRAPH

In order to explore the decoupling problem and the possibility of reducing the dynamic order of the controller we recall the notion of graph representation of a system, introduced by Siljak (1977).

The system graph of nonlinear system (2) is a weighted oriented graph G(N,L), where N is the set of nodes representing the input, state and output variables and L is the set of weighted arcs representing the influences among variables.

More precisely, the weight of an arc (u_i, x_k) is given by g_{ki} where g_{ki} is the k-th element of g_i; the weight of an arc (x_k, x_h) is given by $\partial f_h/\partial x_k$ where f_h is the h-th element of f; the weight of an arc (x_h, y_j) is given by $\partial h_j/\partial x_h$, where h_j is the j-th element of h. L is constituted by the nonzero-weighted arcs only. The system graph of the two-link planar robot with elastic joints is shown in Fig. 1 (arcs in heavy= minimal graph defined below).

Let $d(u_i,y_j)$ be the minimal number of arcs of G forming an oriented path from u_i to y_j and let $d_j = \min d(u_i,y_j)$; define length of a path the number of its arcs. The minimal graph G_M is the subgraph of G constituted by all input-output paths of length d_j ending in y_j, for each $j = 1,\ldots,\ell$. G_M gives a complete information on the dynamic structure with respect to the decoupling property. Call weight of a path the product of the weights of the arcs forming the path. The entries a_{ji} of the decoupling matrix are given by the sum of the weights of the paths joining u_i to y_j in G_M (Kasinski and Levine, 1983). In Fig. 1 the minimal graph reveals that the decoupling matrix has only one nonzero column, being thus singular.

Consider now, in terms of system graph, the effects of adding chains of integrators to the inputs of the system. Connect one integrator to input u_i i.e. $u_i = \xi$, $\dot{\xi} = v_i$. Modifications occur in the graph only at arcs ending in state nodes directly connected with u_i. More precisely, in the new system graph the weight of an arc(x_k,x_h), with x_h connected to u_i, becomes $\partial f_h/\partial x_k + \xi\partial g_{hi}/\partial x_k$. New arcs are created if $\partial f_h/\partial x_k = 0$ but $\partial g_{hi}/\partial x_k \neq 0$. As a matter of fact, the addition of an integrator modifies consistently the structure of the graph. Further additions of integrators to the input u_i leave the graph unchanged.

Figure 1 shows that the singularity of the decoupling matrix for the two-link arm is due to the fact that both minimal paths start from input u_2. We may try to extend the system graph with properly connected integrators so as to build a nonsingular decoupling matrix $\hat{A}(\hat{x})$; therefore, to bring into play input u_1 we increase the length of the paths starting from the second input. Adding two cascaded integrators to input u_2, the decoupling matrix becomes full but is still singular due to the weights on the minimal paths. Moreover, the cascaded addition of further integrators does not change this situation because it does not change the weights of the minimal paths.

At this point one possibility in order to modify the graph structure is to apply first a feedback transformation, which use the results of the previous section, and then to work on the obtained graph with the above dynamic extension.

DECOUPLING DYNAMIC CONTROLLER

We saw that during each step of the Algorithm a particular static feedback is computed. Whenever the system is statically decoupable, the feedback obtained at the last step is a decoupling one. In our case (see (13)) we found:

$$u = \alpha(x)+\beta(x)w = \begin{bmatrix} 0 \\ -f_6/g_{62} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1/g_{62} \end{bmatrix} w. \quad (14)$$

The application of this feedback law to the robot obviously does not achieve a decoupled structure. However, in the system

$$\begin{aligned} \dot{x} &= [f(x)+g(x)\alpha(x)]+[g(x)\beta(x)]w = \tilde{f}(x)+\tilde{g}(x)w \\ y &= h(x) \end{aligned} \quad (15)$$

so obtained (see Appendix 2) the first output is decoupled from input w_1, as shown by the system graph (solid arcs only) in Fig. 2. As a matter of fact, the north-west entry of the decoupling matrix will always be zero; therefore, it is straightforward to see that four integrators added to the second input give a nonsingular decoupling matrix.

In fact the following dynamic extension

$$\begin{aligned} w_1 &= \bar{v}_1 \\ w_2 &= \xi_1,\ \dot{\xi}_1 = \xi_2,\ \dot{\xi}_2 = \xi_3,\ \dot{\xi}_3 = \xi_4,\ \dot{\xi}_4 = \bar{v}_2 \end{aligned} \quad (16)$$

applied to system (15) leads to the new system

$$\begin{aligned} \dot{\hat{x}} &= \bar{f}(\hat{x}) + \bar{g}(\hat{x})\bar{v} \\ y &= \bar{h}(\hat{x}) = h(x) \end{aligned} \quad (17)$$

where the extended state $\hat{x}$ is defined as

$$\hat{x} = [x_1 x_2 \ldots x_8 \xi_1 \ldots \xi_4]^T$$

and the vector fields $\bar{f}, \bar{g}_i$ are given respectively by:

$$\bar{f}(\hat{x}) = [x_5 x_6 x_7 x_8 f_5 \xi_1 (\tilde{f}_7+\tilde{g}_{72}\xi_1)(\tilde{f}_8+\tilde{g}_{82}\xi_1)\xi_2\xi_3\xi_4\ 0]^T$$

$$\bar{g}(\hat{x}) = \begin{bmatrix} \bar{g}_1(\hat{x})^T \\ \bar{g}_2(\hat{x})^T \end{bmatrix}^T = \begin{bmatrix} 0\ 0\ 0\ 0\ g_{51}\ 0 \ldots 0\ 0 \\ 0\ 0 \quad \ldots \quad 0\ 1 \end{bmatrix}^T$$

In Fig. 2 the dashed arcs represent the added integrators and the crossbarred arcs are those whose weights are modified by the dynamic extension. The decoupling matrix $\bar{A}(\hat{x})$ for system (17) is:

$$\bar{A}(\hat{x}) = L_{\bar{g}} L_{\bar{f}}^5 h = \begin{bmatrix} 0 & 1 \\ g_{51}\dfrac{\partial\tilde{f}_7}{\partial x_1}\dfrac{\partial\tilde{f}_8}{\partial x_3} & \tilde{g}_{82} \end{bmatrix}$$

We can compute now the decoupling feedback for the extended system:

$$\bar{v} = \bar{\alpha}(\hat{x}) + \bar{\beta}(\hat{x})v. \quad (18)$$

Applying (4) and using the explicit expressions of the terms involved (see Appendix 2) we obtain:

$$\bar{\beta}(\hat{x}) = \bar{A}(\hat{x})^{-1} = \begin{bmatrix} -D(A_3\cos x_4 + A_2) & DA_2 \\ 1 & 0 \end{bmatrix}$$

where $D = N_1 N_2 JRZ_1 JRZ_2/K_1 K_2$. Since $L_{\bar{f}}^6 h_1 = 0$, we have further

$$\bar{\alpha}(\hat{x}) = -\bar{A}(\hat{x})^{-1}\cdot\begin{bmatrix} L_{\bar{f}}^6 h_1 \\ L_{\bar{f}}^6 h_2 \end{bmatrix} = \begin{bmatrix} -DA_2 L_{\bar{f}}^6 h_2 \\ 0 \end{bmatrix}$$

where $L_{\bar{f}}^6 h_2$ has a rather long and complex expression, composed by trigonometric polynomials. Use of symbolic and algebraic manipulation systems such as MACSYMA or REDUCE is indicated for the easy derivation and simplification of this term.

The obtained controller is a dynamical system of the form (5) with inputs v_1, v_2, outputs u_1, u_2 and 4-dimensional state $\xi = (\xi_1\xi_2\xi_3\xi_4)$, described by equation of the form

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ \dot{\xi}_3 &= \xi_4 \\ \dot{\xi}_4 &= v_1 \\ u_1 &= \bar{\alpha}_1(x,\xi) + \bar{\beta}_{11}(x)v_1 + \bar{\beta}_{12}v_2 \\ u_2 &= -\frac{f_6(x)}{g_{62}(x)} + \frac{1}{g_{62}(x)}\xi_1 \end{aligned} \quad (19)$$

with $\bar{\alpha}_1 = -DA_2(L_{\bar{f}}^6 h_2)$, $\bar{\beta}_{11} = -D(A_3\cos x_4 + A_2)$, $\bar{\beta}_{12} = DA_2$.

The order of the controller has been reduced from twelve to four by the joint application of static pre-feedback (14), dynamic extension (16) and static decoupling feedback (18) from the extended state. The proposed dynamic controller exhibits a further nice property. Static feedback decoupling creates a closed-loop system which has an unobservable part with a possibly nonlinear dynamics of dimension $\hat{n}^* = \hat{n} - \sum_{j=1}^{\ell} (\hat{\rho}_j + 1)$, see Isidori and co-workers (1981); the remaining part of the system is equivalent to a linear controllable and observable subsystem. In our case we have $\hat{\rho}_1 = \hat{\rho}_2 = 5$, i.e. two chains of six integrators, $\hat{n} = \dim \hat{x} = 12$ and hence $\hat{n}^* = 0$, so that the decoupling law is also a linearizing one for the extended system. Thus, the composition of the control law (19) with the robot arm equations (2) yield a dynamical system whose external behavior is decoupled and which is diffeomorphic to a linear and controllable system. As a matter of fact, in the coordinates

$$z_1=h_1, z_2=L_{\tilde{f}}h_1, z_3=L_{\tilde{f}}^2h_1, z_4=L_{\tilde{f}}^3h_1, z_5=L_{\tilde{f}}^4h_1, z_6=L_{\tilde{f}}^5h_1,$$

$$z_7=h_2, z_8=L_{\tilde{f}}h_2, z_9=L_{\tilde{f}}^2h_2, z_{10}=L_{\tilde{f}}^3h_2, z_{11}=L_{\tilde{f}}^4h_2, z_{12}=L_{\tilde{f}}^5h_2$$

the closed-loop system is described by the equations

$$\dot{z} = \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix} z + \begin{bmatrix} B_1 & \\ & B_2 \end{bmatrix} v \ , \ y = \begin{bmatrix} C_1 & \\ & C_2 \end{bmatrix} z$$

with the triple (A_i, B_i, C_i), $i = 1,2$, in Brunovski canonical form.

We conclude that in the examined elastic robot we obtain, as a byproduct of the decoupling, the full state linearization. This allows to assign all the dynamic behavior by standard techniques.

CONCLUSIONS

In this paper we have shown how nonlinear model matching theory can be applied for the dynamic decoupling control of industrial robots with joint elasticity. The existence of a decoupling controller is guaranteed for the planar two-link robot with elastic joints. The model matching approach leads to a twelve-order dynamic controller; the resulting closed-loop system matches the input-output behavior of a prescribed decoupled linear system but includes an unobservable part with a possibly nonlinear dynamics. However, the analysis of the robot model structure allows both to reduce the order of the controller down to four and to fully linearize the closed-loop state dynamics.

We note also that similar results may be obtained by means of an algorithm for dynamic state-feedback decoupling recently proposed by Descusse and Moog (1985).

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APPENDIX 1

In this Appendix we describe the Algorithm for the computation of the structure at the infinity of a nonlinear system (Krener, 1985). We recall that if $\lambda: R^n \to R$ is a smooth function, its differential $d\lambda$ is the $1 \times n$ row vector with j-th component given by $\partial\lambda/\partial x_j$.

From the components $h_1, \ldots, h_\ell$ of the map h one construct first of all the (x-dependent) subspace (of row vectors)

$$\Omega_0(x) = \text{span}\,\{dh_1(x), \ldots, dh_\ell(x)\}$$

Suppose $\Omega_0(x)$ has dimension $s_0 \leq \ell$ in a neighborhood of a point x^o. Then there exists an $s_0 \times 1$ column vector γ_0, whose entries $\gamma_{01}, \ldots, \gamma_{0s_0}$ are entries of h, with the properties that the differentials $d\gamma_{01}, \ldots, d\gamma_{0s_0}$ are linearly independent at all x in a neighborhood of x^o. The algorithm consists of a finite number of iterations, each one defined as follows.

Iteration (k). Consider the $s_k \times m$ matrix $A_k(x)$ whose (i,j)-entry is $d\gamma_{ki}(x)g_j(x)$. Suppose that in a neighborhood of x^o the rank of $A_k(x)$ is constant and equal to r_k. Then it is possible to find r_k rows of $A_k(x)$ which, for all x in a neighborhood of x^o, are linearly independent. Let $P_k^T = [P_{k1}^T \mid P_{k2}^T]$

be an $s_k \times s_k$ permutation matrix, such that the r_k rows of $P_{k1}A_k(x)$ are linearly independent. Let $B_k(x)$ be an s_k-vector whose i-th element is $d\lambda_{ki}(x)f(x)$. As a consequence of the assumptions on P_{k1}, the equations

$$\begin{aligned} P_{k1}A_k(x)\alpha(x) &= -P_{k1}B_k(x) \\ P_{k1}A_k(x)\beta(x) &= K \end{aligned} \qquad \text{(A1)}$$

(where K is a matrix of real numbers, of rank r_k) may be solved for α and β, an m-vector and an $m \times m$ invertible matrix whose entries are real-valued smooth functions defined in a neighborhood of x^o. Set $\tilde{f} = \tilde{g}_o = f+g\alpha$ and $\tilde{g}_i = (g\beta)_i$, $1 \leq i \leq m$.

Consider the set of functions

$$\Lambda_k = \{\lambda = L_{\tilde{g}_i}\lambda_{kj} : 1 \leq j \leq s_k,\ 0 \leq i \leq m\}$$

and the two (x-dependent) subspaces (of row vectors)

$$\Omega_k(x) = \text{span}\{d\lambda_{k1}(x),\dots,d\lambda_{ks_k}(x)\}$$

$$\Omega'_k(x) = \text{span}\{d\lambda(x) : \lambda \in \Lambda_k\}$$

Set $\Omega_{k+1}(x) = \Omega_k(x)+\Omega'_k(x)$.

Suppose $\Omega_{k+1}(x)$ has constant dimension $s_{k+1}(\geq s_k)$ in a neighborhood of x^o. Let $\lambda_{k+1,1},\dots,\lambda_{k+1,s_{k+1}}$ be entries of λ_k and/or elements of Λ_k such that the differentials $d\lambda_{k+1,1},\dots,d\lambda_{k+1,s_{k+1}}$ are linearly independent at all x in a neighborhood of x^o. Define the s_{k+1}-vector λ_{k+1} whose i-th entry is the function $\lambda_{k+1,i}$. This concludes the k-th iteration.

At each stage of the algorithm two integers are considered

$$s_k = \dim \Omega_k(x), \quad r_k = \text{rank } A_k(x).$$

Since $s_k \leq s_{k+1} \leq n$, a dimensionality argument shows that there exists an integer k^* such that $s_k = s_{k^*}$, $r_k = r_{k^*}$ for all $k \geq k^*$. The sequence $\{r_0, r_1, \dots\}$ provides the so-called structure at the infinity associated with the triplet (f,g,h).

APPENDIX 2

We report here the dynamic model of a two-link robot arm with joint elasticity, whose possible configurations lie in a vertical plane:

$$\dot{x} = f(x) + \sum_{i=1}^{2} g_i(x)u_i\ , \quad y = h(x) = [x_2\ x_4]^T$$

with

$$f(x) = [x_5\ x_6\ x_7\ x_8\ f_5\ f_6\ f_7\ f_8]^T$$

$$g(x) = \begin{bmatrix} g_1(x)^T \\ g_2(x)^T \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & g_{51} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & g_{62} & g_{72} & g_{82} \end{bmatrix}^T$$

where

$$g_{51} = G_1$$

$$g_{62} = A_2/\omega_1$$

$$g_{72} = (A_3^2\cos^2 x_4 - A_1A_2 + A_2^2)/JRZ_2\omega_1$$

$$g_{82} = -\omega_2/\omega_1$$

$$f_5 = (N_1x_2 - x_1)G_1K_1/N_1^2$$

$$\begin{aligned} f_6 = \{&[K_1(N_1x_2-x_1)+N_1A_5\cos x_2 + N_1\omega_3]N_2^2A_2 \\ &+[K_2A_2(N_2x_4-x_3)-(K_2(N_2x_4-x_3)+N_2\omega_3)N_2\omega_2]N_1 \\ &-[x_6^2\omega_2+A_2x_8(2x_6+x_8)]N_1N_2^2A_3\sin x_4\}/N_1N_2^2\omega_1 \end{aligned}$$

$$\begin{aligned} f_7 = \{&[x_6^2\omega_2+A_2x_8(2x_6+x_8)]N_1N_2^2A_3JRZ_2\sin x_4 \\ &-[K_1(N_1x_2-x_1)+N_1A_5\cos x_2+N_1\omega_3]N_2^2A_2JRZ_2 \\ &-[K_2(A_3^2\cos^2 x_4+A_2^2-A_1A_2)(x_3-N_2x_4) \\ &+(K_2(x_3-N_2x_4)-N_2\omega_3)N_2\omega_2JRZ_2]N_1\}/N_1N_2^2\omega_1JRZ_2 \end{aligned}$$

$$\begin{aligned} f_8 = \{&[x_6^2(2\omega_2 + A_1-JRZ_2-2A_2)+x_8(2x_6+x_8)\omega_2]A_3N_1N_2^2\sin x_4 \\ &-[K_1(N_1x_2-x_1)+N_1A_5\cos x_2 + N_1\omega_3]N_2^2\omega_2 \\ &-[(K_2(x_3-N_2x_4)-N_2\omega_3)N_2(2\omega_2+A_1-JRZ_2-2A_2) \\ &-K_2(x_3-N_2x_4)\omega_2]N_1\}/N_1N_2^2\omega_1. \end{aligned}$$

We defined for compactness the following terms:

$$\omega_1 = A_3^2\cos^2 x_4 + A_2(JRZ_2 + A_2 - A_1)$$

$$\omega_2 = A_3\cos x_4 + A_2$$

$$\omega_3 = A_4\cos(x_2 + x_4).$$

At joint i, N_i is the reduction ratio of the gear box, K_i is the elastic constant, JRZ_i is the inertia of the rotor. The constants $A_1, A_2, \dots, A_5$ and G_1 include all the data of the robot (mass and inertia of links and rotors, length of links and their mass center).

Finally we collect here the expressions of the vector fields $\tilde{f}, \tilde{g}_i$ computed with the Algorithm:

$$\tilde{f}(x) = f(x)+g(x)\alpha(x) = [x_5\ x_6\ x_7\ x_8\ f_5\ 0\ \tilde{f}_7\ \tilde{f}_8]^T$$

$$\tilde{g}(x) = g(x)\beta(x) = \begin{bmatrix} 0 & 0 & 0 & 0 & g_{51} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \tilde{g}_{72} & \tilde{g}_{82} \end{bmatrix}^T$$

where

$$\tilde{g}_{72} = g_{72}/g_{62} = [(A_3^2/A_2)\cos^2 x_4 + A_2 - A_1]/JRZ_2$$

$$\tilde{g}_{82} = g_{82}/g_{62} = -(1+(A_3/A_2)\cos x_4)$$

$$\begin{aligned} \tilde{f}_7 = f_7 - f_6\tilde{g}_{72} = {} &(N_2A_2K_1x_1 - N_1K_2\omega_2x_3)/A_2N_1N_2JRZ_2 \\ &+\{A_3\sin x_4[x_6^2\omega_2+A_2x_8(x_8+2x_6)] \\ &+\omega_2[\omega_3+K_2x_4] \\ &-A_2[\omega_3+A_5\cos x_2+K_1x_2]\}/A_2JRZ_2 \end{aligned}$$

$$\tilde{f}_8 = f_8 - f_6\tilde{g}_{82} = -[A_3N_2x_6^2\sin x_4 + N_2\omega_3 + K_2(N_2x_4-x_3)]/N_2A_2.$$

We note explicitly that $\tilde{f}_6 = 0$ and that $\tilde{f}_8$ is a function of x_2, x_3, x_4 and x_6 only; this is reflected in the system graph of Fig. 2.

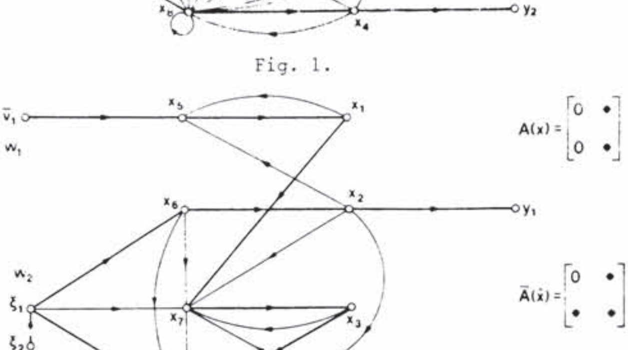

Fig. 1.

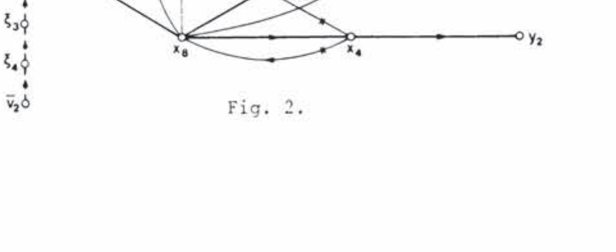

Fig. 2.