

## Wheeled Robots



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### Abstract

The use of mobile robots in applications is steadily increasing, both in the industrial and the service domains. Most mobile robots achieve locomotion using wheels. As a consequence, they are subject to differential constraints that are nonholonomic, i.e., non-integrable. This article reviews the kinematic models of wheeled robots arising from these constraints and discusses their fundamental properties and limitations from a control viewpoint. An overview of the main approaches for trajectory planning and feedback motion control is provided.

### Keywords

Wheeled robots · Nonholonomic constraints · Differential flatness · Nonlinear controllability · Smooth stabilizability

## Introduction

Although all robots are by definition capable of movement, the expression *mobile robots* is mainly used to indicate robots that can displace

their own base by means of some locomotion mechanism. Most often, this consists of a set of wheels. The main advantage of mobile robots over fixed-base manipulators is their virtually unlimited workspace. As a consequence, such robots are increasingly being used in both the industrial and the service domains, and in general in all applications which require increased capabilities of autonomous motion.

From a mechanical viewpoint, a *wheeled robot* essentially consists of a rigid body (*base*, or *chassis*) equipped with a system of wheels. This basic arrangement may be complicated, for example, by attaching to the base one or more trailers, or by mounting a manipulator on the base (*mobile manipulator*).

Any wheeled vehicle is subject to kinematic constraints that in general reduce its local mobility, while leaving intact the possibility of reaching arbitrary configurations by appropriate maneuvers. For example, any driver knows by experience that, while it is impossible to move instantaneously a car in the direction orthogonal to its heading, it is still possible to park it anywhere, at least in the absence of obstacles. This peculiar feature makes wheeled mobile robots very challenging from the control viewpoint, and in fact some recent developments in nonlinear control were motivated and inspired by the study of these systems.

Here, we will consider only mobile robots that are equipped with conventional wheels, either orientable or fixed (as the front or rear wheels of a car, respectively). Omnidirectional mobile robots



**Wheeled Robots, Fig. 1** The Roomba (here, model 880) series of vacuum cleaning robots produced by iRobot features differential-drive kinematics

realized using, e.g., Mecanum wheels are not covered in this article. Indeed, the local mobility of these vehicles is unrestricted, and therefore no special control treatment is necessary.

The most popular wheel arrangement for mobile robots is the *differential drive*, in which two fixed wheels whose axes of rotation coincide are controlled by separate actuators (see Fig. 1). One or more passive (*caster*) wheels are usually added for statical balance. This wheeled robot is the most agile, as it can rotate on the spot by applying equal and opposite angular speeds to the wheels. A kinematically equivalent arrangement is the *synchro drive*, in which three orientable wheels are synchronously driven by two motors through mechanical coupling; the first motor provides traction, whereas the second controls the common orientation of the wheels.

Other possible kinematic structures include the *tricycle* (one steering and two fixed wheels) and the *car-like* (two steering and two fixed wheels). Vehicles of this type are however less

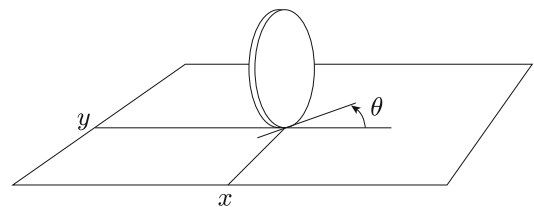
common in robotics, due partly to their reduced maneuverability (they have a nonzero turning radius) and partly to their increased mechanical complexity. For example, both these vehicles require a specific device (*differential*) for distributing traction torque to the driving wheels.

## Modeling

The starting point for modeling wheeled mobile robots is the single wheel. This may be represented as an upright disk rolling on the ground. Its configuration is described by three generalized coordinates: the Cartesian coordinates  $(x, y)$  of the contact point with the ground, measured in a fixed reference frame, and the orientation  $\theta$  of the disk plane with respect to the  $x$  axis (see Fig. 2). The configuration vector is therefore  $q = (x \ y \ \theta)^T$ . The *pure rolling* constraint, expressed as

$$(\sin \theta - \cos \theta) \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = 0, \quad (1)$$

entails that in the absence of slipping, the velocity of the contact point has zero component in the direction orthogonal to the wheel plane. The angular speed of the wheel around the vertical axis is instead unconstrained. The kinematic constraint (1) is *nonholonomic*, i.e., it cannot be integrated to a geometric constraint; this may be easily shown using Frobenius Theorem, a well-known differential geometry result on integrability of differential forms. An important consequence of its being nonholonomic is that constraint (1) does not imply a loss of accessibility in the configuration space of the wheel.



**Wheeled Robots, Fig. 2** Generalized coordinates for a single wheel

In a single-body vehicle equipped with multiple wheels, the  $n$ -dimensional configuration vector  $q$  consists of the Cartesian coordinates of a representative point on the robot, the orientation of all independently orientable wheels, plus the orientation of the body if there are fixed wheels. By writing one pure rolling constraint like (1) for each independent wheel, either orientable or fixed, and expressing it in the chosen generalized coordinates, one obtains a set of  $k$  constraints in the form

$$A^T(q)\dot{q} = 0. \quad (2)$$

Kinematic constraints of this form (i.e., linear in the generalized velocities) are called *Pfaffian*. In wheeled mobile robots, Pfaffian constraints are in general completely nonholonomic.

The  $k$  Pfaffian constraints (2) reduce the number of degrees of freedom (i.e., independent instantaneous motions) of the robot to  $m = n - k$ . In particular, at each configuration  $q$  the generalized velocities  $\dot{q}$  must belong to the  $m$ -dimensional null space of matrix  $A^T(q)$ :

$$\dot{q} = \sum_{j=1}^m g_j(q)u_j = G(q)u, \quad (3)$$

where vector fields  $g_1(q), \dots, g_m(q)$  are a basis of  $\mathcal{N}(A^T(q))$  and  $u = (u_1 \dots u_m)^T$  is a coefficient vector. Kinematically admissible trajectories are the solutions of (3), which is called *kinematic model* of the wheeled mobile robot. This model can be seen as a nonlinear dynamical system, with  $q$  as state and  $u$  as input. In particular, system (3) is driftless and has more state variables than control inputs.

For example, consider the *unicycle*, a somewhat ideal mobile robot equipped with a single, orientable wheel. The configuration vector for this robot is  $q = (x \ y \ \theta)^T$ , the same as the single wheel, and the vehicle is subject to the rolling constraint (1). One possible kinematic model of the form (3) for the unicycle is then

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix} v + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \omega, \quad (4)$$

where  $v = \pm\sqrt{\dot{x}^2 + \dot{y}^2}$  and  $\omega = \dot{\theta}$  represent, respectively, the driving and steering velocity of the wheel. Both the differential-drive and the synchro-drive robots are kinematically equivalent to the unicycle, i.e., their kinematic model can be put in the form (3) by properly defining  $q$  and  $u$ .

Similarly to what is done for robot manipulators, dynamic models of wheeled mobile robots may be derived following the Euler-Lagrange method. The main difference is the presence of the nonholonomic Pfaffian constraints, which give rise to reaction forces expressed via Lagrange multipliers (Neimark and Fufaev 1972).

## Structural Properties

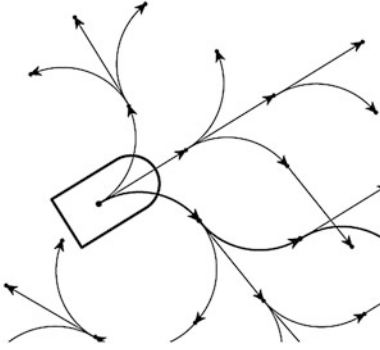
The nonholonomic nature of wheeled mobile robots has precise consequences in terms of structural properties of the kinematic model (3).

The first, and most important, is that in spite of the reduced number of degrees of freedom, wheeled robots are *controllable* in their configuration space; i.e., given two arbitrary configurations, there always exists a kinematically admissible trajectory (with the associated velocity inputs) that transfers the robot from one to the other (Fig. 3). Since the kinematic model (3) is driftless, this fact can be verified using a well-known result (Chow Theorem), according to which the system is controllable if and only if the accessibility rank condition holds:

$$\dim \bar{\Delta} = n, \quad (5)$$

where  $\bar{\Delta}$  denotes the involutive closure of distribution  $\Delta = \{g_1, \dots, g_m\}$  under the Lie bracket operation. In turn, this is guaranteed to be true in view of the nonholonomy of constraints (2). For example, since the Lie bracket of the two input vector fields in (4) is always linearly independent from them, the kinematic model of the unicycle is controllable.

However, the controllability of wheeled mobile robots is intrinsically nonlinear. In



**Wheeled Robots, Fig. 3** In spite of its restricted local mobility, a nonholonomic wheeled robot can reach any point in its configuration space

fact, the linear approximation of (3) at any configuration turns out to be uncontrollable due to the reduced number of inputs. This indicates that no linear feedback can stabilize the system at a given configuration. The situation is actually worse: for nonholonomic robots, there exist no continuous time-invariant feedback law that achieves stabilization at a point. This can be established on the basis of a celebrated result on smooth stabilizability of control systems (Brockett 1983). This negative result does not apply to time-varying stabilizing controllers, which may thus be continuous in  $q$ .

Another (related) complication of wheeled mobile robots is that in general they do not admit *universal* controllers, i.e., feedback control laws that can asymptotically stabilize *arbitrary* state trajectories, either persistent or not (Lizárraga 2004). Therefore, for these systems one needs in principle different controllers for solving trajectory tracking and point regulation problems.

All the above limitations of nonholonomic systems are established with reference to the kinematic model, but of course they are passed on to dynamic models. Altogether, they contribute to making wheeled mobile robots much more difficult to control than, for example, robotic manipulators, which are linearly controllable, smoothly stabilizable, and admit universal controllers.

## Trajectory Planning

Trajectory planning for wheeled robots is a non-trivial problem, because not all trajectories are feasible – once again, a consequence of non-holonomy. This leads to the necessity of *maneuvering*, i.e., performing certain specific movements, in order to execute transfer motions.

Most kinematic models of wheeled mobile robots exhibit a property known as *differential flatness* (Fliess et al. 1995): namely, there exists a set of outputs  $z$ , called *flat* outputs, such that the state  $q$  and the control inputs  $u$  can be expressed algebraically as a function of  $z$  and its time derivatives up to a certain order  $\sigma$ :

$$q = \varphi(z, \dot{z}, \ddot{z}, \dots, z^{(\sigma)}) \quad (6)$$

$$u = \gamma(z, \dot{z}, \ddot{z}, \dots, z^{(\sigma)}) \quad (7)$$

As a consequence, the state trajectory  $q(t)$  and control history  $u(t)$  associated to a given output trajectory  $z(t)$  are uniquely determined. For example, the unicycle admits  $z = (x \ y)^T$  as flat outputs. In fact, once a Cartesian trajectory is assigned for the contact point with the ground, the wheel orientation  $\theta(t)$  is constrained to be tangent to the trajectory; the associated control input  $v$  and  $\omega$  are then uniquely and algebraically computable from  $q(t)$ .

Differential flatness is particularly useful for planning. In fact, assume that we want to transfer a wheeled mobile robot from an initial configuration  $q_i$  to a final configuration  $q_f$ . One then computes the corresponding values  $z_i$  and  $z_f$  of the flat outputs, plus the appropriate boundary conditions, and uses any interpolation scheme (e.g., polynomial interpolation) to plan the trajectory of  $z$ . The evolution of the generalized coordinates  $q$ , together with the associated control inputs  $u$ , can then be computed algebraically from (6)–(7). The resulting configuration space trajectory will automatically satisfy the nonholonomic constraints (2).

Another approach to nonholonomic trajectory planning relies on the possibility of putting the equations of most wheeled robots into a canonical format known as a 2-input *chained form*

$$\begin{aligned}
 \dot{z}_1 &= w_1 \\
 \dot{z}_2 &= w_2 \\
 \dot{z}_3 &= z_2 w_1 \\
 &\vdots \\
 \dot{z}_n &= z_{n-1} w_1
 \end{aligned} \tag{8}$$

by means of a feedback transformation, i.e., a change of coordinates  $z = \alpha(q)$  coupled with an input transformation  $w = \beta(q)u$ . In particular, this is always possible for kinematic models like (3) when  $n \leq 4$  and  $m = 2$  (e.g., unicycle or car-like robots). Once the system is cast in the form (8), one may use sinusoidal open-loop controls at integrally related frequencies to drive all variables sequentially to their final values (Murray and Sastry 1993). This approach is particularly interesting from a theoretical viewpoint because such control maneuvers achieve motion in the direction of the Lie brackets of the input vector fields.

Note that differential flatness and chained-form transformability are equivalent properties for 2-input nonholonomic mobile robots.

## Feedback Control

The motion control problem for wheeled mobile robots is generally formulated with reference to the kinematic model (3). For example, in the case of the unicycle (4) this means that the control inputs are directly  $v$  and  $\omega$ , the driving and steering velocities. There are essentially two reasons for taking this simplifying assumption.

First, the kinematic model (3) fully captures the essential nonlinearity of single-body wheeled robots, which stems from their nonholonomic nature. This is another fundamental difference with respect to the case of robotic manipulators, in which the main source of nonlinearity is inertial coupling among multiple bodies. Second, in mobile robots it is typically impossible to command directly the wheel torques, because there are low-level wheel control loops integrated

in the hardware or software architecture. These loops will accept as input a reference value for the wheel angular speed, which is then reproduced as accurately as possible by standard regulation actions (e.g., PID controllers). In this situation, the actual inputs available for high-level control are precisely these reference velocities.

Two basic control problems can be considered:

- *Trajectory tracking*: the robot must asymptotically track a desired Cartesian trajectory  $(x_d(t), y_d(t))$ .
- *Point stabilization*: the robot must asymptotically reach a desired configuration  $q_d$ .

From a practical point of view, the most relevant of these problems is arguably the first. This is because mobile robots must be able to operate in unstructured workspaces that invariably contain obstacles. Clearly, forcing the robot to move along (or close to) a trajectory planned in advance reduces considerably the risk of collisions. The point stabilization problem, however, is more difficult and therefore particularly interesting from a scientific perspective. In a certain sense, the relative difficulty of the two problems is reminiscent of human car driving: learning to drive a car along a road is relatively easy, whereas parking poses a greater challenge.

## Trajectory Tracking

Several methods are available to drive a wheeled mobile robot in feedback along a desired trajectory. A straightforward possibility is to compute first the linear approximation of the system along the desired trajectory (which, unlike the approximation at a configuration, results to be controllable) and then stabilize it using linear feedback. Only local convergence, however, can be guaranteed with this approach. For the kinematic model of the unicycle, global asymptotic stability may be achieved by suitably morphing the linear control law into a nonlinear one (Canudas de Wit et al. 1993).

In robotics, a popular approach for trajectory tracking is input-output linearization via static

feedback. In the case of a unicycle, consider as output the Cartesian coordinates of a point  $B$  located ahead of the wheel, at a distance  $b$  from the contact point with the ground. The linear mapping between the time derivatives of these coordinates and the velocity control inputs turns out to be invertible provided that  $b$  is nonzero; under this assumption, it is therefore possible to perform an input transformation via feedback that converts the unicycle to the parallel of two simple integrators, which can be globally stabilized with a simple proportional controller (plus feedforward). This simple approach works reasonably well. However, if one tries to improve tracking accuracy by reducing  $b$  (so as to bring  $B$  closer to the ground contact point) the control effort quickly increases.

Trajectory tracking with  $b = 0$  (i.e., for the actual contact point on the ground) can be achieved using dynamic feedback linearization (Oriolo et al. 2002). In particular, this method provides a one-dimensional dynamic compensator that transforms the unicycle into a parallel of two double integrators, which is then globally stabilized with a proportional-derivative controller (plus feedforward). In contrast to static feedback linearization, no residual zero dynamics is present in the transformed system. However, the dynamic compensator has a singularity when the unicycle driving velocity is zero. This is expected, because otherwise the tracking controller would represent a universal controller. Note that dynamic feedback linearizability using the  $x, y$  outputs is related to them being flat – the two properties are equivalent.

### Point Stabilization

The impossibility of stabilizing a nonholonomic mobile robot using continuous pure-state feedback has generated two main directions of research to solve the problem:

- *discontinuous* feedback, i.e., time-invariant control laws  $u = \gamma(q)$ , where  $\gamma$  is discontinuous precisely at the configuration that one seeks to stabilize;

- *time-varying* feedback, in the form  $u = \gamma(q, t)$  where  $\gamma$  may or may not be continuous at the desired configuration.

For the unicycle, a well-known stabilizing controller belonging to the first category was designed by Aicardi et al. (1995) by formulating the problem in polar coordinates centered at the goal and then using a Lyapunov-like analysis to establish asymptotic convergence. The controller, once rewritten in original coordinates, turns out to be discontinuous at the goal (not surprisingly). Although this rules out proper stability in the sense of Lyapunov, this controller is effective in that it produces rather natural approach trajectories to the goal.

Continuous time-varying stabilizers in the sense of Lyapunov exist (Samson 1993) but have mainly theoretical interest due to their provably slow (polynomial) rate of convergence; this is a direct consequence of the fact that the linear approximation of the system is not controllable. A more effective approach is to give up (Lipschitz-) continuity at the desired configuration. As shown by M'Closkey and Murray (1997) and Morin and Samson (2000), this allows to design control laws that guarantee a modified form of exponential convergence to the goal.

Most of the aforementioned control designs – both for trajectory tracking and point stabilization – were first developed with reference to the unicycle robot but can be carried out on chained forms, thereby providing an effective extension to other kinematic models, e.g., the car-like robot.

### Summary and Future Directions

Wheeled mobile robots are increasingly present in applications. Over the last two decades, significant results have been reached in terms of modeling, planning, and control of these systems, and the field is now considered to be well-established, at least from an application point of view. Nevertheless, a number of research directions are still open, including the following:

- *Planning and control for non-flat systems:* Relatively harmless wheeled robots (such as a unicycle towing more than one off-hooked trailer) are not flat.
- *Robustness:* The design of controllers that can effectively withstand disturbances and model perturbations is still an active area of research.
- *Localization:* Feedback control requires accurate measurements of the configuration variables, which in mobile robots cannot be reliably reconstructed from on-board sensors (odometric data). Integration of exteroceptive sensing is essential to this end.
- *Vision-based control:* As an alternative to localization-based methods, the feedback loop may be closed directly in the image plane, with significant advantages in terms of simplicity and robustness.
- *Multi-robot systems:* The problem is to control the motion of multiple mobile robots in order to perform a cooperative motion task, e.g., formation control.

## Cross-References

- ▶ [Differential-Geometric Methods in Nonlinear Control](#)
- ▶ [Feedback Linearization of Nonlinear Systems](#)
- ▶ [Feedback Stabilization of Nonlinear Systems](#)
- ▶ [Lie-Algebraic Methods in Nonlinear Control](#)
- ▶ [Vehicle Dynamics Control](#)

## Recommended Reading

For background material on nonlinear controllability, including the necessary concepts of differential geometry, see Sastry (2005). General introductions to mobile robots can be found in Siegwart and Nourbakhsh (2004), Choset et al. (2005), Morin and Samson (2008), Siciliano et al. (2009), and Tzafestas (2013). A classification of wheeled mobile robots based on the number, placement and type of wheels was proposed by Bastin et al. (1996). A detailed extension of some of the planning and control techniques reviewed in this article to the case of car-like

kinematics is given in De Luca et al. (1998). A detailed treatment of control problems for mobile robots can be found in Samson et al. (2016). A framework for the stabilization of non-flat nonholonomic robots was presented by Oriolo and Vendittelli (2005). Recent work aimed at designing practical universal controllers was carried out by Morin and Samson (2009).

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