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Aggregation in Sraffa's Simple Production Model

By

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1. Introduction

Often in economics, models having clear microeconomic foundations lead to properties extending far beyond their original scope, involving sometimes the economy as a whole.

Whether this is the unveiling of a macroeconomic truth or the result of a particular formulation is a key question to address before drawing conclusions of theoretical nature. As the variables of a model are usually available in more or less grouped form, one way to make that assessment specific is to investigate the invariance of a macroproperty with respect to the level of model aggregation. This, in turn, raises the question of specifying the set of possible aggregates upon which a given model retains its consistency.

Aggregation theory (Ijiri, 1971) studies conditions under which some or all the relationships described by a micromodel continue to hold in the macromodel. This concept arises in such diverse disciplinary areas as information theory (Marschak, 1964), control theory (Aoki, 1971) mathematical programming (Dantzig, 1976; Geoffrion, 1977) and statistics (W. Fisher, 1958, 1979; Chipman, 1975).

In economics, aggregation is of special concern in capital theory, a topic of long, partially settled controversies in the history of economic thought.

Generally speaking, aggregability conditions either put restrictions on the shape of the production function (F. Fisher, 1969, 1983) or limit the ways in which production factors ought to be grouped (Brown and Chang, 1976). Although some results are avail-

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able for general production functions, the most extensively studied case is when this function is of Leontief type.

Starting with the pioneering works of Hatanaka (1952), McManus (1956) and Malinvaud (1954), Theil (1957) studied the prediction bias due to imperfect aggregation. In the same context, Morimoto (1970) defined a first order aggregation bias and derived sufficient conditions for its vanishing. Ara (1959) studied the effects of aggregation on the stability of a dynamic input-output model.

In this note, the study of aggregation is extended to the Sraffian simple production model. Although this model formally contains a linear correspondence between labour inputs and prices, the main relationship it describes is how the price vector and the uniform wage rate vary with respect to changes in the uniform profit rate. As this relationship is non-linear, aggregation criteria derived for traditional input-output models are inapplicable, despite formal similarities.

A second, possibly more important theoretical motivation is the explicit presence of a macro-relation, the factor-price frontier, in a model which has microeconomic origins. The question of concern is the stability of the frontier with respect to diverse aggregation schemes.

Thirdly, the Sraffian reduction procedure to the standard system, when interpreted in the light of the present theory may provide a plausible and alternative aggregation scheme with respect to the existing ones as well as providing a rationale for the aggregation of a larger class of linear models.

After reviewing in Sec. 2 the mathematical formalism of aggregation and its application to linear models, conditions of fully consistent aggregation for the Sraffian model are studied in Sec. 3. In Sec. 4 the role of basic and nonbasic sectors in aggregation is analysed. Partially consistent aggregation is discussed in Sec. 5 and in Sec. 6 two aggregation schemes are outlined in detail. Numerical examples are reported in Sec. 7. Conclusions and potential extensions are contained in Sec. 8. Mathematical notation is standard except for the symbol diag (x) which, for a vector x, denotes the diagonal matrix whose (i, i)-th element is x_i. Definitional equalities are marked by $\triangleq$. A small square □ signals the end of a proof.

2. Aggregation in Linear Models

Let the spaces X, Y collect the values of the exogenous, endogenous variables of a disaggregated or micromodel A defined as a mapping A: $X \rightarrow Y$. Suppose X and Y are aggregated into some macrovariable spaces W and Z by the rules B: $X \to W$, and B^*: $Y \to Z$.

A macromodel C (associated to A) is a mapping between macrovariables C: $W \to Z$.

Fully consistent aggregation obtains when data can be aggregated *before* being processed at no loss of information that is, for *all* $x \in X$, $B^* \cdot A\, x = C \cdot B\, x$ (Ijiri, 1971).

If the endogenous variables are the same in both models B^* can be assumed to be the identity and *fully consistent aggregation* requires $z = Cw$. When applied to linear models on finite dimensional spaces ($X = \mathbb{R}^n$, $Y = \mathbb{R}^m$) we obtain the matrix equation

$$CBx = Ax \quad \text{for all} \quad x \in \mathbb{R}^n. \tag{2.1}$$

Equation (2.1) has a solution C if and only if

$$N\,(B) \subseteq N\,(A) \tag{2.2}$$

where $N\,(A)$ denotes the null space of a matrix A (Ijiri, 1971, p. 769).

Let B^+ be the pseudoinverse of B.[1]

Then $N\,(B) = \{x \in \mathbb{R}^n : x = (I - B^+ B)\, s,\ \ s \in \mathbb{R}^n\}$ and the full consistency condition (2.2) is equivalent to $A\,(I - B^+ B)\, s = 0$, $\forall\, s \in \mathbb{R}^n$, or

$$A = AB^+ B. \tag{2.3}$$

If this condition is satisfied, then a fully consistent aggregated model exists. Its coefficient matrix is $C = AB^+ + S\,(I - BB^+)$ with S arbitrarily chosen as one can see by direct substitution into (2.1).

Inspection of (2.3) shows that for aggregation to be fully consistent every row of A must be a left eigenvector of B^+B associated to the unit eigenvalue, a condition seldom verified in practice.

A less demanding criterion is that of *partial consistency*, in which *some* information loss is allowed. This can arise in any of the following situations.

i) Consistency is only required for values of the exogenous variables belonging to a subspace of $\mathbb{R}^n$ as in the case, for instance, of balanced changes in final demand of multisector models (Paris, 1975) or of unchanged relative prices in Hicks' (1946) commodity aggregation theorem.

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[1] That is, a matrix satisfying (Boullion and Odell, 1971, p. 1):

$$(BB^+)' = BB^+, \quad (B^+B)' = B^+B, \quad BB^+B = B, \quad B^+BB^+ = B^+.$$

ii) Consistency is only required for variables "filtered" by an operator H (see Ijiri, 1971, p. 772). In particular, H may degenerate into a vector and be interpreted as a decision or cost function, as for instance in the case of price vectors p and p^0, solutions of the micro- and macromodels, violating (2.1) but yielding the same value of the objective function in the economic optimization problem (Geoffrion, 1977).

iii) More generally, it is required that aggregation keep certain model properties unchanged. Rosenblatt (1967) established conditions on the aggregation matrix under which non-negativity of the micromodel is maintained. Other properties may concern equilibrium, stability or vitality of the production process.

In the case of the static Leontief model, the micromodel is $q=[I-P]^{-1}d$, with P the input-output coefficient matrix, q and d n-dimensional vectors denoting outputs and final demands. Denote the macromodel by $q^0=[I^0-P^0]^{-1}d^0$ and suppose that d must be aggregated into d^0 via the m by n matrix T, with $d^0=Td$. Let now $\hat{q}^0=Tq$. Full consistency requires $\hat{q}^0=q^0$ for all $d\in\mathbb{R}^n$ and this leads to the necessary and sufficient Hatanaka (1952) condition $TP=P^0T$. (See also Chipman, 1976, Sec. 2.4.)

Several properties of this matrix equation will be used in the rest of the paper. These are summarized in the following

Proposition 2.1. *The equation $TP=P^0T$ in the unknown T either has no nontrivial solutions or has infinitely many solutions. The latter case obtains if and only if matrices P and P^0 have at least one eigenvalue in common.*

Proof. See Gantmacher (1977, Vol. 1, Chpt. 8, Sec. 1).

Proposition 2.2. *Let $TP=P^0T$ with T full (row) rank.*

i) *If a right eigenvector of P is not in the null space of T, the corresponding eigenvalue of P is an eigenvalue of P^0;*

ii) *All eigenvalues of P^0 are eigenvalues of P.*

Proof. See Appendix.

Proposition 2.3. *Let $TP=P^0T$ with T, P elementwise non-negative and P indecomposable. The dominant eigenvalue of P coincides with that of P^0.*

Proof. See Appendix.

3. Fully Consistent Aggregation of the Sraffa Model

The aggregation concepts of the previous section can be extended to the simple production model studied by Sraffa (1960):

$$(1+r)\ Xp + wl = \text{diag}\ (x)\ p,$$
$$\mathbf{1}'\ l = \mathbf{1}'\ [\text{diag}\ (x) - X]\ p = 1.$$

Here, vectors $l, x, p \in \mathbb{R}^n$ collect sectoral labour shares, output quantities and relative prices of a multisector economy, where each sector produces a single output. The input quantities employed to produce x_i units of the i-th commodity are collected in the i-th row of matrix X, which is assumed indecomposable.

The uniform wage rate w, and the uniform profit rate r, as well as all prices, are expressed in units of the net product's value, which is taken as numeraire. Vector $\mathbf{1}$ has unit components,[2] and the prime denotes transpose.

Assuming all quantities to be produced in positive amounts, the model may be expressed in terms of a matrix A of production coefficients ($A \triangleq X\ \text{diag}\ (x)^{-1}$) and a vector π of sectoral output values ($\pi \triangleq \text{diag}\ (x)\ p$) as

$$(1+r)\ A\ \pi + wl = \pi, \tag{3.1}$$
$$\mathbf{1}'\ l = \mathbf{1}'\ [I - A]\ \pi = 1. \tag{3.2}$$

In the Appendix we prove the following

Proposition 3.1. *Given the model (3.1—3.2) with* $l \geq 0$, $A \geq 0$ *indecomposable* $\sum_{i=1}^{n} a_{ij} \leq 1\ \forall j$, *there exists an* $R > 0$ *such that for any* $r \in [0, R]$ *the solution* $\pi(r)$, $w(r)$ *is unique and* $\pi(r)$ *is strictly positive.*

Assuming the profit rate r as independent variable, the solution can be put in the form (see Appendix)

$$w\ (r) = \frac{1}{1 + r\mathbf{1}'\ A\ [I + (1+r)\ A + (1+r)^2\ A^2 + \ldots]\ l} \tag{3.3}$$

$$\pi\ (r) = \frac{[I + (1+r)\ A + (1+r)^2\ A^2 + \ldots\ldots]\ l}{1 + r\mathbf{1}'\ A\ [I + (1+r)\ A + (1+r)^2\ A^2 + \ldots]\ l} \tag{3.4}$$

where the series converge for $r < R = \frac{1-\lambda(A)}{\lambda(A)}$ with $\lambda(A)$ dominant

[2] Clearly vector $\mathbf{1}$ has dimension n in this case. We shall use the notation $\mathbf{1}_n$ when confusion is possible.

eigenvalue of matrix A. When n microsectors are aggregated into $m<n$ macrosectors, each of them will employ a weighted sum of the labour shares of the component microsectors. This results in an aggregation rule $l^0=Tl$, where T $(m\times n)$ has a block diagonal structure T = block diag $\{t_i'\} = \begin{bmatrix} t_1' & . & 0 \\ & \cdot & \\ 0 & . & t_m' \end{bmatrix}$ where $t_i' = [t_{i,1} \dots t_{i,n_i}]$ contains the n_i weights of the i-th macrosector, with $\sum_{i=1}^{m} n_i = n$.

The equations of the model in aggregated form are

$$(1+r)\, A^0 \pi^0 + w^0\, l^0 = \pi^0, \tag{3.5}$$

$$\mathbf{1}'\, l^0 = \mathbf{1}'\, [I - A^0]\, \pi^0 = 1, \tag{3.6}$$

where all variables bear a 0 except for r, which is common to both models ($r^0 = r$).

From what we have seen in Sec. 2, full consistency of the aggregated model requires

$$l^0 = Tl \Rightarrow \pi^0 = T\pi$$

$$l^0 = Tl \Rightarrow w^0 = w$$

for any feasible value of the exogenous variables l and r.

The first condition refers to the linear part of the model and, not surprisingly, leads to the same condition found in the static Leontief case, namely

$$TA = A^0 T. \tag{3.7}$$

The second concerns the non-linear part and it requires further restrictions.

Theorem 3.1. *Necessary and sufficient conditions for (3.5), (3.6) to be a fully consistent aggregation of (3.1), (3.2) are*

i) $TA = A^0 T$;

ii) *T has unitary weights.*

Proof. For the necessity, let $\pi^0 = T\pi$, $w^0 = w$ in (3.5).

Letting $l^0 = Tl$ we get

$$(1+r)\, A^0 T\pi + wTl = T\pi$$

which subtracted from (3.1) pre-multiplied by T yields

$$(TA - A^0 T)\, \pi = 0, \quad \forall\, l \geq 0.$$

Since there is a one to one correspondence between l and π for any $t \in [0, R]$, this condition implies i).

Letting $l^0 = Tl$ in (3.6) we get

$$\mathbf{1}_m' l^0 = \mathbf{1}_m' Tl = 1, \quad \forall l \geq 0$$

which subtracted from (3.2) implies ii).

For the sufficiency, consider the solution to (3.5), (3.6) and substitute $l^0 = Tl$

$$w^0(r) = \frac{1}{1 + r\mathbf{1}_m' A^0 [I + (1+r) A^0 + (1+r)^2 A^{02} + \ldots] l^0}$$

$$= \frac{1}{1 + r\mathbf{1}_m' [A^0 T + (1+r) A^{02} T + (1+r)^2 A^{03} T + \ldots] l}$$

where the series converges for $r \in [0, R^0)$, $R^0 = \frac{1 - \lambda(A^0)}{\lambda(A^0)}$.

Due to i) and Proposition 2.3, $R^0 = R$.

It is easy to see from $A^0 T = TA$ that $A^{0k} T = TA^k$ for all k (see first proof in Appendix) which used in the series above yields

$$w^0(r) = \frac{1}{1 + r\mathbf{1}_m' TA [I + (1+r) A + (1+r)^2 A^2 + \ldots] l}.$$

Using ii), $\mathbf{1}_m' T = \mathbf{1}_n'$ so that $w^0(r) = w(r)$. In the same way using (3.4) and i) we get $\pi^0(r) = T\pi(r)$ for $r \in [0, R]$. □

Leontief full consistency with unitary weights is therefore equivalent to Sraffa full consistency. When the attention is focused on a *given* Sraffa model rather than on the entire class of Sraffa models we may assume l to be a fixed vector and consider only r as the exogenous variable. Fully consistent aggregation conditions are given in this case by the following

Theorem 3.2. *Necessary and sufficient conditions for (3.5), (3.6) to be a fully consistent aggregation of (3.1), (3.2) with fixed l are:*

i) $[TA - A^0 T] \pi(r) = 0$;

ii) $[\mathbf{1}_n' - \mathbf{1}_m' T] l = 0$;

iii) $[\mathbf{1}_n' - \mathbf{1}_m' T] A \pi(r) = 0$.

Proof. Repeat the steps above. □

These conditions appear quite restrictive and in practice difficult to check. However they do not require T to have unitary weights.

The implications of (3.7) with unitary weights will now be analysed and compared to the case of arbitrary weights.

Assume, with no loss of generality, that the first n_1 microsectors are to be aggregated into a single macrosector leaving the others unchanged, so T is partitioned into

$$T = \begin{bmatrix} \mathbf{1}_{n_1}' & 0' \\ 0 & I \end{bmatrix}. \tag{3.8}$$

Accordingly, matrices A and A^0 are partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad A^0 = \begin{bmatrix} A_{11}^0 & A_{12}^0 \\ A_{21}^0 & A_{22}^0 \end{bmatrix}.$$

From (3.7) it follows

a) $A_{22} = A_{22}^0$;

b) $\mathbf{1}_{n_1}' A_{12} = A_{12}^0$; (3.9)

c) $\mathbf{1}_{n_1}' A_{11} = [A_{11}^0 \ldots A_{11}^0]$ (A_{11}^0 is a scalar);

d) $A_{21} = A_{21}^0 \mathbf{1}_{n_1}'$.

Condition a) is trivial. Condition b) simply states that if input j is supplied in α units to sector h and β units to sector k, it will have to be supplied in $\alpha + \beta$ units to the aggregate of h and k.

Rather more restrictive appear conditions c) and d). The first permits to aggregate only microsectors having the same input-output ratio. In Sraffa's terminology, as it will be clear in Sec. 4, this requires the subsystem A_{11} to be in standard proportions. The last condition requires that the outputs of the aggregated microsectors be inputs to the non-aggregated sectors in equal proportions. When weights are allowed to take on arbitrary values a similar reasoning allows to derive conditions analogous to a)—d). This is done in the Appendix. Clearly, unitary weights are a possible choice. The following simple examples clarify the various situations.

Consider the aggregation of the first two sectors in the cases

$$A_1 = \begin{bmatrix} 0.4 & 0.3 & 0.2 \\ 0.1 & 0.2 & 0.1 \\ 0.2 & 0.2 & 0.3 \end{bmatrix}; \quad A_2 = \begin{bmatrix} 0.4 & 0.3 & 0.2 \\ 0.1 & 0.1 & 0.1 \\ 0.2 & 0.2 & 0.3 \end{bmatrix}; \quad A_3 = \begin{bmatrix} 0.1 & 0.16 & 0.04 \\ 0.5625 & 0.10 & 0.3 \\ 0.15 & 0.08 & 0.2 \end{bmatrix}.$$

Aggregation of A_1 is fully consistent if and only if we use unitary weights and leads to (add the first two rows and average the first two columns so obtained)

$$A_1^0 = \begin{bmatrix} 0.5 & 0.3 \\ 0.2 & 0.3 \end{bmatrix}.$$

A_2 violates c), A_3 violates c) and d) and the aggregation of their first two sectors with unitary weights is not fully consistent. However, whereas A_2 violates (3.7) for any choice of the weights, A_3 does satisfy it for

$$T = \begin{bmatrix} 1.5 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and it aggregates into } \begin{bmatrix} 0.4 & 0.3 \\ 0.1 & 0.2 \end{bmatrix}.$$

This is an instance where conditions of Theorem 3.1 are indeed more restrictive. Loosely speaking, however, matrices satisfying the assumptions of Theorem 3.1 with unitary weights are as many as those which satisfy them with arbitrary weights.

Finally, we remark that a weighted aggregation satisfying i) but not ii) in Theorem 3.1, (that is a fully consistent aggregation in the Leontief sense) would still leave relative prices unchanged in the Sraffa model, since vector π^0 preserves the direction of $T\pi$. However its amplitude is varied by a factor $w^0(r)/w(r)$. The bias introduced by a weighted aggregation only affects the non-linear part of the model.

4. Full Consistency in the Decomposable Case

In the case of a decomposable system, we shall assume that sectors are re-numbered (if necessary) so that A is in canonic form

$$A = \begin{bmatrix} A_{ss} & A_{sr} \\ 0 & A_{rr} \end{bmatrix}$$

with A_{rr} indecomposable. By calling basic sectors those indexed by r, non-basic the others, we have then

Proposition 4.1. *Given a decomposable matrix A with at least two basic sectors, the aggregation of one non-basic sector with one basic sector cannot be fully consistent.*

Proof (by contradiction).

Let b be the number of basic sectors and $n-b$ the number of non-basic sectors. Assume A to be in canonic form. If a basic sector j $(j>n-b)$ and a non-basic sector i $(i \leq n-b)$ are aggregated condition (3.9 d) requires for the elements of A

$$a_{ki} = a_{kj} \quad k = 1, \ldots, n; \quad k \neq i; \quad k \neq j.$$

Since $a_{ki} = 0$ for $k = n-b+1, \ldots, n$ column j of A_{rr} must have zero entries except, possibly, a_{ij}. But then A_{rr} is decomposable, contrary to our assumption. □

When one or more non-basic sectors are to be aggregated with more basic sectors we have

Proposition 4.2. *A necessary condition for the fully consistent aggregation of one or more non-basic sectors with basic sectors is that the aggregate includes all basic sectors.*

Proof. Use the argument of proposition 4.1 for each basic non-basic sector pair. □

If the non-basic block A_{ss} is itself decomposable, a final block-triangular structure can be reached by proper renumbering

$$\begin{bmatrix} A_{11} & A_{12} \ldots\ldots & A_{1r} \\ & A_{22} \ldots\ldots & A_{2r} \\ & & \vdots \\ & & A_{rr} \end{bmatrix}$$

where A_{ii} are now indecomposable square matrices of order b_i, $\sum_{i=1}^{r} b_i = n$.

By defining a sector i of level l if it belongs to the block $(r-l)$, [3] the basic sectors will be of level zero.

A generalization of Proposition 4.2 then leads to the

Corollary. *A necessary condition for the fully consistent aggregation of one or more sectors of level l' with sectors of level $l'' < l'$ is that the aggregate includes all sectors of level l''.*

As a consequence of decomposability and conditions (3.9) a)—d) we have, finally

Proposition 4.3. *If all sectors of level $r-1$ (i. e., A_{11}) form a block in standard proportions,[4] they can be aggregated into a single sector with full consistency.*

Proof. With reference to conditions 3.9 a)—d) we see that a), b), d) are trivially satisfied while c) is equivalent to assuming standard proportions. □

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[3] That is, if $\sum_{j=1}^{r-l-1} b_j < i \leq \sum_{j=1}^{r-l} b_j$.

[4] That is, if the columns of A_{11} have constant sum.

For example, matrix

$$\begin{bmatrix} 0.3 & 0.1 & 0.2 & 0.4 & 0.1 \\ 0 & 0.2 & 0.1 & 0 & 0.1 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0.2 & 0.2 & 0.01 & 0.1 & 0.01 \\ 0 & 0.1 & 0.01 & 0 & 0.1 \end{bmatrix}$$

becomes, after renumbering sectors 1, 2, 3, 4, 5 by 2, 4, 5, 1, 3

$$\begin{bmatrix} 0.1 & 0.2 & 0.01 & 0.2 & 0.01 \\ 0.4 & 0.3 & 0.1 & 0.1 & 0.2 \\ 0 & 0 & 0.1 & 0.1 & 0.01 \\ 0 & 0 & 0.1 & 0.2 & 0.1 \\ 0 & 0 & 0 & 0 & 0.5 \end{bmatrix}.$$

The first diagonal block is in standard proportions and can be replaced by a single sector.

Thus a fully consistent aggregation with unitary weights is

$$A^0 = \begin{bmatrix} 0.5 & 0.11 & 0.3 & 0.21 \\ 0 & 0.1 & 0.1 & 0.01 \\ 0 & 0.1 & 0.2 & 0.1 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}.$$

Notice that the results of this section (with a slight generalization of Proposition 4.3) hold for any choice of the weights in T and are applicable to linear models of general form.

5. Partially Consistent Aggregation

The conditions for a fully consistent aggregation are quite restrictive and, in practice, they are seldom verified.

Consequently, Eq. (3.7) for a fixed A should be replaced by

$$TA = A^0 T + E. \tag{5.1}$$

When $E \neq 0$, the specification TAT^+ for A^0 yields a partially consistent aggregation in the sense that A^0 minimizes the Euclidean norm of matrix E (Boullion and Odell, 1971, p. 42).

When T has full rank, T^+ can be written as $T'(TT')^{-1}$, and for unitary weights,

$$T^+ = \text{block diag}\left\{\frac{1}{n_i}\mathbf{1}_{n_i}\right\} \text{ where } \sum_{i=1}^{m} n_i = n.$$

The aggregated model is

$$(1+r)\ TAT^{+}\ \pi^0 + w^0\ l^0 = \pi^0, \tag{5.2}$$

$$\mathbf{1}'\ l^0 = \mathbf{1}'\ [I - TAT^{+}]\ \pi^0 = 1. \tag{5.3}$$

A measure of productivity, that is the ability to produce each output in excess to its use as input is given by $\varrho = 1 - \max_j \sum_{i=1}^{n} a_{ij}$, a quantity ranging between zero and one.[5]

A first consequence of partial consistency is that the productivity of the original system is not necessarily preserved.

Proposition 5.1. *The productivity of a partially consistent aggregated system with unitary weights is greater or equal to the productivity of the original system.*

Proof. Let A be aggregated into $A^0 = TAT^{+}$ with unitary weights. Then

$$1 - \max_j \sum_{i=1}^{m} a_{ij}{}^0 = 1 - ||A^0|| = 1 - ||TAT^{+}||.$$

Since $||TAT^{+}|| \le ||T||\ ||A||\ ||T^{+}||$ and

$$||T|| = ||T^{+}|| = 1,$$

$$\varrho^0 = 1 - \max_j \sum_{i=1}^{m} a_{ij}{}^0 \ge 1 - \max_j \sum_{i=1}^{n} a_{ij} = \varrho. \qquad \square$$

As the productivity of the original system implies that of the aggregated system, the latter has unique positive solutions in $r \in [0, R^0)$ with $R^0 = (1 - \lambda\ (A^0))/\lambda\ (A^0)$

$$w^0\ (r) = \frac{1}{1 + r\ \mathbf{1}'\ TAT^{+}\ l^0 + r\ (1+r)\ \mathbf{1}'\ (TAT^{+})^2\ l^0 + \ldots} \tag{5.4}$$

$$\pi^0\ (r) = w^0\ (r)\ [l^0 + (1+r)\ TAT^{+}\ l^0 + (1+r)^2\ (TAT^{+})^2\ l^0 + \ldots]. \tag{5.5}$$

To see how these differ from those of the original system, define the aggregation bias as the difference between the functions $w^0\ (r)$ and $w\ (r)$ (factor price frontier) and $\pi^0\ (r)$ and $T\pi\ (r)$ (sectoral output values).

[5] In matrix A rows refer to sectors, columns to commodities. The higher the column sum, the smaller the excess production. If the commodity with maximum column sum is produced in excess all commodities are produced in excess.

A first order[6] approximation to (5.4) is $w_1^0(r) \triangleq 1/(1+r\,\mathbf{1}'\,TAT^+\,l^0)$ and comparing this with the analogous approximation to $w(r)$, $w_1(r) \triangleq 1/(1+r\,\mathbf{1}'\,Al)$ we have

Proposition 5.2. *A sufficient condition for the vanishing of the first order aggregation bias in the factor price frontier under the assumption $A^0 = TAT^+$, where T has full rank, is that*

i) *the aggregation function has unitary weights,*
ii) *the sectors in each aggregate employ equal labour shares.*

Proof. By i) the matrix T^+T has diagonal blocks of the form

$$\frac{1}{n_i}\begin{bmatrix} 1 & \dots & 1 \\ \vdots & & \vdots \\ 1 & \dots & 1 \end{bmatrix}, \quad (n_i \times n_i).$$

By ii) the vector l has components

$$l = \{l_1 \dots l_1\, l_2 \dots l_2 \dots l_m \dots l_m\}.$$

Therefore l is a right eigenvector associated to the unit eigenvalue of T^+T or

$$T^+\,Tl = l.$$

Letting $l^0 = Tl$ in $w_1^0(r)$ and recalling that $\mathbf{1}_m'\,T = \mathbf{1}_n'$ the equality $w_1^0(r) = w_1(r)$ follows. □

Defining likewise a first order approximation to the vector $\pi^0(r)/w^0(r)$, $\pi_1^0(r) \triangleq [I+(1+r)\,TAT^+]\,l^0$ and comparing this to the analogous approximation to $T\pi(r)/w(r)$, $T\pi_1(r) \triangleq T\,[I+ +(1+r)\,A]\,l$, we have

Corollary. *A sufficient condition for the vanishing of the first order aggregation bias in the values of the aggregated outputs is that*

i) *the aggregation function has unitary weights,*
ii) *the sectors in each aggregate employ equal labour shares.*

Replacing the notion of output [labour] with that of endogenous [exogenous] variable, the corollary holds for general linear models aggregated with unitary weights.

[6] That is, of order $0(\|TAT^+\|)$. Notice that $\|(TAT^+)^k\| \leq \|TAT^+\|^k \leq \|T\|^k\,\|A\|^k\,\|T^+\|^k$. If A is productive, $\|A\| < 1$ and powers of TAT^+ go to zero faster than $\|A\|^k$. Presumably, the faster the higher the degree of aggregation.

6. Preserving the Maximum Profit Rate in Aggregation

We briefly review the Sraffian construction of the standard system from the viewpoint of aggregation theory.

Given a Sraffa model, Eqs. (3.1), (3.2), denote by q a left eigenvector of A associated to the dominant eigenvalue $\lambda(A)$. In the assumption of indecomposable A, the standard system associated to (3.1), (3.2) is

$$(1+r)\,\tilde{A}\,\tilde{\pi} + \tilde{w}\tilde{l} = \tilde{\pi}, \tag{6.1}$$

$$\mathbf{1}'\,\tilde{l} = \mathbf{1}'\,[I-\tilde{A}]\,\tilde{\pi} = 1 \tag{6.2}$$

where $\tilde{A} \triangleq \text{diag}\,(q)\,A\,\text{diag}\,(q)^{-1}$, $\tilde{\pi} \triangleq \text{diag}\,(q)\,\pi$, $\tilde{l} \triangleq \text{diag}\,(q)\,l$.

Assume q is normalized with $q'\,l=1$ so that the components q_i are the Sraffa multipliers.

As is known, the relationship between r and $\tilde{w}$ is linear and there exists a composite commodity whose value is independent of r. Furthermore matrix $\tilde{A}$ has columns adding up to $\lambda(A)$, since $\mathbf{1}'\,\tilde{A} = \mathbf{1}'\,\text{diag}\,(q)\,A\,\text{diag}\,(q)^{-1} = q' A\,\text{diag}\,(q)^{-1} = \lambda(A)\,q'\,\text{diag}\,(q)^{-1} = = \lambda(A)\,\mathbf{1}'$.

Therefore, choosing $T=\mathbf{1}'$ as aggregation function the full consistency condition of Theorem 3.1, $T\tilde{A} = \tilde{A}^0 T$, is satisfied.

For choices of the aggregation function T different from $\mathbf{1}'$ the aggregation of the standard system $\tilde{A}$ into $\tilde{A}^0 = T\tilde{A}T^+$ does not usually guarantee full consistency.

However, the partially consistent aggregation resulting from the choice $A^0 = TAT^+$ presents an interesting feature when A is in standard proportions $(A=\tilde{A})$ and T has unitary weights. We have in such a case $\mathbf{1}_m'\,\tilde{A}^0 = \mathbf{1}_m'\,T\tilde{A}T^+ = \mathbf{1}_n'\,\tilde{A}T^+ = \lambda(\tilde{A})\,\mathbf{1}_n'\,T^+ = \lambda(\tilde{A})\,\mathbf{1}_m'$ which shows that the aggregated system $\tilde{A}^0$ preserves the dominant eigenvalue, $\lambda(\tilde{A}^0) = \lambda(\tilde{A}) = \lambda(A)$.

Furthermore, columns of $\tilde{A}^0$ still add up to $\lambda(A)$ so the aggregated system is still in standard proportions.

We can rephrase this result with

Proposition 6.1. *Let $\tilde{A}$ be the coefficient matrix of a standard system. The partially consistent aggregation $T\tilde{A}T^+$ with unitary weights transforms standard systems into standard systems.*

This leads us to study the properties of a partially consistent aggregation of the original system A in which the weights in the aggregation function T are just the multipliers q_i. We shall refer to this aggregation scheme as *one-step procedure*.

Proposition 6.2. *The aggregation of the Sraffian model (3.1), (3.2) in the form (3.5), (3.6) with* $\hat{A}^0 = \hat{T} A \hat{T}^+$ *where*

$$\hat{T} = \begin{bmatrix} q_1 q_2 \dots q_{n_1} & & & \\ & q_{n_1+1} \dots q_{n_1+n_2} & & \\ & & \dots\dots & \\ & & & q_{n-n_m+1} \dots q_n \end{bmatrix}$$

and q_i $(i = 1, 2, \dots, n)$ *are the components of the normalized left eigenvector associated to the dominant eigenvalue* $\lambda(A)$*, preserves the maximum profit rate.*

Proof. We have

$$\hat{A}^0 = \hat{T} A \hat{T}^+ = T \operatorname{diag}(q)\, A \operatorname{diag}(q)^{-1} \operatorname{diag}(q)\, \hat{T}^+ =$$
$$= T\tilde{A} \operatorname{diag}(q)\, \hat{T}^+ \qquad (6.7)$$

where T is given by (6.6) with unitary weights. Let now

$$W \triangleq \begin{bmatrix} \frac{q_1^2}{\sum_{i=1}^{n_1} q_i^2} & & & \\ \vdots & & & \\ \frac{q_{n_1}^2}{\sum_{i=1}^{n_1} q_i^2} & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \frac{q_{n-n_m+1}^2}{\sum_{i=n-n_m+1}^{n} q_i^2} \\ & & & \vdots \\ & & & \frac{q_n^2}{\sum_{i=n-n_m+1}^{n} q_i^2} \end{bmatrix}.$$

One has $\hat{A}^0 = T\tilde{A}W$ and since $\mathbf{1}_n{}'\tilde{A} = \lambda(A)\,\mathbf{1}_n{}'$ we get

$$\mathbf{1}_m{}'\, T\tilde{A}W = \mathbf{1}_n{}'\,\tilde{A}W = \lambda(A)\,\mathbf{1}_m{}' \quad \text{or} \quad \lambda(\hat{A}^0) = \lambda(A)$$

that is $R^0 = \dfrac{1-\lambda(\hat{A}^0)}{\lambda(\hat{A}^0)} = R.$ $\square$

This procedure aggregates the original n-sector system into an m-sector standard system and the factor price frontier becomes a straight line similarly to the Sraffian reduction to a single sector. We should warn, however, that the procedure outlined is *not* the same as reducing *first* the original system to a standard system of equal size and *then* aggregating the latter with unitary weights.

Indeed a procedure of this kind, which will be called *two-step procedure*, yields an aggregate matrix[7]

$$\tilde{A}^0 = T \operatorname{diag}(q)\, A \operatorname{diag}(q)^{-1}\, T^+ = T\tilde{A}T^+. \tag{6.3}$$

Comparing this with $\hat{T}A\hat{T}^+$ shows that $\tilde{A}^0 = \hat{A}^0$ if and only if each column of the matrix $(\operatorname{diag}(q)^{-1} T^+ - \hat{T}^+)$ belongs to the null-space of the matrix $\hat{T}A$. A sufficient condition for this to be true is that $\operatorname{diag}(q)^{-1} T^+ = \hat{T}^+$, which holds if and only if the multipliers q_i of all sectors in the same aggregate are equal.

On the other hand, $\hat{A}^0 = \hat{T}A\hat{T}^+$ is the best approximate minimum norm solution to the full consistency equation $\hat{T}A = A^0\hat{T}$.

Therefore procedure (6.3), though it preserves the maximum profit rate yields

$$||\tilde{E}|| \triangleq ||\hat{T}A - \tilde{A}^0\,\hat{T}|| \geq ||\hat{T}A - \hat{A}^0\,\hat{T}|| \triangleq ||\hat{E}||. \tag{6.4}$$

In any case, inequality (6.4) does not imply $||\tilde{e}|| \triangleq ||\hat{T}\pi - \tilde{\pi}^0|| > > ||\hat{T}\pi - \hat{\pi}^0|| \triangleq ||\hat{e}||$, where $\pi, \hat{\pi}^0$ and $\tilde{\pi}^0$ are respectively the sectoral values in the original system, in the system aggregated with $\hat{A}^0$ and with $\tilde{A}^0$. In other words, the criterion of best approximating the full consistency condition not necessarily implies the lowest possible bias in the aggregated sectoral values.

Notice again that the two aggregation procedures outlined above, and related results, are immediately extendable to linear models of general form, including the Leontief case which is obtained by simply fixing a value for r: preservation of the maximum profit rate corresponds to preservation of vitality in the static case and of stability in the dynamic case. In the next section two simple examples will be given, where the pseudoinverse solution $\hat{A}^0$ is in turn better or worse than the solution $\tilde{A}^0$ from the viewpoint of biases in sectoral values.

[7] The one-step and two-step procedures coincide when the original system is already in standard proportion ($\operatorname{diag}(q) = I$).

7. A Numerical Example

Consider a five-sector economy described by[8]

$$A = \begin{bmatrix} 0.1 & 0.3 & 0.0125 & 0.015 & 0.04 \\ 0.2 & 0.15 & 0.1 & 0.05 & 0.1 \\ 0.1 & 0.1 & 0.05 & 0.15 & 0.2 \\ 0.3 & 0.06 & 0.06 & 0.2 & 0.3 \\ 0.06 & 0.12 & 0.03 & 0.035 & 0.05 \end{bmatrix} \qquad l = \begin{bmatrix} 0.03 \\ 0.0675 \\ 0.37375 \\ 0.4125 \\ 0.11625 \end{bmatrix}$$

which will be assumed as our micromodel. The associated dominant eigenvalue, maximum rate of profit and Sraffa multipliers are respectively, $\lambda(A) = 0.54199$, $R = 0.84505$, $q' = [1.96615,\ 2.25664,\ 0.70238,\ 0.87111,\ 1.43524]$.

We consider a three-sector macromodel obtained by aggregating the first three microsectors with one of the following schemes

a) unitary weights,

b) weights set equal to Sraffa multipliers:

b1) one-step procedure: $\hat{A}^0 = \hat{T}A\hat{T}^+$,

b2) two-step procedure: $\tilde{A}^0 = T\tilde{A}T^+$.

The resulting macromatrices are

$$A^0 = \begin{bmatrix} 0.370833 & 0.215 & 0.34 \\ 0.14 & 0.2 & 0.3 \\ 0.07 & 0.035 & 0.05 \end{bmatrix} \qquad \text{(case } a\text{)}$$

$$\hat{A}^0 = \begin{bmatrix} 0.409026 & 0.284328 & 0.309904 \\ 0.070727 & 0.2 & 0.182083 \\ 0.062235 & 0.057666 & 0.05 \end{bmatrix} \qquad \text{(case } b_1\text{)}$$

$$\tilde{A}^0 = \begin{bmatrix} 0.404685 & 0.284328 & 0.309904 \\ 0.076830 & 0.2 & 0.182083 \\ 0.060474 & 0.057666 & 0.05 \end{bmatrix} \qquad \text{(case } b_2\text{)}.$$

Case a): unitary weights

The dominant eigenvalue is $\lambda(A^0) = 0.55031$, $(R^0 = 0.81714)$, different as expected from $\lambda(A)$.[9] In Fig. 1 the aggregated sectoral values $T\pi$ are shown as a function of r (solid lines) together with the values π^0 calculated from the macromodel.

[8] Data are taken from an example of the German economy reported in Beutel (1983).

[9] Notice that the productivity measure of A is $\varrho(A) = 0.24$ while that of A^0 is $\varrho(A^0) = 0.31 > \varrho(A)$ as established in Prop. 5.1.

Aggregation biases are evident over the whole range of r. In Fig. 2 the factor-price frontier resulting from the two models is drawn. In this numerical example the aggregated model exhibits a nearly linear frontier.

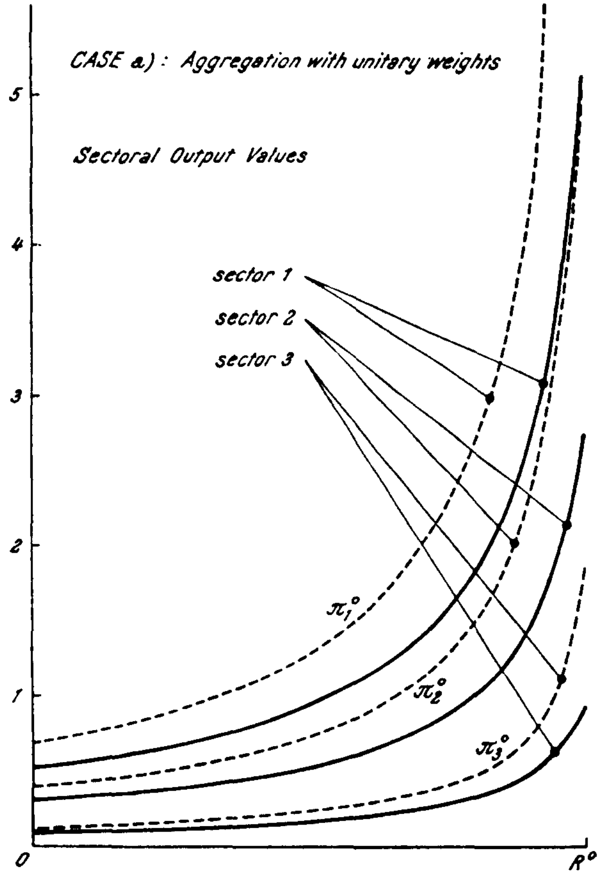

Fig. 1. Aggregation of a 5 sector model into 3 sectors with unitary weights. Comparison between reference values (solid lines) and values in the aggregated model (dashed lines)

Case b): weights equal to Sraffa multipliers

The dominant eigenvalue is $\lambda(\tilde{A}^0) = \lambda(\hat{A}^0) = 0.54199 = \lambda(A)$ and the factor-price frontier is linear. In Fig. 3 on p. 186 are reported the aggregation biases of the sectoral values as functions of r in case b1), while in Fig. 4 the same is done for case b2).

In both cases the biases are much lower than in the case of unitary weights. Furthermore the one-step procedure performs better than the two-step procedure when measured with either the Euclidean or the max deviation norms.

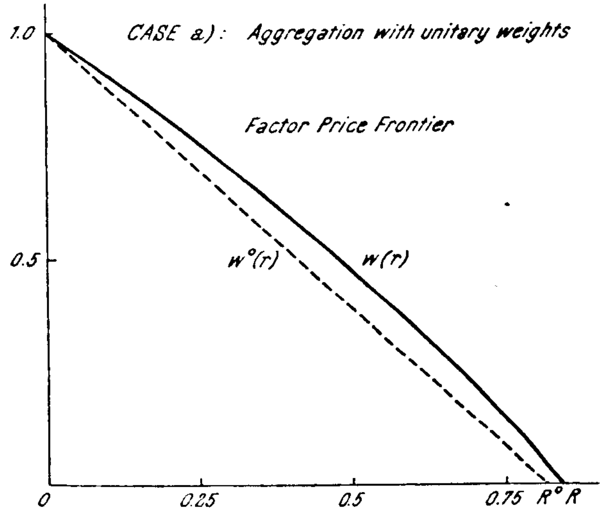

Fig. 2. Aggregation of a 5 sector model into 3 sectors with unitary weights. Comparison of the factor-price frontiers

This last result, however, is reversed if we aggregate as a second example, the second, third and fourth sectors instead of the first three. In this case matrices $\hat{A}^0$ and $\tilde{A}^0$ become

$$\hat{A}^0 = \begin{bmatrix} 0.1 & 0.216566 & 0.054796 \\ 0.398189 & 0.252502 & 0.437190 \\ 0.043798 & 0.072922 & 0.05 \end{bmatrix}$$

$$\tilde{A}^0 = \begin{bmatrix} 0.1 & 0.110076 & 0.054796 \\ 0.398189 & 0.366819 & 0.437190 \\ 0.043798 & 0.065096 & 0.05 \end{bmatrix}.$$

The resulting aggregation biases (Figs. 5 and 6 on p. 187) indicate a superiority of the two-step procedure despite a greater deviation from the full consistency condition. Indeed when E is measured in the Euclidean norm we have

$$||\tilde{E}||^2 = ||\hat{T}A - \tilde{A}^0\,\hat{T}||^2 = 0.168 \quad \text{(two-step procedure)},$$

$$||\hat{E}||^2 = ||\hat{T}A - \hat{A}^0\,\hat{T}||^2 = 0.112 \quad \text{(one-step procedure)}.$$

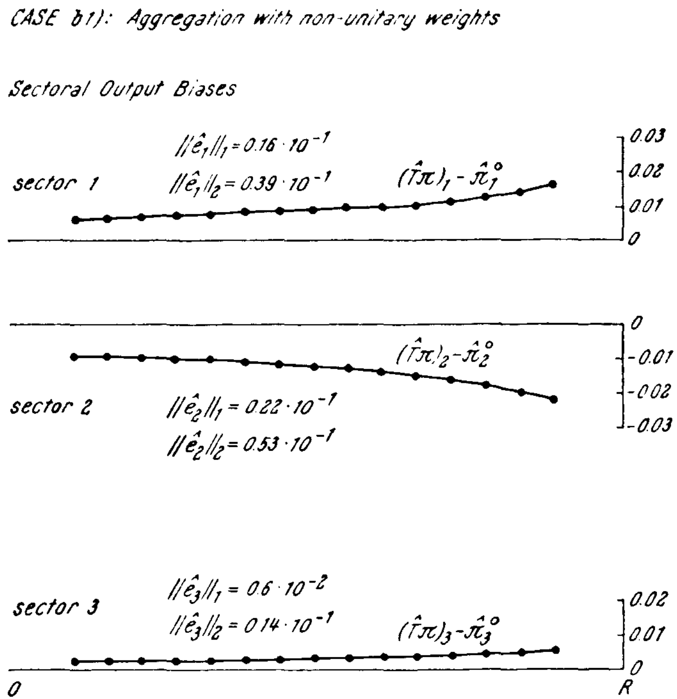

Fig. 3. Aggregation biases in sectoral values: one-step procedure. (First example)
$\|\cdot\|_1$ = max deviation norm; $\|\cdot\|_2$ = Euclidean norm

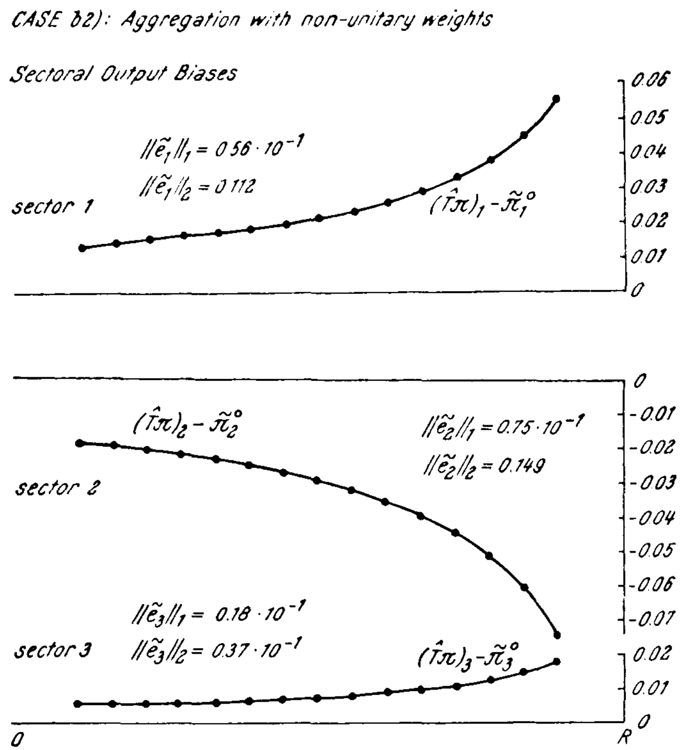

Fig. 4. Aggregation biases in sectoral values: two-step procedure. (First example)

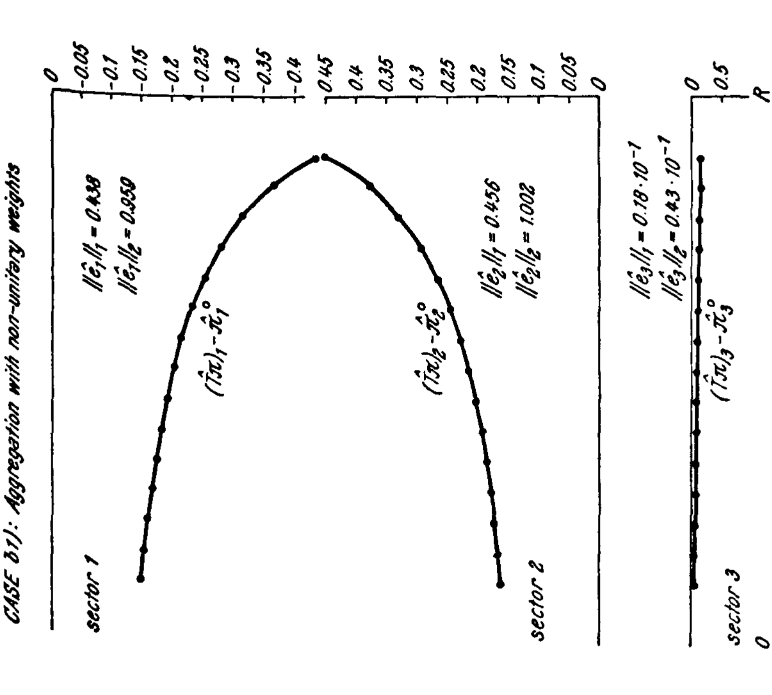

Fig. 5. Aggregation biases in sectoral values: one-step procedure.
(Second example)

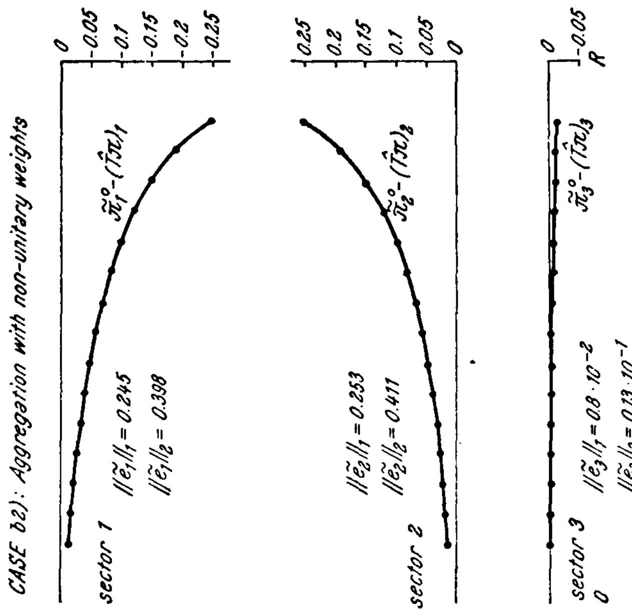

Fig. 6. Aggregation biases in sectoral values: two-step procedure.
(Second example)

8. Conclusions

After reviewing the notion of aggregation for linear models, conditions of fully consistent aggregation have been studied for the Sraffian simple production model.

General conditions of full consistency are satisfied if and only if

a) each microsector to be aggregated in the same macro-sector

— employs commodities produced by other macrosectors in equal proportions,

— exhibits identical proportions between net output and input;

b) the aggregation matrix T has unitary weights.

For aggregation with fixed labour input Theorem 3.2 gives very stringent necessary and sufficient conditions of full consistency.

On the other hand, condition i) of Theorem 3.1, the well known Hatanaka condition for the Leontief model, only ensures that sectoral output values in the aggregated system when compared to those of the original system are unchanged. However, the factor price frontier is not preserved and absolute sectoral output values are distorted by $w^0(r)/w(r)$, a quantity varying with r.

In the decomposable case, fully consistent aggregation requires that a macrosector containing a basic and a non-basic sector include all basic sectors.

Partially consistent aggregation has been considered in the sense of minimizing the deviation from the fully consistent condition satisfaction. When unitary weights are employed this usually results in an artificial productivity increase. On the other hand under unitary weights, an equal labor share condition is found sufficient for the vanishing of first order aggregation biases in endogenous variables.

The Sraffian construction of the standard system has been reviewed in the light of aggregation theory leading to the conclusion that the set of possible aggregates over which a Sraffa model retains full consistency must be in standard proportions. This, in turn, suggests two procedures of partially consistent aggregation in which the maximum rate of profit is preserved and the factor-price frontier is made linear. Although both possibly superior to aggregation with unitary weights, as supported by numerical evidence in simple cases, the relative merits of one over the other is an open question.

Appendix

Proof of Proposition 2.2. Let $\sigma(P)$ be the eigenvalue set of matrix P and consider first the case of distinct eigenvalues

$$\sigma(P)=\{\lambda_1 \ldots \lambda_n\}, \quad \lambda_i \neq \lambda_j \quad \text{for} \quad i \neq j.$$

i) From $TP=P^0T$ with P^0 of order m, it follows that for any right eigenvector v_i of P

$$TPv_i=\lambda_i Tv_i=P^0 Tv_i.$$

If $Tv_i \neq 0$, λ_i is thus also an eigenvalue of P^0.

ii) A sufficient condition for an eigenvalue of P^0 to belong to $\sigma(P)$ is that an independent m-ple exists among the n vectors Tv_i, $i=1,\ldots,n$. In fact, assuming without loss of generality the first $Tv_1,\ldots,Tv_m$ being linearly independent, all the m eigenvalues of P^0 are simply found from the identities

$$P^0(Tv_i)=(P^0T)v_i=(TP)v_i=T(Pv_i)=T(\lambda_i v_i)=\lambda_i(Tv_i),$$

for $i=1,\ldots,m$, which shows that $\sigma(P^0)\subset\sigma(P)$. The full rank assumption on T guarantees that such an m-ple indeed exists.

For a multiple eigenvalue λ with order of multiplicity k, let v satisfy

$$(P-\lambda I)^{k-1}v \neq 0, \quad (P-\lambda I)^k v=0.$$

Such a v does indeed exist.

i) From $TP=P^0T$ follows $TP^2=P^0TP=P^{02}T$ and thus $TP^k=P^{0k}T$ for all k. Hence it is easy to verify that

$$T(P-\lambda I)^k v=(P^0-\lambda I^0)^k Tv$$

(expand the polynomials in λ and make repeated use of the above property).

If $Tv\neq 0$ there exists an index $j \leq k$ such that

$$(P^0-\lambda I)^{j-1}Tv \neq 0, \quad (P^0-\lambda I)^j Tv=0$$

and matrix P^0 preserves the eigenvalue λ up to multiplicity j.

ii) With T full (row) rank, the same reasoning of the first part of the proof shows that all eigenvalues of P^0 belong to $\sigma(P)$. □

Proof of Proposition 2.3. The dominant eigenvalue $\lambda(P)$ is associated to a strictly positive eigenvector v due to indecomposability of P. Since $T \neq 0$ is a non-negative matrix, $Tv \neq 0$. Hence

$$TPv = \lambda(P)\, Tv = P^0\, Tv = \lambda(P^0)\, Tv$$

and $\lambda(P^0) = \lambda(P)$. □

Proof of Proposition 3.1. The existence of an economically meaningful solution to Eqs. (3.1), (3.2) will be proved by means of a constructive (and, to our knowledge, original) procedure which permits to write the solution to the Sraffa model in the closed form of Eqs. (3.3) and (3.4).

Rewrite (3.1) as

$$\left[\frac{1}{1+r} I - A\right] \pi = \frac{w l}{1+r}.$$

By Frobenius theorem (Gantmacher, 1977, vol. 2) the matrix in square brackets has a non-negative inverse in the interval $\lambda(A) < \frac{1}{1+r} \leq 1$, where $\lambda(A)$ is the strictly positive dominant eigenvalue of A. A unique positive solution $\pi(r)$ exists for

$$0 \leq r < \frac{1-\lambda(A)}{\lambda(A)} \triangleq R.$$

To calculate the solution $w(r)$ let

$$B = I - (1+r)\, A, \quad u = rl, \quad v' = \mathbf{1}' A. \tag{A.1}$$

Premultiply (3.1) by $\mathbf{1}'$ and use (3.2) to get

$$w = 1 - r\mathbf{1}' A\pi \tag{A.2}$$

which substituted into (3.1) and using (A.1) yields

$$[B + uv']\,\pi = l.$$

From the Sherman-Morrison identity (Wolfe, 1978, p. 283)

$$[B + uv']^{-1} = B^{-1} - \frac{B^{-1} uv' B^{-1}}{1 + v' B^{-1} u}$$

we get

$$\pi(r) = [I - (1+r)\, A]^{-1} l - \frac{r\,[I-(1+r)\,A]^{-1}\, l\, \mathbf{1}' A\, [I-(1+r)\,A]^{-1} l}{1 + r\mathbf{1}' A\, [I-(1+r)\,A]^{-1} l} =$$

$$= \frac{[I-(1+r)\,A]^{-1} l}{1 + r\mathbf{1}' A\, [I-(1+r)\,A]^{-1} l}$$

and from (A.2)

$$w(r) = \frac{1}{1 + r\mathbf{1}' A (I - (1+r) A]^{-1} l}. \tag{A.3}$$

Notice that $\pi(r)$ can be expressed as

$$\pi(r) = w(r) [I - (1+r) A]^{-1} l. \tag{A.4}$$

The solutions $w(r)$, $\pi(r)$ are strictly positive for $r \in [0, R)$ due to the indecomposability of A.

When $r = R$ the inverse matrix in (A.3) does not exist. Since $w(r)$ is continuous in $r = R$ we can take the limit

$$\lim_{r \to R} w(r) = w(R)$$

and this limit is zero because $(1+r) \to \frac{1}{\lambda(A)}$ as $r \to R$.

Setting $r = R$ in (3.1) we have

$$[I - (1+R) A] \pi = 0$$

or

$$[\lambda(A) I - A] \pi = 0$$

so π is the right eigenvector of A associated to the dominant eigenvalue, a unique and strictly positive vector due to the indecomposability of A.

Finally expanding in Neumann series the inverses in (A.3) and (A.4), (3.3) and (3.4) follow. □

Necessary conditions for full consistency with non unitary weights. Assuming arbitrary weights $t_1 \dots t_{n_1}$ matrix T analogous to (3.8) becomes

$$T = \left[\begin{array}{c|c} t_1 \dots t_{n_1} & 0 \\ 0 & I \end{array}\right] = \left[\begin{array}{c|c} \mathbf{1}'_{n_1} \operatorname{diag}(t_i) & 0 \\ \hline 0 & I \end{array}\right].$$

Therefore $TA = A^0 T$ implies

a′) $A_{22} = A_{22}^0$

b′) $\mathbf{1}'_{n_1} \operatorname{diag}(t_i) A_{12} = A_{12}^0$

c′) $\mathbf{1}'_{n_1} \operatorname{diag}(t_i) A_{11} = [t_1 A_{11}^0 \;\; t_2 A_{11}^0 \dots t_{n_1} A_{11}^0]$

(A_{11}^0 is a scalar)

d′) $A_{21} = A_{21}^0 \mathbf{1}'_{n_1} \operatorname{diag}(t_i)$.

By simple inspection these conditions are not less restrictive than (3.9 a—d). Notice that condition d′), as d), requires the submatrix A_{21} to be of rank one.

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