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Smooth Trajectory Planning for XY$n\overline{\text{R}}$ Planar Underactuated Robots

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Abstract

We consider the trajectory planning problem for the class of XY$n\overline{R}$ planar underactuated robots, having the first two (rotational or prismatic) joints actuated and the last n rotational joints passive. Under the assumption that each passive link is attached at the center of percussion of the previous passive link, the dynamic model assumes a simplified form and we show how to recursively design a dynamic feedback that completely linearizes the system equations. This result allows to plan smooth rest-to-rest motions using polynomial interpolation. As an example, we report the numerical results obtained for trajectory planning of an RR2$\overline{R}$ robot.

1 Introduction

Robots with passive joints are underactuated mechanical systems, i.e., having N degrees of freedom (dof's) and $M < N$ command inputs [1]. The dynamics of the $N - M$ passive joints imposes second-order differential constraints (which may even be integrable [2]) that limit the feasible set of system trajectories. Therefore, the trajectory planning and associated control problems for robots with passive joints are difficult and unsolved in the general case.

For robots with two dof's and a single actuator, such as the planar R$\overline{\text{R}}$ (a bar denotes a passive joint), there exists no closed-form generator of feasible trajectories. Either a numerical planning strategy is then used (e.g., based on time scaling [3]), or a feedback control approach is directly applied [4, 5].

Special attention has been paid to XY$\overline{\text{R}}$ planar robots, having the first two actuated joints of any kind (prismatic or rotational) and a third rotational passive joint. In [6], a trajectory planning algorithm is determined through the composition of translational and rotational motions of the last link, while in [7] the existence of a linearizing (or flat) output is used to solve the trajectory planning and tracking problems via dynamic feedback linearization. In both cases, the so-called center of percussion (CP) of the passive link plays a central role.

More in general, one can check whether a Lagrangian system with degree of underactuation equal to one (i.e., $N - M = 1$) is flat [8] and thus determine an associated motion planning strategy.

There are barely planning or control results for robots with $N - M > 1$ passive joints. In [9] it has been shown that a chain of n coupled planar rigid bodies subject to two cartesian force inputs at one end is flat when each body is hinged at the CP of the previous one, being the CP of the last body the linearizing output. Interestingly, this can be seen as the dynamic counterpart of the nonholonomic n-trailer wheeled mobile robot with zero hooking [10]. In our terms, the above system is actually an XY$n\overline{\text{R}}$ robot and one can try to generalize the trajectory planning results holding for the XY$\overline{\text{R}}$ robot. In fact, the algorithm of [6] has been adapted to the XY$n\overline{\text{R}}$ robot in [11]. This method, however, needs to decompose the global motion into a long sequence of translational and rotational phases for each passive link. Moreover, this approach imposes limitations in the design of a tracking controller for the piecewise planned trajectory.

In this paper, we extend the approach presented in [7] to the trajectory planning of XY$n\overline{\text{R}}$ robots moving in the absence of gravity, under the same hinging hypothesis of [9, 11]. In particular, we exploit the intrinsic recursive nature of the dynamic system (the existence of special points whose acceleration is related to the orientation of the passive links) in order to explicitly design a dynamic feedback linearizing law. In this way, we can determine a single smooth robot motion that joins any initial and desired robot configurations, using a simple polynomial interpolation strategy. We also determine the singularities that should be avoided during planning. As a side but relevant result, after dynamic linearization, it is easy to achieve stable tracking of a nominal trajectory under perturbed conditions by using a single linear feedback law.

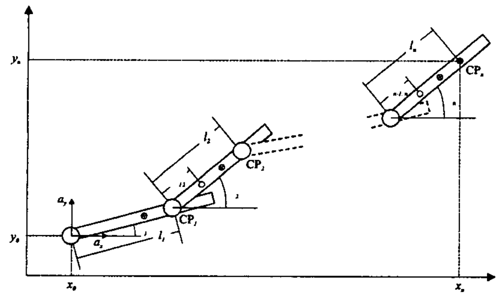

Figure 1: The generic XYn$\overline{\text{R}}$ underactuated robot

The paper is organized as follows. In Sect. 2, we derive the general model of an XYn$\overline{\text{R}}$ robot with n passive joints moving on a horizontal plane (i.e., without gravity), and then specialize it to the particular hinging of passive links. An intrinsic dynamical property of XYn$\overline{\text{R}}$ robots is presented in Sect. 3. This is used in Sect. 4 in order to obtain dynamic feedback linearization of the system in a recursive fashion, from which the associated trajectory planning strategy directly follows. As an example, numerical results for a rest-to-rest smooth motion of an RR2$\overline{\text{R}}$ robot are reported in Sect. 5.

2 Dynamic model

A picture of an XYn$\overline{\text{R}}$ planar underactuated robot is shown in Fig. 1, where the first two actuated joints (that may be prismatic or rotational) have not been sketched.

Let $q = (x_0, y_0, \theta)$ be a set of generalized coordinates, where (x_0, y_0) are the cartesian coordinates of the first passive joint and $\theta = (\theta_1, \ldots, \theta_n)$ are the (absolute, i.e., w.r.t. the x-axis) orientations of the n links associated to the passive joints. This choice simplifies model analysis, capturing all the following cases of interest: RRn$\overline{\text{R}}$, RPn$\overline{\text{R}}$, PRn$\overline{\text{R}}$ and PPn$\overline{\text{R}}$.

Without loss of generality, we consider the accelerations $(a_x, a_y) = (\ddot{x}_0, \ddot{y}_0)$ at the base of the first passive link as the (only) inputs to the system[1], thus focusing our analysis on the dynamics of passive joints.

2.1 Dynamics of passive joints

We use the notation introduced in [12] to write the equation of motion of the n passive joints. To this end, for $j = 1, 2, \ldots, n$, let m_j, d_j and I_j be respectively the mass, the distance between the joint j and the j-th link center of mass, and the moment of inertia of the j-th link. Moreover, let l_j be the distance between joints j and $j+1$, for $j = 1, 2, \ldots, n-1$. Assume that $l_j > 0$ and $d_j > 0$. For compactness, let l_{jk} be defined by

$$l_{jk} = \begin{cases} l_j & (j < k) \\ d_j & (j = k) \\ 0 & (j > k) \end{cases} \qquad (1)$$

and $s_j = \sin\theta_j$, $c_j = \cos\theta_j$, $s_{jk} = \sin(\theta_j - \theta_k)$, $c_{jk} = \cos(\theta_j - \theta_k)$, $j, k = 1, \ldots, n$. The position of the center of mass of the k-th link is given by

$$r_k = \begin{bmatrix} x_0 + \sum_{j=1}^{n} l_{jk} c_j \\ y_0 + \sum_{j=1}^{n} l_{jk} s_j \end{bmatrix}. \qquad (2)$$

The total kinetic energy T is given by

$$\begin{aligned} T &= \frac{1}{2}\sum_{k=1}^{n} m_k \left(\dot{x}_0^2 + \dot{y}_0^2\right) + \frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{n} M_{jk} c_{jk} \dot{\theta}_k \dot{\theta}_j \\ &+ \sum_{k=1}^{n} m_k \sum_{j=1}^{n} l_{jk} \left(c_j \dot{y}_0 - s_j \dot{x}_0\right) \dot{\theta}_j, \end{aligned}$$

where we set

$$M_{jk} = \begin{cases} I_j + \sum_{\ell=1}^{n} m_\ell l_{j\ell}^2 & (j = k) \\ \sum_{\ell=1}^{n} m_\ell l_{j\ell} l_{k\ell} & (j \neq k). \end{cases} \qquad (3)$$

Since the n joints are passive, the associated Lagrangian equations of motion can be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_i}\right) - \frac{\partial T}{\partial \theta_i} = 0, \qquad i = 1, \ldots, n. \qquad (4)$$

Performing computations [13], the passive joints dynamics become

$$\sum_{k=1}^{n}\left(M_{ik} c_{ik} \ddot{\theta}_k + m_k l_{ik}\left(c_i \ddot{y}_0 - s_i \ddot{x}_0\right) + M_{ik} s_{ik} \dot{\theta}_k^2\right) = 0 \qquad (5)$$

for $i = 1, \ldots, n$. Note that it is $M_{ik} = M_{ki}$.

2.2 Model simplification

Divide the i-th eq. (5) by $\sum_{k=1}^{n} m_k l_{ik} > 0$. The resulting (scaled) inertia matrix of the system is no longer symmetric and the following quantities appear in the dynamic equations: $M_{ij} / (\sum_{k=1}^{n} m_k l_{ik}) = l_j$ for $i > j$, and $M_{ij} / (\sum_{k=1}^{n} m_k l_{ik}) \doteq \lambda_{ij}$ for $i < j$, from which it also is $l_i > \lambda_{ij}$, $i, j = 1, \ldots, n$.

The center of percussion of the i-th link (CP$_i$) is located on the link body at a distance $K_i = \frac{I_i + m_i d_i^2}{m_i d_i}$ from the i-th joint. If the mechanical design of the robot is such that the $(i+1)$-th link is hinged at the CP$_i$ of the i-th link (with $i = 1, \ldots, n-1$), the diagonal elements of the scaled inertia matrix assume a special form. In fact, imposing $l_i = K_i$, it is [13]: $M_{ii} / (\sum_{k=1}^{n} m_k l_{ik}) = l_i$.

[1] By a preliminary nonlinear static feedback and a change of coordinates one can always represent the dynamics of the proximal actuated joints by means of the cartesian acceleration $(\ddot{x}_0, \ddot{y}_0)$ at the base of the first passive link [13].

For compactness, we shall denote by $l_n = K_n$ the distance of the CP_n of the last link from joint n.

Under this hypothesis, eqs. (5) can be rewritten as

$$\sum_{j\leq i} l_j(c_{ij}\ddot{\theta}_j + s_{ij}\dot{\theta}_j^2) + \sum_{j>i} \lambda_{ij}(c_{ij}\ddot{\theta}_j - s_{ij}\dot{\theta}_j^2) - s_i\ddot{x}_0 + c_i\ddot{y}_0 = 0 \qquad (6)$$

for $i = 1, \ldots, n$, or, in compact matrix form, as

$$B_u\ddot{\theta} + B_{au}^T \begin{bmatrix} \ddot{x}_0 \\ \ddot{y}_0 \end{bmatrix} + C_u\dot{\theta}^2 = 0, \qquad (7)$$

where we set $\ddot{\theta} = [\ddot{\theta}_1, \ldots, \ddot{\theta}_n]^T$ and $\dot{\theta}^2 = [\dot{\theta}_1^2, \ldots, \dot{\theta}_n^2]^T$. Later on, we shall use also the factorization $C_u(\theta)\dot{\theta}^2 = S_u(\theta, \dot{\theta})\dot{\theta}$.

3 Model analysis

We present a dynamical property of system (7). Consider the points $P_i = (x_i, y_i)$, $(i = 1, 2, \ldots, n)$, whose coordinates are

$$\begin{bmatrix} x_i \\ y_i \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \sum_{j\leq i} l_j \begin{bmatrix} c_j \\ s_j \end{bmatrix} + \sum_{j>i} \lambda_{ij} \begin{bmatrix} c_j \\ s_j \end{bmatrix}. \quad (8)$$

Points P_i $(i = 1, 2, \ldots, n)$ are related to the whole configuration of the robot, but while P_n and P_{n-1} are located on the kinematic structure of the robot, both on the n-th passive link, all the others are external, as shown in Fig. 2 for an $\mathrm{XY5\overline{R}}$ robot. Note also that P_n coincides always with CP_n. Differentiating eq. (8) twice, we obtain the acceleration of point P_i

$$\begin{bmatrix} \ddot{x}_i \\ \ddot{y}_i \end{bmatrix} = \begin{bmatrix} \ddot{x}_0 \\ \ddot{y}_0 \end{bmatrix} + \sum_{j\leq i} l_j \left(\begin{bmatrix} -s_j \\ c_j \end{bmatrix} \ddot{\theta}_j + \begin{bmatrix} -c_j \\ -s_j \end{bmatrix} \dot{\theta}_j^2 \right) + \sum_{j>i} \lambda_{ij} \left(\begin{bmatrix} -s_j \\ c_j \end{bmatrix} \ddot{\theta}_j + \begin{bmatrix} -c_j \\ -s_j \end{bmatrix} \dot{\theta}_j^2 \right). \quad (9)$$

Solving for $(\ddot{x}_0, \ddot{y}_0)$ and substituting in eq. (6), it is

$$s_i\ddot{x}_i = c_i\ddot{y}_i, \qquad i = 1, 2, \ldots, n, \qquad (10)$$

so that the linear acceleration of each point P_i is oriented as the i-th link (i.e., by θ_i). We denote point P_i as *link related acceleration point* (LRAP).

LRAP's present a backward recursive form, so that $(\ddot{x}_i, \ddot{y}_i)$ can be written in terms of the acceleration $(\ddot{x}_{i+1}, \ddot{y}_{i+1})$ of P_{i+1} and the dynamics of θ_{i+1} as

$$\begin{bmatrix} \ddot{x}_i \\ \ddot{y}_i \end{bmatrix} = \begin{bmatrix} \ddot{x}_{i+1} \\ \ddot{y}_{i+1} \end{bmatrix} + (l_{i+1} - \lambda_{i,i+1}) \begin{bmatrix} s_{i+1}\ddot{\theta}_{i+1} + c_{i+1}\dot{\theta}_{i+1}^2 \\ s_{i+1}\dot{\theta}_{i+1}^2 - c_{i+1}\ddot{\theta}_{i+1} \end{bmatrix}, \quad (11)$$

or, conveniently, in terms of the CP_n acceleration as

$$\begin{bmatrix} \ddot{x}_i \\ \ddot{y}_i \end{bmatrix} = \begin{bmatrix} \ddot{x}_n \\ \ddot{y}_n \end{bmatrix} + \sum_{j>i} (l_j - \lambda_{ij}) \begin{bmatrix} s_j\ddot{\theta}_j + c_j\dot{\theta}_j^2 \\ s_j\dot{\theta}_j^2 - c_j\ddot{\theta}_j \end{bmatrix}. \quad (12)$$

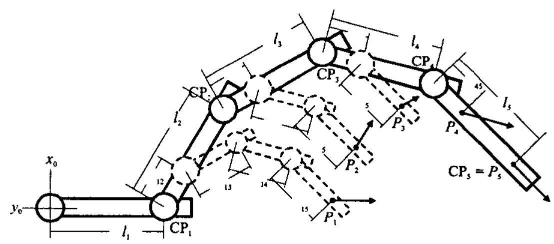

Figure 2: LRAP's of an $XY5\overline{R}$ robot

Finally, due to eqs. (10) we can also write

$$\begin{bmatrix} \ddot{x}_i \\ \ddot{y}_i \end{bmatrix} = \begin{bmatrix} c_i \\ s_i \end{bmatrix} \zeta_i^{(0)}, \qquad i = n, n-1, \ldots, 2, 1 \quad (13)$$

where $\zeta_i^{(0)} = \sqrt{\ddot{x}_i^2 + \ddot{y}_i^2}$ can be evaluated recursively, knowing $\ddot{x}_n$ and $\ddot{y}_n$, and using eq. (11) or eq. (12).

4 Dynamic feedback linearization

The previous considerations about LRAP's show that, knowing the dynamics of the center of percussion CP_n of the last link, one can determine recursively the dynamics of all the passive joints of the robot. In the following, we use this property of CP_n to achieve full linearization of the system, performing a dynamic extension of the state space in a recursive fashion.

Substituting eq. (7) in eq. (9) for $i = n$, the acceleration of CP_n can be written in compact form as

$$\begin{bmatrix} \ddot{x}_n \\ \ddot{y}_n \end{bmatrix} = (I_2 - B_{au}LB_u^{-1}B_{au}^T) \begin{bmatrix} \ddot{x}_0 \\ \ddot{y}_0 \end{bmatrix} + \left(\dot{B}_{au}L - B_{au}LB_u^{-1}S_u\right)\dot{\theta} \doteq \hat{A} \begin{bmatrix} \ddot{x}_0 \\ \ddot{y}_0 \end{bmatrix} + \hat{B}\dot{\theta} \qquad (14)$$

where I_2 is the 2×2 identity matrix and $L = \mathrm{diag}\{l_i\}$. Rotating the acceleration inputs by θ_1

$$\begin{bmatrix} \ddot{x}_0 \\ \ddot{y}_0 \end{bmatrix} = \begin{bmatrix} c_1 & -s_1 \\ s_1 & c_1 \end{bmatrix} \begin{bmatrix} {}^1\ddot{x}_0 \\ {}^1\ddot{y}_0 \end{bmatrix} \doteq R_1 \begin{bmatrix} {}^1\ddot{x}_0 \\ {}^1\ddot{y}_0 \end{bmatrix}, \quad (15)$$

the second column of matrix $A = \hat{A}R_1$ is zero (see [13]), so that matrix $\hat{A}$ is singular and the acceleration of CP_n actually depends only on ${}^1\ddot{x}_0$, i.e., the acceleration input component along the first passive link. Recalling eq. (13), we can write the CP_n acceleration as

$$\begin{bmatrix} \ddot{x}_n \\ \ddot{y}_n \end{bmatrix} = \begin{bmatrix} c_n \\ s_n \end{bmatrix} \zeta_n^{(0)} \qquad (16)$$

where $\zeta_n^{(0)} = \sqrt{\ddot{x}_n^2 + \ddot{y}_n^2}$ is a function of $(\theta, \dot{\theta})$ and the input ${}^1\ddot{x}_0$, through eqs. (14) and (15). Differentiating eq. (16), the $(k+2)$-th derivative of CP_n position is

$$\begin{bmatrix} x_n^{[k+2]} \\ y_n^{[k+2]} \end{bmatrix} = R_n \begin{bmatrix} \zeta_n^{(k)} \\ \xi_n^{(k)} \end{bmatrix} \tag{17}$$

where we denoted by $z^{[i]}$ the i-th time derivative of a function $z(t)$, R_n represents the rotation matrix by θ_n, and we have set

$$\begin{aligned} \zeta_n^{(k)} &= \dot{\zeta}_n^{(k-1)} - \dot{\theta}_n \xi_n^{(k-1)} \\ \xi_n^{(k)} &= \dot{\xi}_n^{(k-1)} + \dot{\theta}_n \zeta_n^{(k-1)}, \end{aligned} \tag{18}$$

with $\zeta_n^{(0)}$ evaluated as in eq. (16) and $\xi_n^{(0)} = 0$. The introduction of the new variables $\zeta_n^{(k)}$ involves a dynamic extension by an integrator on the input $\zeta_n^{(k-1)}$ plus a static feedback that depends on the robot state and on the added states $\zeta_n^{(0)}, \ldots, \zeta_n^{(k-1)}$. On the other hand, the definition of variable $\xi_n^{(k)}$ involves a pure static feedback from the original robot state and from the added states $\zeta_n^{(0)}, \ldots, \zeta_n^{(k-1)}$. As a consequence, the dynamic linearization algorithm adds one integrator for each output derivative (starting from the second one). At the $2(n+1)$-th derivative (see eq. (17) with $k = 2n$), $\xi_n^{(2n)}$ is a function of the original robot state, of the states $(\zeta_n^{(0)}, \ldots, \zeta_n^{(2n-1)})$ added during the dynamic extension, and of the input ${}^1\ddot{y}_0$ [13]. Therefore, introducing two new command inputs (u_x, u_y), we use the input-output decoupling law

$$\begin{bmatrix} \zeta_n^{(2n)} \\ \xi_n^{(2n)} \end{bmatrix} = R_n^T \begin{bmatrix} u_x \\ u_y \end{bmatrix} \tag{19}$$

from which we obtain

$$\begin{bmatrix} x_n^{[2(n+1)]} \\ y_n^{[2(n+1)]} \end{bmatrix} = \begin{bmatrix} u_x \\ u_y \end{bmatrix}, \tag{20}$$

i.e., two chains of $2(n+1)$ input-output integrators. Since the dimension of the original robot state is $2(n+2)$ and the added states $\zeta_n^{(k)}$ $(k = 0, \ldots, 2n-1)$ are $2n$, there is nothing left beyond the input-output dynamics (20). We have thus obtained *full* linearization and input-output decoupling of the system.

The designed dynamic linearizing feedback with inputs (u_x, u_y) and outputs $(\ddot{x}_0, \ddot{y}_0)$ is thus

$$\begin{aligned} \dot{\zeta}_n^{(0)} &= \zeta_n^{(1)} \\ \dot{\zeta}_n^{(k)} &= \zeta_n^{(k+1)} + \dot{\theta}_n \xi_n^{(k)} \quad k = 1, \ldots, 2n-1 \\ \xi_n^{(k+1)} &= \dot{\xi}_n^{(k)} + \dot{\theta}_n \zeta_n^{(k)} \quad k = 1, \ldots, 2n-1 \\ \zeta_n^{(2n)} &= c_n u_x + s_n u_y \\ \xi_n^{(2n)} &= -s_n u_x + c_n u_y \\ {}^1\ddot{x}_0 &= f(\theta, \dot{\theta}, \zeta_n^{(0)}) \\ {}^1\ddot{y}_0 &= g(\theta, \dot{\theta}, \zeta_n^{(0)}, \ldots, \zeta_n^{(2n-1)}, \xi_n^{(2n)}) \\ \ddot{x}_0 &= c_1\, {}^1\ddot{x}_0 - s_1\, {}^1\ddot{y}_0 \\ \ddot{y}_0 &= s_1\, {}^1\ddot{x}_0 + c_1\, {}^1\ddot{y}_0 \end{aligned} \tag{21}$$

where f and g are the inverse functions that return the value of ${}^1\ddot{x}_0$ —obtained from $\zeta_n^{(0)}$ using eqs. (14–16)— and ${}^1\ddot{y}_0$ —determined from eq. (19)— respectively, from the actual states of the robot, from the states $\zeta_n^{(0)}, \ldots, \zeta_n^{(2n-1)}$ inside the dynamic feedback law (21), and from the new inputs (u_x, u_y) to the system.

Assume that a trajectory for the CP_n position output has been assigned, together with its derivatives. In order to calculate the mapping from the linearizing outputs to the extended system state, we will take advantage of the recursive properties of the LRAP's. Note first that, from eq. (13), the $(k+2)$-th derivatives of the position P_i of the i-th LRAP is

$$\begin{bmatrix} x_i^{[k+2]} \\ y_i^{[k+2]} \end{bmatrix} = R_i \begin{bmatrix} \zeta_i^{(k)} \\ \xi_i^{(k)} \end{bmatrix} \tag{22}$$

where

$$\begin{aligned} \zeta_i^{(k)} &= \dot{\zeta}_i^{(k-1)} - \dot{\theta}_i \xi_i^{(k-1)} \\ \xi_i^{(k)} &= \dot{\xi}_i^{(k-1)} + \dot{\theta}_i \zeta_i^{(k-1)} \end{aligned} \tag{23}$$

with $\zeta_i^{(0)}$ given by eqs. (11–13) and $\xi_i^{(0)} = 0$, for $i = n, \ldots, 1$. As suggested by eq. (11), the derivatives (22) can be evaluated knowing the dynamics of θ_{i+1}. In particular, one can compute, for θ_i, with backward recursion $i = n, \ldots, 1$, the behavior of the following variables

$$\begin{aligned} \theta_i &= \mathrm{atan2}\{\mathrm{sign}(\zeta_i^{(0)})\ddot{y}_i, \mathrm{sign}(\zeta_i^{(0)})\ddot{x}_i\} \\ \zeta_i^{(k)} &= c_i x_i^{[k+2]} + s_i y_i^{[k+2]} \qquad k \geq 0 \\ \xi_i^{(k)} &= -s_i x_i^{[k+2]} + c_i y_i^{[k+2]} \quad k \geq 1 \\ \dot{\theta}_i &= \xi_i^{(1)} / \zeta_i^{(0)}. \end{aligned} \tag{24}$$

In order to proceed back recursively and determine the dynamics of joint θ_{i-1}, one needs also to evaluate

$$\ddot{\theta}_i = \frac{\xi_i^{(2)} - 2\dot{\theta}_i \zeta_i^{(1)}}{\zeta_i^{(0)}} \tag{25}$$

$$\theta_i^{[3]} = \frac{\xi_i^{(3)} - 3\zeta_i^{(2)}\dot{\theta}_i - 2\xi_i^{(1)}\dot{\theta}_i^2 - 3\zeta_i^{(1)}\ddot{\theta}_i}{\zeta_i^{(0)}} \tag{26}$$

required to compute $(x_i^{[k]}, y_i^{[k]})$, $(k \geq 2)$ [see eq. (12)]. Using eqs. (12) and (22–26), there is a mapping from $x_n(t), y_n(t)$ and their time derivatives up to the $2(n+1)$ degree to the states $\theta(t)$, $\dot{\theta}(t)$ and $\zeta_n^{(0)}(t), \ldots, \zeta_n^{(2n-1)}(t)$. Finally, using eq. (8) with $i = n$ and its first derivative, we map the linearizing output also to $x_0(t)$, $y_0(t)$ and $\dot{x}_0(t)$, $\dot{y}_0(t)$, completing the state transformation.

From eqs. (24–26) it is easy to see that the coordinate mapping suffers from singularity problems.

In particular, if the acceleration of the i-th LRAP vanishes (i.e., $\zeta_i^{(0)} = 0$) at some time instant, the derivatives $\theta_i^{[k]}, k \geq 0$, will not be defined. Using backward recursion, one can show [13] that $\zeta_i^{(0)} \neq 0$, for all $i = n-1, \ldots, 1$, provided that the acceleration $\zeta_n^{(0)}$ of the CP_n output never vanishes during motion. Therefore, if a trajectory for the CP_n is planned so that $|\zeta_n^{(0)}| \neq 0$ during the whole motion, singularities are always avoided.

5 Case study: the RR2$\overline{\text{R}}$ robot

We now show explicitly the linearizing feedback for an RR2$\overline{\text{R}}$ planar robot with the first two joints (X=Y=R) actuated and the last $n = 2$ passive joints. Using eqs. (7), the dynamic model of the passive joints is rewritten as

$$
\begin{aligned}
l_1\ddot{\theta}_1 + \lambda_{12}c_{12}\ddot{\theta}_2 - s_1\ddot{x}_0 + c_1\ddot{y}_0 + \lambda_{12}s_{12}\dot{\theta}_2^2 &= 0 \\
l_1c_{12}\ddot{\theta}_1 + l_2\ddot{\theta}_2 - s_2\ddot{x}_0 + c_2\ddot{y}_0 - l_1s_{12}\dot{\theta}_1^2 &= 0.
\end{aligned}
$$

Particularizing eqs. (21), the 4-th order dynamic linearizing feedback is:

$$
\begin{aligned}
\dot{\zeta}_2^{(0)} &= \zeta_2^{(1)} \\
\dot{\zeta}_2^{(1)} &= \zeta_2^{(2)} - \dot{\theta}_2\xi_2^{(1)} \\
\dot{\zeta}_2^{(2)} &= \zeta_2^{(3)} - \dot{\theta}_2\xi_2^{(2)} \\
\dot{\zeta}_2^{(3)} &= \zeta_2^{(4)} - \dot{\theta}_2\xi_2^{(3)} \\
\zeta_2^{(4)} &= c_2u_x + s_2u_y \\
\xi_2^{(4)} &= -s_2u_x + c_2u_y \\
{}^1\ddot{x}_0 &= \frac{1}{c_{12}}\left(\frac{l_2 - \lambda_{12}c_{12}^2}{l_2 - \lambda_{12}}\zeta_2^{(0)} + l_2\dot{\theta}_2^2\right) + l_1\dot{\theta}_1^2 \\
{}^1\ddot{y}_0 &= \frac{l_1c_{12}^2}{\mu\zeta_2^{(0)}}\left(\xi_2^{(4)} - \Phi\zeta_2^{(0)} - \Psi - \dot{\theta}_2\zeta_2^{(3)}\right) \\
\ddot{x}_0 &= c_1\,{}^1\ddot{x}_0 - s_1\,{}^1\ddot{y}_0 \\
\ddot{y}_0 &= s_1\,{}^1\ddot{x}_0 + c_1\,{}^1\ddot{y}_0
\end{aligned}
$$

in which

$$
\begin{aligned}
\xi_2^{(1)} &= \dot{\theta}_2\zeta_2^{(0)} \\
\xi_2^{(2)} &= \ddot{\theta}_2\zeta_2^{(0)} + 2\dot{\theta}_2\zeta_2^{(1)} \\
\xi_2^{(3)} &= \theta_2^{[3]}\zeta_2^{(0)} + 3\ddot{\theta}_2\zeta_2^{(1)} + 3\zeta_2^{(2)}\dot{\theta}_2 + 2\xi_2^{(1)}\dot{\theta}_2^2 \\
\ddot{\theta}_2 &= -\frac{s_{12}}{c_{12}}\left(\frac{\zeta_2^{(0)}}{l_2 - \lambda_{12}} + \dot{\theta}_2^2\right) \doteq -\frac{s_{12}}{c_{12}}\mu \\
\theta_2^{[3]} &= -\frac{1}{c_{12}^2}\mu\left(\dot{\theta}_1 - \dot{\theta}_2\right) - \frac{s_{12}}{c_{12}}\dot{\mu}
\end{aligned}
$$

where we introduced the function μ, with $\dot{\mu} = \zeta_2^{(1)}/(l_2 - \lambda_{12}) - 2\dot{\theta}_2\mu s_{12}/c_{12}$, and defined:

$$
\Phi = -\frac{1}{c_{12}^2}[2(\dot{\theta}_1 - \dot{\theta}_2)[\dot{\mu} - \ddot{\theta}_2(\dot{\theta}_1 - \dot{\theta}_2)] + \mu\delta] - \frac{s_{12}}{c_{12}}\ddot{\mu}
$$

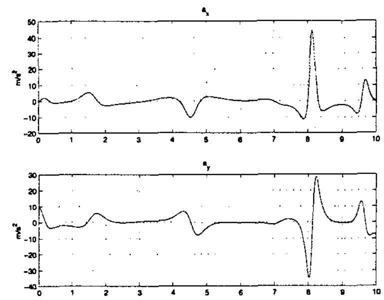

Figure 3: Trajectory planning: inputs $\ddot{x}_0$ and $\ddot{y}_0$

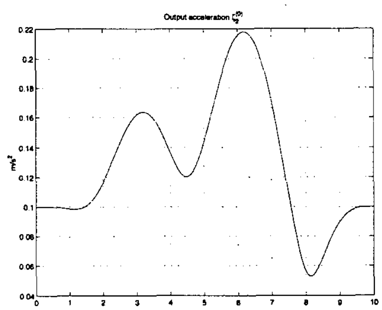

Figure 4: Trajectory planning: $\zeta_2^{(0)}$

$$
\begin{aligned}
\Psi &= 4\theta_2^{[3]}\zeta_2^{(2)} + 6\ddot{\theta}_2\zeta_2^{(2)} + 3\zeta_2^{(3)}\dot{\theta}_2 + 8\dot{\theta}_2\ddot{\theta}_2\xi_2^{(1)} + 4\dot{\theta}_2^2\xi_2^{(2)} \\
\delta &= \frac{s_{12}}{c_{12}}\left(\frac{l_1 + \lambda_{12}c_{12}}{l_1(l_2 - \lambda_{12})}\zeta_2^{(0)} + \dot{\theta}_2^2\right).
\end{aligned}
$$

From the expressions of ${}^1\ddot{x}_0$ and ${}^1\ddot{y}_0$, singularities occur when the third and fourth links become orthogonal ($c_{12} = 0$). Specifically, a motion starting or ending with a configuration having the two passive links orthogonal is not feasible.

We present the numerical results obtained for a rest-to-rest motion from $(x_{0s}, y_{0s}, \theta_{1s}, \theta_{2s}) = (1,1,0,\pi/8)$ [m,m,rad,rad] to $(x_{0g}, y_{0g}, \theta_{1g}, \theta_{2g}) = (1,2,0,\pi/4)$ [m,m,rad,rad] in $T = 10$ s. The first two (actuated) links have length $l_{a1} = 3.5$ m and $l_{a2} = 2.5$ m, while the last two (passive) links are uniform thin rods of unitary mass and length. Therefore, $l_1 = l_2 = 2/3$ m, and $\lambda_{12} = 2/7$ m. We set $\zeta_{2s}^{(0)} = \zeta_{2g}^{(0)} = 0.1$, in order to avoid dynamic singularities, and $\zeta_{2s}^{(i)} = \zeta_{2g}^{(i)} = 0$, $i = 1, 2, 3$. The nominal trajectory is planned using 11-th order polynomials for the CP_2 coordinates (x_2, y_2), by imposing the proper boundary conditions (initial and final values of (x_2, y_2) and of their derivatives up to the fifth order).

The acceleration inputs $(\ddot{x}_0, \ddot{y}_0)$ to the system are shown in Fig. 3. The high values of $(\ddot{x}_0, \ddot{y}_0)$ around

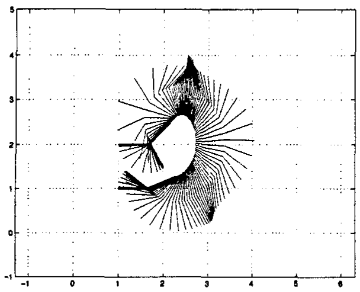

Figure 5: Passive link motion

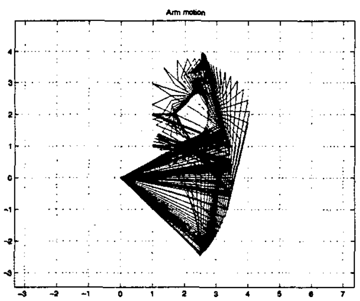

Figure 6: RR2$\overline{R}$ arm motion

$t = 8$ s come from rapid rotations of the passive joints, and correspond to a decrease of $\zeta_2^{(0)}$ (see Fig. 4). However, the (positive) evolution of $\zeta_2^{(0)}$ is always bounded away from zero and dynamic singularities are thus avoided $\forall t = [0, T]$. A stroboscopic view of the cartesian motion of the two passive links is shown in Fig. 5, while Fig. 6 shows the motion of the complete arm. Note that the passive links never become orthogonal during the trasfer.

6 Conclusions

We have considered the problem of smooth trajectory planning for XY$n\overline{R}$ planar underactuated robots, with n passive rotational joints and in the absence of gravity. The system has been fully linearized through dynamic feedback under the assumption that each passive joint is hinged at the center of percussion of the previous passive link, and exploiting the recursive acceleration properties of the system. Rest-to-rest trajectory planning is then easily carried out, suitably avoiding dynamic singularities.

Current work involves inclusion of gravity and trajectory tracking problems. We are also exploring the modifications needed when removing the special hinging assumption at the passive links, at least for the case of $n = 2$ passive joints.

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