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Motion Planning under Gravity for Underactuated Three-Link Robots*

Alessandro De Luca **Giuseppe Oriolo**

Dipartimento di Informatica e Sistemistica
Università degli Studi di Roma "La Sapienza"
Via Eudossiana 18, 00184 Roma, Italy
{deluca,oriolo}@dis.uniroma1.it

Abstract

We present a method for planning motions of three-link planar robots with a passive rotational third joint in the presence of gravity. These underactuated mechanisms can be fully linearized and input-output decoupled by means of a nonlinear dynamic state feedback, provided that a physical singularity is avoided. The linearizing output is the position of the center of percussion of the third link. Based on this, one can plan motions joining in finite time any initial and desired final state; in particular, transfers between inverted equilibria and swing-up maneuvers are easily obtained. Simulation results are reported for a 3R robot.

1 Introduction

A remarkable research activity has been lately devoted to the study of underactuated robotic systems, i.e., second-order mechanical systems with less control inputs than degrees of freedom [1]. From a general control perspective, significant results can be found in [2] and [3]. However, there is no general theory for planning feasible trajectories (and controlling them by feedback) for these systems; successful solutions have been obtained only in specific cases.

In particular, the motion planning problem for two-dof mechanical systems with a single actuator and a passive joint is still unsolved; feasible motions can be only generated as a byproduct of feedback control strategies. Stabilization of a planar 2R robot with a passive elbow joint in zero gravity has been addressed in [4], in which an oscillatory feedback is designed via Poincaré map analysis, and [5], where an iterative steering technique, consisting of the repeated application of open-loop commands, is used to achieve robust convergence to a desired configuration. Such non-conventional approaches are needed because smooth stabilization of the system is not possible.

The case of 2R planar robots with a single actuator in the presence of gravity has been considered, e.g., in [6, 7] (Acrobot, passive first joint) and [8] (Pendubot, passive second joint). Since the approximate linearization of these systems is controllable, they are in principle easy to control, at least locally. However, due to the gravitational drift, the region of the state space where the robot can be kept in equilibrium is reduced, and consists of two disjoint manifolds. Moving between these two requires appropriate swing-up maneuvers, whose synthesis has been so far tackled by energy and/or passivity-based control techniques.

Another class of underactuated robot that has been studied in some detail are three-link planar arms with a passive rotational third joint in the absence of gravity. Arai *et al.* [9] showed how to plan rest-to-rest motions through a sequence of elementary maneuvers consisting of pure translations of the third link or pure rotations around its center of percussion (CP). Under the assumption that the first two joints are prismatic, Imura *et al.* [10] proved that the system can be transformed into second-order chained form via static state feedback. In [11] we have built upon these works, showing that the position of the third link CP becomes a linearizing output under the action of a dynamic state feedback. On the resulting linear and decoupled closed-loop system, motion planning can be performed using smooth trajectories that interpolate, in a given finite time, any initial and desired final state.

In this paper, we show how this powerful approach can be directly extended to account for the presence of gravity. Using a second-order dynamic feedback

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compensator, we obtain an equivalent linear and decoupled system: each coordinate of the CP is independently driven by an auxiliary input through a chain of four integrators. Motion planning between two given states reduces then to a mere interpolation problem for the CP. In particular, robot transfers between inverted equilibria and swing-up maneuvers can be easily generated, as will be illustrated by simulations.

2 Three-link planar robot dynamics

The generic dynamic model of a three-link planar robot moving in a vertical plane with a passive rotational third joint is written in the usual way as

$$B(q)\ddot{q} + c(q,\dot{q}) + g(q) = T(q)\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix}, \tag{1}$$

where $q = (q_1, q_2, q_3)$ is any set of generalized coordinates, and τ_1 and τ_2 are the available inputs on the first two joints. The 3×2 matrix $T(q)$ maps the inputs into generalized forces performing work on q.

We use a set of generalized coordinates that simplifies both model analysis and control design, while capturing all the cases of interest, viz., RRR (3R), RPR, PRR, and PPR. Let $q = (x, y, \theta)$, where (x, y) are the cartesian coordinates of the base of the third link and θ its orientation w.r.t. the x-axis. Letting $s\theta = \sin\theta$ and $c\theta = \cos\theta$, the dynamic model becomes

$$\left[\begin{array}{c|c} B_a(x,y) & \begin{array}{c}-m_3 d_3\, s\theta\\ m_3 d_3\, c\theta\end{array} \\ \hline -m_3 d_3\, s\theta \quad m_3 d_3\, c\theta & I_3 + m_3 d_3^2\end{array}\right]\left[\begin{array}{c}\ddot{x}\\ \ddot{y}\\ \hline \ddot{\theta}\end{array}\right] + \left[\begin{array}{c} c_a(q,\dot{q})\\ \hline 0\end{array}\right] + \left[\begin{array}{c} g_a(x,y)\\ \hline g_0 m_3 d_3\, c\theta\end{array}\right] = \left[\begin{array}{c}F_x\\ F_y\\ \hline 0\end{array}\right], \tag{2}$$

where I_3, m_3, and d_3 are respectively the baricentral inertia, mass, and distance of the center of mass from its base for the third link, $g_0 = 9.81$ m/s^2, and (F_x, F_y) are cartesian forces. The last equation in (2) is a second-order nonholonomic constraint that should be satisfied during the motion.

2.1 Partial feedback linearization

To make the analysis independent from the nature of the first two joints, we perform first a partial linearization via static feedback of eq. (2). Let

$$\begin{bmatrix}F_x\\ F_y\end{bmatrix} = \hat{B}_a(q)\begin{bmatrix}a_x\\ a_y\end{bmatrix} + c_a(q,\dot{q}) + g_a(x,y),$$

where (a_x, a_y) are cartesian accelerations of the third link base and

$$\hat{B}_a(q) = B_a(x,y) - \frac{m_3 d_3^2}{I_3 + m_3 d_3^2}\begin{bmatrix} s^2\theta & -s\theta\, c\theta\\ -s\theta\, c\theta & c^2\theta\end{bmatrix}$$

is nonsingular being the Schur complement of diagonal element b_{33} of the positive definite inertia matrix B.

The resulting equations are:

$$\begin{aligned}\ddot{x} &= a_x\\ \ddot{y} &= a_y\\ \ddot{\theta} &= \frac{1}{K}\left(s\theta\, a_x - c\theta\,(a_y + g_0)\right),\end{aligned} \tag{3}$$

where $K = (I_3 + m_3 d_3^2)/m_3 d_3$ is precisely the distance of the center of percussion (CP) from the third link base. If uniform mass distribution is assumed for the third link, one gets $K = 2\,\ell_3/3$ (ℓ_3 is the link length).

3 Linearization via dynamic feedback

Model (3) can be converted to a linear controllable system by means of nonlinear dynamic feedback and change of coordinates. To this end, one uses the input-output decoupling algorithm [12], that proceeds by differentiating the output until an auxiliary input appears in a nonsingular way. This may require the intermediate addition on the input channels of integrators, which become states of the dynamic compensator.

Define the CP position as the output:

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}x\\ y\end{bmatrix} + K\begin{bmatrix}c\theta\\ s\theta\end{bmatrix}. \tag{4}$$

Following the same procedure as in [11], one finds that the dynamic controller

$$\dot{\xi} = \eta - 2\dot{\theta}\alpha_2 \tag{5}$$

$$\dot{\eta} = \alpha_1 \tag{6}$$

$$\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} = \begin{bmatrix}1 & -\frac{3g_0 c\theta}{\xi - K\dot{\theta}^2 + g_0 s\theta}\\ 0 & \frac{K}{\xi - K\dot{\theta}^2 + g_0 s\theta}\end{bmatrix}\left(R^T(\theta)\begin{bmatrix}v_1\\ v_2\end{bmatrix} - \begin{bmatrix}n_1\\ n_2\end{bmatrix}\right) \tag{7}$$

$$\begin{bmatrix}a_x\\ a_y\end{bmatrix} = R(\theta)\begin{bmatrix}\xi\\ \alpha_2\end{bmatrix}, \tag{8}$$

where $R(\theta)$ is the standard rotation matrix and

$$\begin{aligned} n_1 &= \dot{\theta}^2\left(K\dot{\theta}^2 - \xi - 4g_0\, s\theta\right) - 3\frac{g_0^2}{K}\, c^2\theta,\\ n_2 &= 2\eta\,\dot{\theta} + 6g_0\,\dot{\theta}^2\, c\theta + \frac{g_0}{K}\, c\theta\left(K\dot{\theta}^2 - \xi - g_0\, s\theta\right),\end{aligned}$$

transforms the original system in two input-output chains of four integrators

$$\begin{bmatrix}y_1^{[4]}\\ y_2^{[4]}\end{bmatrix} = \begin{bmatrix}v_1\\ v_2\end{bmatrix}, \tag{9}$$

with (v_1, v_2) the auxiliary input, under the regularity assumption $\gamma \triangleq \xi - K\dot{\theta}^2 + g_0 s\theta \neq 0$.

Since the sum (8) of the relative degrees of the two outputs in the above linearized system equals the dimension (6) of the robot state plus the dimension (2) of the compensator state (ξ, η), full state linearization has been achieved [12]. The new set of coordinates consists of the output function (4) and

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} + K\dot{\theta} \begin{bmatrix} -s\theta \\ c\theta \end{bmatrix} \tag{10}$$

$$\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} = R(\theta) \begin{bmatrix} \xi - K\dot{\theta}^2 \\ -g_0\, c\theta \end{bmatrix} \tag{11}$$

$$\begin{bmatrix} y_1^{[3]} \\ y_2^{[3]} \end{bmatrix} = R(\theta) \left\{ \begin{bmatrix} \eta \\ 0 \end{bmatrix} + \dot{\theta} \begin{bmatrix} 3g_0\, c\theta \\ \xi - K\dot{\theta}^2 + g_0\, s\theta \end{bmatrix} \right\}. \tag{12}$$

The inverse transformation from these linearizing coordinates to the robot and compensator states is:

$$\begin{aligned} \theta &= \text{ATAN2}\left\{\text{sign}(\gamma)(\ddot{y}_2 + g_0), \text{sign}(\gamma)\ddot{y}_1\right\} \\ \dot{\theta} &= \frac{c\theta\, y_2^{[3]} - s\theta\, y_1^{[3]}}{s\theta(\ddot{y}_2 + g_0) + c\theta\, \ddot{y}_1} \\ \xi &= c\theta\, \ddot{y}_1 + s\theta\, \ddot{y}_2 + K\dot{\theta}^2 \\ \eta &= c\theta\, y_1^{[3]} + s\theta\, y_2^{[3]} - 3g_0\, c\theta\, \dot{\theta} \\ \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - K \begin{bmatrix} c\theta \\ s\theta \end{bmatrix} \\ \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} &= \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} - K\dot{\theta} \begin{bmatrix} -s\theta \\ c\theta \end{bmatrix}, \end{aligned}$$

and is regular iff $\gamma \neq 0$. From eq. (11), we note that $\gamma^2 = \ddot{y}_1^2 + (\ddot{y}_2 + g_0)^2$, and thus the regularity condition can be checked before actually computing θ, $\dot{\theta}$, and ξ. Physically, $\gamma \neq 0$ means that the linear acceleration ξ of the third-link base along the link axis should not equate the sum of the centrifugal acceleration $K\dot{\theta}^2$ (arising from an instantaneous rotation around the CP) and of the projection of the gravitational acceleration along the link.

4 Motion planning

Planning a feasible motion on the equivalent representation (9) can be formulated as a smooth interpolation problem for the two outputs $y_1(t)$ and $y_2(t)$.

At time $t = 0$, the robot starts from a generic state $(x_s, y_s, \theta_s, \dot{x}_s, \dot{y}_s, \dot{\theta}_s)$ and should reach, at $t = T$, a goal state $(x_g, y_g, \theta_g, \dot{x}_g, \dot{y}_g, \dot{\theta}_g)$. To obtain the boundary conditions for $y_1(t)$, $y_2(t)$ and their derivatives, we use eqs. (4) and (10–12), where $\xi(0) = \xi_s$, $\xi(T) = \xi_g$, $\eta(0) = \eta_s$, and $\eta(T) = \eta_g$ can be chosen arbitrarily.

For example, a swing-up maneuver from the downward equilibrium ($\theta_s = -90°$, $\dot{x}_s = \dot{y}_s = \dot{\theta}_s = 0$) to the inverted equilibrium ($\theta_g = 90°$, $\dot{x}_g = \dot{y}_g = \dot{\theta}_g = 0$) of the third link requires as boundary conditions for the first output

$$\begin{bmatrix} y_{1s} \\ \dot{y}_{1s} \\ \ddot{y}_{1s} \\ y_{1s}^{[3]} \end{bmatrix} = \begin{bmatrix} x_s \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} y_{1g} \\ \dot{y}_{1g} \\ \ddot{y}_{1g} \\ y_{1g}^{[3]} \end{bmatrix} = \begin{bmatrix} x_g \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

and for the second output

$$\begin{bmatrix} y_{2s} \\ \dot{y}_{2s} \\ \ddot{y}_{2s} \\ y_{2s}^{[3]} \end{bmatrix} = \begin{bmatrix} y_s - K \\ 0 \\ -\xi_s \\ -\eta_s \end{bmatrix}, \quad \begin{bmatrix} y_{2g} \\ \dot{y}_{2g} \\ \ddot{y}_{2g} \\ y_{2g}^{[3]} \end{bmatrix} = \begin{bmatrix} y_g + K \\ 0 \\ \xi_g \\ \eta_g \end{bmatrix}.$$

A straightforward solution to this interpolation problem is to use polynomials of seventh degree:

$$y_i(t) = \sum_{j=1}^{7} a_{ij} \lambda^j, \qquad i = 1, 2, \tag{13}$$

with normalized time $\lambda = t/T$. The open-loop commands to system (9) will be $(i = 1, 2)$:

$$v_i(t) = \frac{1}{T^4} \left(840 a_{i7} \lambda^3 + 360 a_{i6} \lambda^2 + 120 a_{i5} \lambda + 24 a_{i4} \right).$$

The expressions of coefficients a_{ij} (for $i = 1, 2$ and $j = 4, \ldots, 7$) can be found in [11].

If, for the swing-up maneuver under consideration, the x coordinate of the start and goal configurations are the same ($x_s = x_g$), the interpolation scheme (13) will generate $y_1(t) \equiv 0$, not allowing the third link to undergo the required rotation of 180° and leading the system to the singularity $\gamma = 0$. In this case, an intermediate state $(x_m, y_m, \theta_m, \dot{x}_m, \dot{y}_m, \dot{\theta}_m, \xi_m, \eta_m)$, to be reached at some instant $T_m \in (0, T)$, should be added. For example, choosing $\theta_m = 0$ (horizontal third link), we would have as intermediate boundary conditions for the two outputs

$$\begin{bmatrix} y_{1m} \\ \dot{y}_{1m} \\ \ddot{y}_{1m} \\ y_{1m}^{[3]} \end{bmatrix} = \begin{bmatrix} x_m + K \\ \dot{x}_m \\ \xi_m - K\dot{\theta}_m^2 \\ \eta_m + 3g_0\, \dot{\theta}_m \end{bmatrix}, \quad \begin{bmatrix} y_{2m} \\ \dot{y}_{2m} \\ \ddot{y}_{2m} \\ y_{2m}^{[3]} \end{bmatrix} = \begin{bmatrix} y_m \\ \dot{y}_m + K\dot{\theta}_m \\ -g_0 \\ \xi_m - K\dot{\theta}_m^2 \end{bmatrix}$$

and motion planning is split in two similar interpolation problems (called phase I and II).

The selection of initial and final compensator state (ξ_s, η_s) and (ξ_g, η_g) (as well as its intermediate value (ξ_m, η_m), if needed) affects the boundary conditions, and thus the generated motion inside the chosen class of interpolating functions. In particular, the compensator states should be chosen so as to avoid the singularity $\gamma = 0$ during the whole motion. In the two-phase swing-up maneuver, we have at times $t = 0$, $t = T_m$, and $t = T$:

$$\gamma_s = \xi_s - g_0, \quad \gamma_m = \xi_m - K\dot{\theta}_m^2, \quad \gamma_g = \xi_g + g_0.$$

Since all variables in the expression of γ are continuous in $t \in [0, T]$, a necessary condition for avoiding the singularity during the motion is that the above three quantities have the same sign. An additional degree of freedom in our motion planner, which may be exploited for guaranteeing $\gamma(t) \neq 0$, is the possibility of resetting the state η at an intermediate instant T_m, i.e., choosing $\eta(T_m^-) = \eta_m^- \neq \eta_m^+ = \eta(T_m^+)$.

5 Simulation results

To illustrate the performance of the planner, we consider two typical case studies: a transfer between two inverted equilibria and a swing-up maneuver. The third link is an uniform thin rod with $\ell_3 = 1$ m, so that $d_3 = 0.5$ m, $K = 2/3$ m.

Figures 1–3 refer to a motion between inverted equilibria, from $(x_s, y_s, \theta_s) = (0.75, 1, 90°)$ to $(x_g, y_g, \theta_g) = (1.75, 1, 90°)$, with $T = 1$ s and $\xi_s = \xi_g = \eta_s = \eta_g = 0$.

From the third link cartesian motion shown in Fig. 1, we note that the CP is always kept at the same height. The motion of a complete 3R arm (with $\ell_1 = \ell_2 = 1.5$ m), obtained by kinematic inversion, is given in Fig. 2. The nominal torques in Fig. 3 are obtained from the inverse dynamics (1) with the following mass data: $m_1 = 10$, $m_2 = 5$, $m_3 = 1$ (kg).

Results on the same task executed slower ($T = 5$ s) are shown in Figs. 4–5. In this case, the third link hardly leaves the vertical position, mimicking the natural balancing of a stick during a slow translation. The necessary torques are also quite reduced.

In Figs. 6–8 we report the results for a two-phase swing-up maneuver, from $(x_s, y_s, \theta_s) = (0.75, 1, -90°)$ to $(x_g, y_g, \theta_g) = (0.75, 1, 90°)$, with $T = 1.6$ s, $\xi_s = 0$, $\xi_g = -20$ m/s^2, and $\eta_s = \eta_g = 0$. As intermediate zero-velocity state we used $(x_m, y_m, \theta_m) = (0.75, 1, 0°)$, with $T_m = 1$ s, $\xi_m = -10$ m/s^2, and $\eta_m = 0$.

Analyzing in detail Fig. 6, we notice in phase I an initial small oscillation, in which the third link builds up kinetic energy, before the counterclockwise rotation to the horizontal position obtained with a slight upward motion. At the beginning of phase II, the third link inverts its angular velocity and executes a large clockwise swing of 270°.

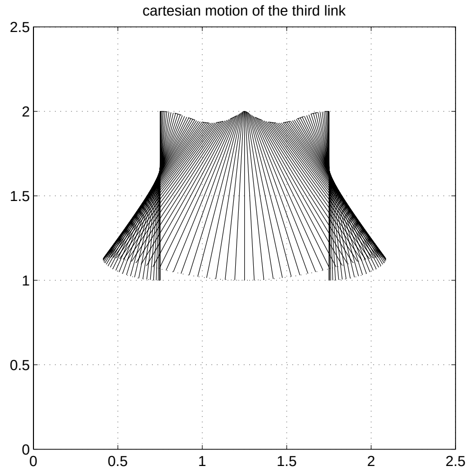

Figure 1: Transfer between inverted equilibria: Third link motion

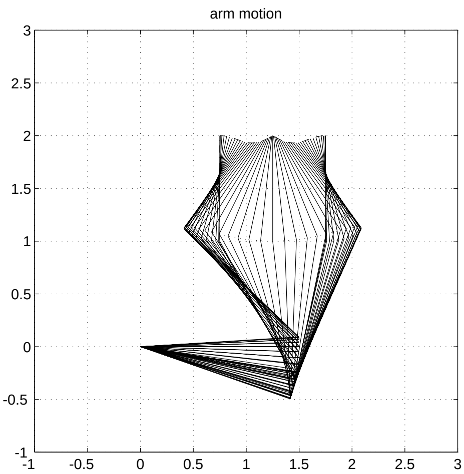

Figure 2: Transfer between inverted equilibria: 3R arm motion

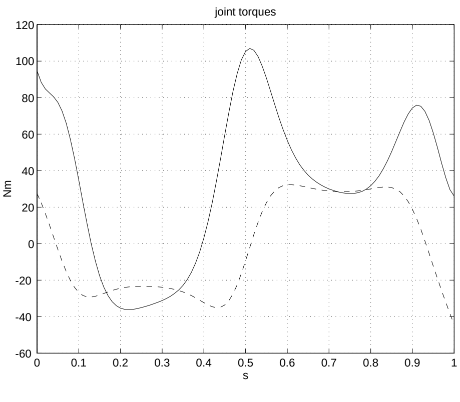

Figure 3: Transfer between inverted equilibria: Torques τ_1 (—) and τ_2 (− −)

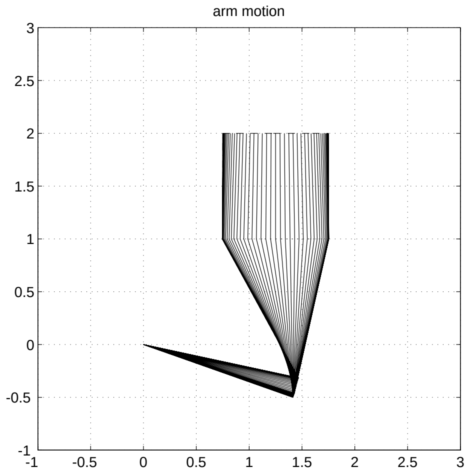

Figure 4: Slower transfer between inverted equilibria: 3R arm motion

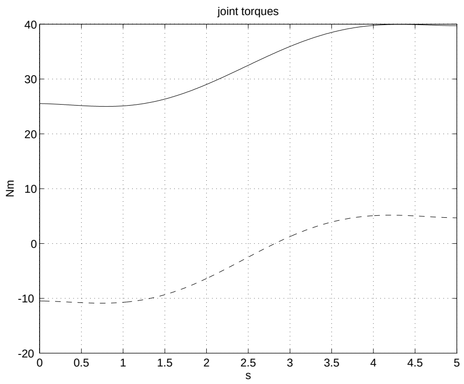

Figure 5: Slower transfer between inverted equilibria: Torques τ_1 (—) and τ_2 (−−)

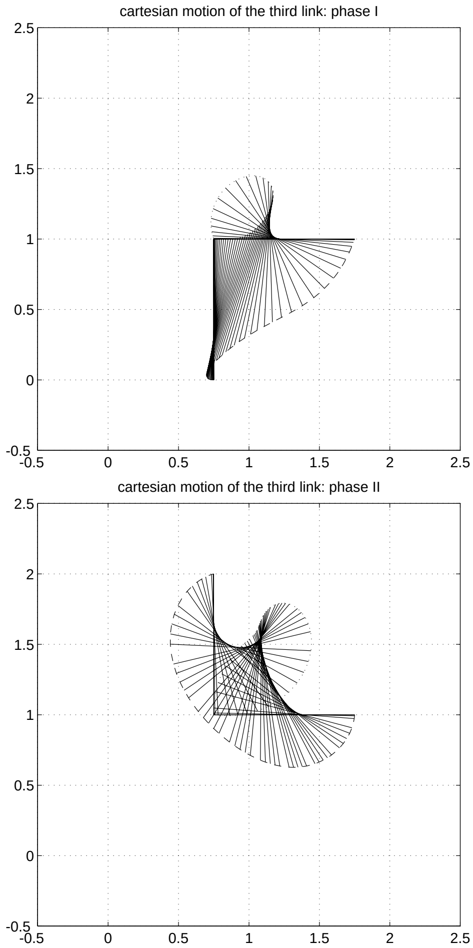

Figure 6: Swing-up maneuver: Third link motion in phase I (above) and in phase II (below)

The nominal torques of the 3R robot in Fig. 8, obtained for $m_1 = m_2 = 1$, $m_3 = 0.5$ (kg), are discontinuous at the phase transition instant. This is because the interpolating polynomials (13) can guarantee boundary continuity only up to the third derivative, while the torque depends on the fourth time derivatives of $y_1(t)$ and $y_2(t)$—see eqs. (7–8) and (9).

6 Conclusions

An effective motion planning method has been presented for the class of three-link planar robots with a passive rotational third joint moving under gravity. The technique is a direct extension of our previous work [11] and uses the relevant property that the position of the center of percussion of the third link becomes a linearizing output within a dynamic feedback linearization scheme. On the linear side of the problem, the design of feasible motions is reduced to independent interpolation problems for the two components of the center of percussion of the third link. Motion planning between equilibria is then easily accomplished, also in the case of swing-up maneuvers.

As a byproduct of this approach, we also mention the possibility [11] of designing exponentially stabilizing feedback controllers along the planned robot trajectories based on standard linear techniques.

The present work can be improved along several directions. A first benefit could be obtained from the separation of path synthesis and timing law generation: instead of using high-order time polynomials $y(t)$, which are prone to wandering, one can fit suitable low-order parametrized functions $y(s)$ between the start and goal configurations, satisfying geometric (directional) boundary conditions, and then use a

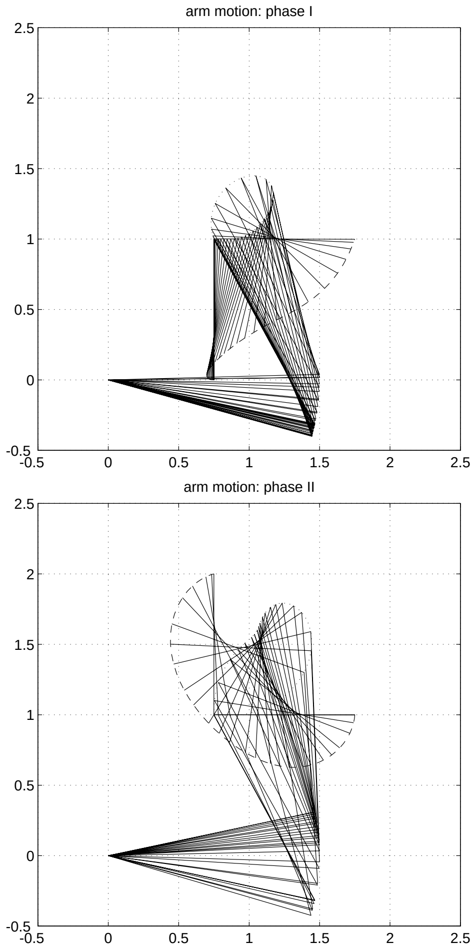

Figure 7: Swing-up maneuver: 3R arm motion in phase I (above) and in phase II (below)

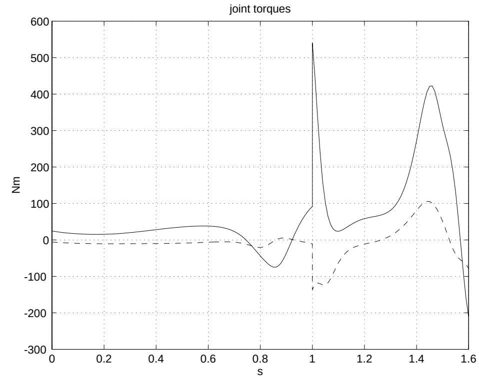

Figure 8: Swing-up maneuver: Torques τ_1 (—) and τ_2 (– –)

scalar time function $s(t)$ to satisfy the remaining dynamic conditions at the start and goal states. Another interesting possibility is to exploit the number of free choices in the proposed approach for optimizing energy or torque consumption during the motion.

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