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DECOUPLING AND FEEDBACK LINEARIZATION OF ROBOTS WITH MIXED RIGID/ELASTIC JOINTS

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SUMMARY

We consider some theoretical aspects of the control problem for rigid link robots having some joints rigid and some with non-negligible elasticity. We start from the reduced model of robots with all joints elastic introduced by Spong, which is linearizable by static feedback. For the mixed rigid/elastic joint case, we give structural necessary and sufficient conditions for input–output decoupling and full-state linearization via static state feedback. These turn out to be very restrictive. However, when a robot fails to satisfy these conditions, we show that a physically motivated dynamic state feedback will always guarantee the same result. The analysis is performed without resorting to the state-space equation format. As a result, the explicit form of the exact linearizing and input–output decoupling controllers is provided directly in terms of the robot dynamic model terms. © 1998 John Wiley & Sons, Ltd.

Key words: robot control; joint elasticity; input–output decoupling; feedback linearization; dynamic state feedback

1. INTRODUCTION

Joint flexibility is present in many current industrial robots. When harmonic drives, belts or long shafts are used as motion transmission elements, a dynamic displacement is introduced between the position of the driving actuators and that of the driven links, which is the output to be controlled. This small deflection concentrated at the robot joints is a major source of oscillatory problems, when accurate trajectory tracking or high sensitivity to cartesian forces is required.

The experimental findings of Reference 1 on the GE P-50 arm first showed that one should include joint elasticity in the model used for control design in order to overcome the above effects. An early study on the modelling of robots with joint elasticity can be found in Reference 2. A more detailed analysis of the model structure was later presented in Reference 3.

Motion control of robots with elastic joints represents a sort of benchmark problem for different control strategies. In general, the use of *linear* controllers for this class of robot arms is limited to quasi-static operations, as in the conventional design of Reference 4, or to regulation tasks—see, e.g. the Lyapunov-based approach of Reference 3. When considering the performance of *nonlinear* control techniques for trajectory tracking, such as feedback linearization or input–output decoupling, the first reported results were of negative flavour. In particular, it was

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shown in Reference 5 that a 3R elbow-type arm with *all* joints being elastic fails to satisfy the necessary conditions for linearization or for decoupling via static state feedback.

Since rigid joints are the limit case of a flexible behaviour, with stiffness going to infinity, the dynamic model of robots with elastic joints lend itself to a singularly perturbed format. Based on the intrinsic two-time-scale nature of the robot dynamics, several *approximate* tracking controllers have been proposed (see, e.g. Reference 6). Performance of these methods is quite satisfactory when the joints are sufficiently stiff.

A solution to the problem of *exact* tracking of smooth trajectories defined for the robot links (i.e. beyond the elasticity) can be obtained via *dynamic state feedback*. In this approach, there is no limitation on joint stiffness, which may be large or very small. The method was proposed first in Reference 7 for a 3R elbow-type arm, while a general approach to dynamic linearization and decoupling of robots with *all* joints being elastic has been described in Reference 8. Recently, it has been proven that robots with elastic joints are invertible nonlinear systems with no zero dynamics,[9] a sufficient condition for full linearization and decoupling via dynamic state feedback.[10] For the resulting linear and decoupled closed-loop system, the trajectory tracking problem is then easily solved.

Despite the above remarkable theoretical properties, the actual design of a linearizing dynamic state-feedback control is still very complex and it has been performed only on simple cases (robots with up to three joints). Furthermore, the physical obstruction asking for dynamic feedback is the presence of inertial coupling terms that are usually of small magnitude. These considerations have opened the way to simpler approaches with a similar provably good performance. A *reduced model* for robots with arbitrary joint elasticity was introduced in Reference 11, assuming that the kinetic energy of the electrical actuators is due only to their own rotor spinning (gyroscopic effects are neglected). This assumption is very reasonable, especially when the reduction gear ratios are large. It was shown in Reference 11 that this reduced model of robots with elastic joints satisfies the conditions for full-state linearization and decoupling via *static state feedback*.

Both dynamic models—the reduced and the complete one—have been used extensively by researchers working on the control of robots with joint elasticity. For instance, the reduced model was used to develop robust control schemes[12] and nonlinear adaptive laws with global convergence properties.[13] On the other hand, the complete model was used for studies on state observers[14] and on regulation and feedforward-based tracking controllers.[15,16] A review of the modelling assumptions and of various proposed control techniques can be found in Reference 17. It should be also pointed out that the distinction between complete and reduced models vanishes for some common robot kinematic arrangements.

In this paper, we consider a different class of robot arms having *some* joints that can be considered completely rigid and *some other* where elasticity is relevant. This mixed picture is the common rule rather than an exception for industrial arms (e.g. in the SCARA family), because the actuator/transmission mechanical design is usually different from joint to joint. Some kinematic arrangements with mixed joints have been studied in Reference 18, where it was found that dynamic state feedback is still needed for obtaining a linear and decoupled closed-loop behaviour. We generalize here the approach to the whole class of robot arms with mixed rigid/elastic joints and study the structural situations that arise from the point of view of applicability of feedback linearization and decoupling control techniques.

Differently from Reference 18, we will assume the same hypothesis of Reference 11 in the modelling phase. The purpose of this choice is twofold: (i) to obtain results based on a model that is as practical as possible; (ii) to show that the need of dynamic compensation for this class of robots is *not* related to the use of a complete or of a reduced model. We shall see that, in the presence of joints of mixed types, static state feedback may or may not suffice to obtain full-state linearization and input–output decoupling. In the latter case, a physically based dynamic feedback law will always solve the problem.

By avoiding the use of a state-space approach (i.e. working directly with the Euler–Lagrange dynamics), we will provide a complete answer to the question whether static or dynamic state feedback is needed, depending on the actual structure of the robot inertia matrix. Moreover, the closed-form expression of the required controllers can be given explicitly in terms of the inertial, centrifugal, Coriolis, and gravity components of the model, making the control implementation and verification steps easier.

The paper is organized as follows. The dynamic modelling issues are presented in Section 2. In Section 3, we show how to compute the nominal torque for reproducing a reference link trajectory for robots with mixed elastic/rigid joints. Linearization and decoupling via static feedback is treated in Section 4, while Section 5 is devoted to the use of dynamic feedback. In the concluding section, extensions and open problems are briefly discussed.

A preliminary version of this work was presented in Reference 19.

2. DYNAMIC MODELLING AND PRELIMINARIES

Consider an open-chained robot arm with n rigid links and all joints being elastic. Let q be the n-vector of link positions and θ the n-vector of actuator (viz., rotor of electrical motor) positions, as reflected through the gear reduction ratios. The difference $q_i - \theta_i$ is the ith joint deformation. The rotors of the actuators are modelled as uniform bodies having their centre of mass on the rotation axis, implying that both the inertia matrix and the gravity term in the dynamic model are independent from the motor position.

The *complete* dynamic model of a robot arm with *all elastic joints* is[3]

$$\begin{bmatrix} B(q) & B_1(q) \\ B_1^{\mathrm{T}}(q) & J \end{bmatrix} \begin{bmatrix} \ddot{q} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} C(q,\dot{q}) + C_1(q,\dot{\theta}) & C_2(q,\dot{q}) \\ C_3(q,\dot{q}) & 0 \end{bmatrix} \begin{bmatrix} \dot{q} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} g(q) \\ 0 \end{bmatrix} + \begin{bmatrix} K(q-\theta) \\ K(\theta - q) \end{bmatrix} = \begin{bmatrix} 0 \\ u \end{bmatrix} \quad (1)$$

where B is the matrix of the rigid arm, J is the constant diagonal inertia matrix of the actuators (reflected through the reduction ratios), matrix B_1 represents inertial couplings between actuators and links, C and C_i $(i = 1,2,3)$ are matrices of Christoffel symbols (related to Coriolis and centrifugal terms), g is the gravity vector of the rigid arm, K is the constant diagonal joint stiffness matrix, and u is the n-vector of torques provided by the actuators (performing work on θ). In general, the dynamic model (1) fails to be linearizable by static state feedback. It has been shown in Reference 9 that this model can always be linearized and, defining as output $y = q$, also input–output decoupled by means of a dynamic state feedback law.

Assuming that the regular part of the kinetic energy of each rotor is due only to its own spinning, the *reduced* dynamic model of a robot arm with *all elastic joints* introduced in Reference 11 is

$$B(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) + K(q-\theta) = 0 \quad (2)$$

$$J\ddot{\theta} + K(\theta - q) = u \quad (3)$$

where the dynamic terms left over in equations (2) and (3) are the same appearing in (1). In the following, we set for compactness $n(q, \dot{q}) = C(q, \dot{q})\dot{q} + g(q)$. Neglecting the inertial term B_1 in the system kinetic energy automatically yields $C_i = 0$ $(i = 1, 2, 3)$.[3] If, for some arrangements of the robot joint axes, the inertial couplings between links and motors vanish ($B_1 = 0$), the complete dynamic model collapses into the reduced one (no approximation needed). Moreover, when the gear reduction ratios become large (say, above hundred motor turns per link turn) there is no practical difference between the dynamic behaviour of equation (1) and of equations (2)–(3). The reduced model (2)–(3) can be transformed into a linear one through a static state feedback and a change of coordinates.[11] Moreover, this feedback transformation gives a decoupled input–output behaviour between the external input v and the output $y = q$, with uniform relative degrees equal to four.

Assume now that r out of the n robot joints are rigid while the remaining $n - r$ are elastic. By reordering the generalized coordinates, we can write the *reduced* dynamic model of a robot arm with *mixed rigid/elastic joints* as follows:

$$\begin{bmatrix} B_{rr}(q) & B_{re}(q) \\ B_{er}(q) & B_{ee}(q) \end{bmatrix} \begin{bmatrix} \ddot{q}_r \\ \ddot{q}_e \end{bmatrix} + \begin{bmatrix} n_r(q, \dot{q}) \\ n_e(q, \dot{q}) \end{bmatrix} + \begin{bmatrix} 0 \\ K_e(q_e - \theta_e) \end{bmatrix} = \begin{bmatrix} u_r \\ 0 \end{bmatrix} \tag{4}$$

$$J_e \ddot{\theta}_e + K_e(\theta_e - q_e) = u_e. \tag{5}$$

The r-vector q_r collects the generalized coordinates of the rigid joints, at which the position of the driven link equals the position of the motor, as reflected through the reduction ratio. The $(n - r)$-vector q_e contains the positions of the links driven by elastic joints, while the motor positions at these joints are collected in the $(n - r)$-vector θ_e. The inertia matrix, Coriolis, centrifugal, and gravity terms have been partitioned according to the dimensions of q_r and q_e. In particular, $B_{er} = B_{re}^T$. Being diagonal blocks of the symmetric inertia matrix, matrices B_{rr} and B_{ee} are always invertible. The torques produced by the motors at rigid and at elastic joints are labelled u_r and u_e, respectively. Finally, the constant diagonal matrices of joint stiffness K_e and actuator inertia J_e have now dimension $n - r$.

In equation (4), the blocks of the arm inertia matrix depend, in general, on the whole link position vector $q = (q_r, q_e)$. The particular arrangement of the robot kinematics may lead to special internal structures, in much the same way as in the model of a fully rigid robot. The structure of the partitioned blocks of the inertia matrix plays a main role in the following analysis.

3. INVERSE DYNAMICS COMPUTATION

By a natural choice of the output, we show first that system (4)–(5) is invertible from the input–output point of view. As a consequence, we can explicitly compute the nominal torque input that enables exact reproduction of a desired smooth output trajectory (the so-called inverse dynamics problem in robotics). The desired trajectory is specified in terms of link motion for the links driven through rigid joints as well as for those driven through elastic joints, i.e. $q^d(t) = (q_r^d(t), q_e^d(t))$.

The first set of equations in (4) can be directly used to determine the nominal torques to be applied at the *rigid* joints

$$u_r^d(t) = B_{rr}(q^d(t))\ddot{q}_r^d(t) + B_{re}(q^d(t))\ddot{q}_e^d(t) + n_r(q^d(t), \dot{q}^d(t)) \tag{6}$$

The desired trajectory $\theta_e^d(t)$ of the motors at the elastic joints is solved from the second set of equation in (4) as

$$\theta_e^d(t) = q_e^d(t) + K_e^{-1}(B_{er}(q^d(t))\ddot{q}_r^d(t) + B_{ee}(q^d(t))\ddot{q}_e^d(t) + n_e(q^d(t), \dot{q}^d(t))) \quad (7)$$

In order to compute the nominal torques to be applied at the *elastic* joints, we need to differentiate twice equation (7) and then replace $\ddot{\theta}_e$ in equation 5, evaluated along the trajectory. Using the shorthand notation $q^{(i)} = d^i q/dt^i$, this gives

$$u_e^d(t) = K_e(\theta_e^d(t) - q_e^d(t)) + J_e K_e^{-1}(B_{er}(q^d(t))q_r^{(4),d}(t) + B_{ee}(q^d(t))q_e^{(4),d}(t) + \alpha_e(q^d(t), \dot{q}^d(t), \ddot{q}^d(t), q^{(3),d}(t))) \quad (8)$$

where (dropping dependencies)

$$\alpha_e = 2(\dot{B}_{er}q_r^{(3)} + \dot{B}_{ee}q_e^{(3)}) + \ddot{B}_{er}\ddot{q}_r + \ddot{B}_{ee}\ddot{q}_e + \ddot{n}_e + K_e\ddot{q}_e$$

Equations (6) and (8) solve the inverse dynamics problem for robots with mixed rigid/elastic joints. The inverse system has no internal dynamics and this computation is purely algebraic. We remark that:

- The reference link trajectories should be smooth enough to guarantee perfect tracking in nominal conditions, when the initial state of the robot is matched with the initial conditions of the trajectory. In particular, the link trajectories associated to *both* rigid and elastic joint types should be, in general, *four times* differentiable. The higher order requirement for $q_r^d(t)$, instead of an expected value of *two*, results from the inertial cross-couplings represented by matrix B_{er}.
- Equations (6)–(8) show that the state trajectory and the input commands associated with a generic output trajectory can be fully determined, without any integration procedure, from the output function and its derivatives. This has been recently labelled as the differential flatness property of a system.[20] In particular, $y = (q_r, q_e)$ is a flat output for robots with mixed rigid/elastic joints. This property corresponds to the robot system being invertible from y and having no zero dynamics (see also Reference 9). The above developments indicate that such a property can be checked without resorting to state-space equations.
- The previous formulas are useful for deriving feedforward terms in a nonlinear regulator approach for trajectory tracking problems, to be combined with a local (linear) stabilizing feedback (see, e.g. Reference 15).

4. DECOUPLING AND LINEARIZATION VIA STATIC FEEDBACK

We analyse next the input–output decoupling properties of the mixed rigid/elastic joint robot model. In doing so, we will find that a *static* state feedback is sufficient to this purpose only for a special class of robots. As a by-product, the decoupling controller will also give linearity of the closed-loop dynamics.

Let the output be defined as

$$y = \begin{bmatrix} q_r \\ q_e \end{bmatrix} \quad (9)$$

Taking twice the derivative of (9) gives the link accelerations. These can be solved directly from equation (4) as

$$\ddot{y} = \begin{bmatrix} \ddot{q}_r \\ \ddot{q}_e \end{bmatrix} = \begin{bmatrix} [B_{rr} - B_{re}B_{ee}^{-1}B_{er}]^{-1} & 0 \\ [B_{ee} - B_{er}B_{rr}^{-1}B_{re}]^{-1}B_{er}B_{rr}^{-1} & 0 \end{bmatrix} \begin{bmatrix} u_r \\ u_e \end{bmatrix} + \gamma(q, \dot{q}, \theta_e) \tag{10}$$

$$= A(q)u + \gamma(q, \dot{q}, \theta_e)$$

where standard formulas for the inverse of a block partitioned matrix have been used. The two inverses in square brackets in equation (10) certainly exist, as Schur complements of the invertible matrices B_{ee} and B_{rr}, respectively, Reference 21, p. 656.

Since the first r rows of $A(q)$ are of full rank none of them is identically zero, being also rows of the *decoupling matrix* $\mathscr{A}(q)$. For the last $n - r$ rows of $A(q)$, we have

$$\text{rank}\{[B_{ee} - B_{er}B_{rr}^{-1}B_{re}]^{-1}B_{er}B_{rr}^{-1}\} = \text{rank}\{B_{er}\} \leqslant \min\{r, n - r\} \tag{11}$$

If $\text{rank}\{B_{er}\} > 0$ at least one of the last $n - r$ rows of $A(q)$ is non-zero, coinciding with a row of the decoupling matrix but being also linearly dependent from the first r rows. Thus, when $B_{er} \neq 0$ the decoupling matrix $\mathscr{A}(q)$ is always *singular*, violating the necessary condition for input–output decoupling by static state feedback.[22]

Consider then the case $B_{er} = 0$. The dynamic equations (4)–(5) become in this case

$$B_{rr}(q)\ddot{q}_r + n_r(q, \dot{q}) = u_r \tag{12}$$

$$B_{ee}(q)\ddot{q}_e + n_e(q, \dot{q}) + K_e(q_e - \theta_e) = 0 \tag{13}$$

$$J_e\ddot{\theta}_e + K_e(\theta_e - q_e) = u_e \tag{14}$$

As a stronger assumption, let the term $n_e(q, \dot{q})$ in equation (13) be independent from $\dot{q}_r$, i.e. $n_e = n_e(q, \dot{q}_e)$. As shown in Appendix A, this happens if and only if the diagonal blocks of the inertia matrix in equations (12)–(13) have a dependence of the form $B_{rr} = B_{rr}(q_r)$ and $B_{ee} = B_{ee}(q_e)$. In turn, this implies also that $n_r = n_r(q, \dot{q}_r)$. Therefore, we assume a complete *inertial separation* between the dynamics of the rigidly driven links and that of the elastically driven links.

Define the control law as

$$u_r = B_{rr}(q_r)v_r + n_r(q, \dot{q}_r) \tag{15}$$

$$u_e = J_eK_e^{-1}(B_{ee}(q_e)v_e + \alpha_s(q, \dot{q}, \ddot{q}_e, q_e^{(3)})) + K_e(\theta_e - q_e) \tag{16}$$

with v_r and v_e as new inputs, and where (dropping dependencies)

$$\alpha_s = (K_e + \ddot{B}_{ee})\ddot{q}_e + 2\dot{B}_{ee}q_e^{(3)} + \ddot{n}_e$$

In equation (16), the following expressions should be used in sequence for eliminating higher-order derivatives:

$$\ddot{q}_e = -B_{ee}^{-1}(q_e)(K_e(q_e - \theta_e) + n_e(q, \dot{q}_e)) \tag{17}$$

$$q_e^{(3)} = -B_{ee}^{-1}(q_e)(\dot{B}_{ee}(q_e)\ddot{q}_e + K_e(\dot{q}_e - \dot{\theta}_e) + \dot{n}_e(q, \dot{q}_e)) \tag{18}$$

Therefore, equations (15)–(16) can be implemented as a static feedback from the robot state of the form $u = \alpha(q, \dot{q}, \theta_e, \dot{\theta}_e) + \beta(q)v$. On the other hand, the assumptions on B_{rr} and B_{ee} are *necessary* to ensure that the term $\dot{n}_e(q, \dot{q}_e)$ in equation (18), and therefore $q_e^{(3)}$, will *not* depend on $\ddot{q}_r$. Such dependence would lead again to a singular decoupling matrix, thus invalidating the static feedback approach.

Applying control (15)–(16), a linear and input–output decoupled closed-loop system is obtained

$$\ddot{q}_r = v_r \tag{19}$$

$$q_e^{(4)} = v_e \tag{20}$$

This result is summarized in the following

Theorem 1

The dynamic model (4)–(5) of robots with mixed rigid/elastic joints, with output (9), can be input–output decoupled and fully linearized by static state feedback if and only if:

(i) $B_{er} = 0$;

(ii) $B_{rr} = B_{rr}(q_r)$ and $B_{ee} = B_{ee}(q_e)$.

Whenever it applies, the required static state feedback law is given by equations (15)–(16).

We remark that:

- The torque input u_r at rigid joints depends only on the external input v_r, while the torque input u_e at elastic joints may or may not depend on v_r, beside v_e. In fact, the actual decoupling matrix of the system is

$$\mathscr{A}(q) = \begin{bmatrix} B_{rr}(q_r) & 0 \\ [*] & J_e K_e^{-1} B_{ee}(q_e) \end{bmatrix}^{-1}$$

  where the block $[*]$ is different from zero if and only if the gravity force components acting on the elastically driven links depends also on q_r, i.e., $g_e = g_e(q_r, q_e)$ (see Appendix A).
- The computed torque technique for full rigid robots and the linearizing static state feedback control of Reference 11 for all elastic joint robots are recovered from equations (15)–(16) in the two extreme cases of $r = n$ and 0, respectively.

It is interesting to check how Theorem 1 applies to some examples of robot arms.

Example 1 (*PRP cylindrical robot*)

The arm inertia matrix takes on the diagonal structure

$$B(q) = \begin{bmatrix} b_{11} & 0 & 0 \\ 0 & b_{22}(q_3) & 0 \\ 0 & 0 & b_{33} \end{bmatrix}$$

When the second and third joints are of the same kind (rigid or elastic), the system can be completely linearized and input–output decoupled by static state feedback.

Example 2 (*3R elbow-type robot*)

The arm inertia matrix has the structure

$$B(q) = \begin{bmatrix} b_{11}(q_2, q_3) & 0 & 0 \\ 0 & b_{22}(q_3) & b_{23}(q_3) \\ 0 & b_{23}(q_3) & b_{33} \end{bmatrix}$$

If the base joint is rigid and the shoulder and elbow joints are elastic (or *vice versa*), this robot satisfies condition (i) of Theorem 1, but not condition (ii). If the shoulder and the elbow joints are of a different kind, both conditions are violated. Static feedback will then always fail for linearization and decoupling purposes.

Example 3 (*2P Cartesian robot*)

Consider a Cartesian robot moving on a horizontal plane, with the two prismatic joint axes forming a twist angle α. Let the first joint be rigid and the second elastic. The (constant) arm inertia matrix is

$$B = \begin{bmatrix} m_1 + m_2 & m_2 \cos\alpha \\ m_2 \cos\alpha & m_2 \end{bmatrix} = \begin{bmatrix} b_{\mathrm{rr}} & b_{\mathrm{re}} \\ b_{\mathrm{er}} & b_{\mathrm{ee}} \end{bmatrix}$$

with m_i being the mass of the ith link. Condition (ii) of Theorem 1 holds indeed, but condition (i) fails except for $\alpha = \pi/2$. On the other hand, the robot equations are already linear in this case

$$b_{\mathrm{rr}}\ddot{q}_1 + b_{\mathrm{re}}\ddot{q}_2 = u_1$$

$$b_{\mathrm{er}}\ddot{q}_1 + b_{\mathrm{re}}\ddot{q}_2 + k_2(q_2 - \theta_2) = 0$$

$$J_2\ddot{\theta}_2 + k_2(\theta_2 - q_2) = u_2$$

This implies that a static state feedback will not suffice to decouple the dynamics. As shown in the next section, a dynamic (here, *linear*) state feedback will allow to obtain this result.

As apparent from the above examples, the necessary and sufficient conditions of Theorem 1 are rather restrictive. The third example raises also the question on the necessity of these conditions for obtaining separately either input–output decoupling or full-state linearization alone. From the arguments proving Theorem 1, we can easily see that condition (i) is certainly necessary for input–output decoupling. On the other hand, nothing can be concluded on the necessity of either condition (i) or condition (ii) (or both) for achieving full-state linearization, independent from a specific output choice (see example in Appendix B).

5. DECOUPLING AND LINEARIZATION VIA DYNAMIC FEEDBACK

Consider the general case in which one or both conditions of Theorem 1 are violated. In these cases the decoupling matrix will always be singular. Input–output decoupling and full-state linearization may still be obtained, but we need to pursue a more general strategy based on *dynamic* state feedback. We will show this in a constructive and physically motivated way.

Define a linear dynamic feedback compensator for the inputs at the rigid joints

$$u_r = K_r(\theta_r - q_r) \tag{21}$$

$$J_r\ddot{\theta}_r + K_r(\theta_r - q_r) = u_{re} \tag{22}$$

with (arbitrary) diagonal matrices $K_r > 0$ and $J_r > 0$, and where the $2r$ states have been conveniently denoted as $\theta_{r,i}$ and $\dot{\theta}_{r,i}$ $(i = 1, \ldots, r)$. The above control structure minics the dynamic behaviour of fictitious elastic joint transmissions, with joint stiffnesses and motor inertias left as control design parameters. It requires a preliminary feedback from the position of those robot links which are driven by rigid joints.

The resulting extended system made by equations (4)–(5) and equations (21)–(22)

$$\begin{bmatrix} B_{rr}(q) & B_{re}(q) \\ B_{er}(q) & B_{ee}(q) \end{bmatrix} \begin{bmatrix} \ddot{q}_r \\ \ddot{q}_e \end{bmatrix} + \begin{bmatrix} n_r(q,\dot{q}) \\ n_e(q,\dot{q}) \end{bmatrix} + \begin{bmatrix} K_r(q_r - \theta_r) \\ K_e(q_e - \theta_e) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{23}$$

$$\begin{bmatrix} J_r & 0 \\ 0 & J_e \end{bmatrix} \begin{bmatrix} \ddot{\theta}_r \\ \ddot{\theta}_e \end{bmatrix} + \begin{bmatrix} K_r(\theta_r - q_r) \\ K_e(\theta_e - q_e) \end{bmatrix} = \begin{bmatrix} u_{re} \\ u_e \end{bmatrix} \tag{24}$$

is formally equivalent to the dynamic model (2)–(3) of robots with all joints elastic, by setting $J = \mathrm{diag}\{J_r, J_e\}$, $K = \mathrm{diag}\{K_r, K_e\}$, and $\theta = (\theta_r, \theta_e)$. Since this model is known to be input–output decoupable and fully linearizable by static state feedback,[11] the original one shares the same features under dynamic state feedback.

The linearizing dynamic controller can be obtained without making explicit use of the state-space format, by differentiating equation (2) (or equation (23)) twice with respect to time and setting $q^{(4)} = v$, a new input, and combining the result with equations (21)–(22). This gives for control $u = (u_r, u_e)$

$$u_r = K_r(\theta_r - q_r) \tag{25}$$

$$\begin{bmatrix} u_{re} \\ u_e \end{bmatrix} = JK^{-1}(B(q)v + \alpha_d(q, \dot{q}, \ddot{q}, q^{(3)})) + K(\theta - q) \tag{26}$$

$$J_r\ddot{\theta}_r + K_r(\theta_r - q_r) = u_{re} \tag{27}$$

where (dropping dependencies)

$$\alpha_d = (K + \ddot{B})\ddot{q} + 2\dot{B}q^{(3)} + \ddot{n}$$

The nominal dependence of α_d in equation (26) on $\ddot{q}$ and $q^{(3)}$ can be transformed into a dependence on $q, \dot{q}, \theta_e$ (the original state of the robot) and on θ_r and $\dot{\theta}_r$ (the state of the dynamic compensator) by means of

$$\ddot{q} = -B^{-1}(q)(K(q - \theta) + n(q, \dot{q})) \tag{28}$$

$$q^{(3)} = -B^{-1}(q)(\dot{B}(q)\ddot{q} + K(\dot{q} - \dot{\theta}) + \dot{n}(q, \dot{q})) \tag{29}$$

used in the proper sequence.

This result is summarized in the following:

Theorem 2

When the conditions of Theorem 1 are violated, the dynamic model (4)–(5) of robots with mixed rigid/elastic joints, with output (9), can be input–output decoupled and fully linearized by a dynamic state feedback of dimension $2r$, where r is the number of rigid joints. The required dynamic state feedback law is given by equations (25)–(27).

Therefore, the role of dynamic feedback in robots with mixed rigid/elastic joints appears to be the balancing of input–output relative degrees of all channels (uniformly equal to four), so that the robot behaves as a fully elastic joint one. We stress, however, that the analysis performed in Section 4 indicates that such a balancing is not needed if the original robot dynamics is inertially separated between rigid joint and elastic joint components.

6. CONCLUSIONS

We have analysed the theoretical aspects of the problem of input–output decoupling and full-state linearization by feedback for robots having some joints rigid and some elastic. It was proved that static state feedback works successfully if and only if *no* couplings are present in the robot inertia matrix between the variables of the rigid joints and those of the elastic joints. When these rather strict conditions are not met, dynamic state feedback will guarantee a linear and decoupled closed-loop dynamics. In this case, the dimension of the dynamic compensator is always equal to $2r$, where r (with $0 < r < n$) is the number of rigid joints.

When needed, the dynamic feedback control method should be considered as dual to a singular perturbation approach. In the latter, the time-scale difference between the rigid and the elastic motion is handled through an approximate analysis so as to have a reduced set of 'rigid' robot equations. Here, dynamic feedback is used to 'soften' the rigid transmissions inducing an elastic-like motion for all robot joints. Decoupling is achieved by slowing down the effects of torques applied at the rigid joints with respect to those acting on the motor side of the elastic joints.

All results were obtained using, for the elastic joint modelling part, the reduced model of Spong[11] which neglects small inertial terms. Indeed, a similar analysis can be performed starting from the complete dynamic model of robots with mixed joints. This extension is not trivial and requires further exploitation of the upper triangular structure of the matrix B_1 representing the actuator-link inertial couplings (see, e.g. Reference 23). In any case, for mechanical systems the direct use of the Euler–Lagrange equations, as done in this paper, simplifies considerably the analysis and synthesis of nonlinear feedback controllers.

Finally, when the conditions of Theorem 1 fail, the question of whether the dynamics of a robot with mixed rigid/elastic joints can be exactly linearized by static state feedback is still open. If we wish to avoid a case-by-case study based on the standard involutivity conditions defined on the system state equations, further investigation on the dynamic properties of the robot model is needed.

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APPENDIX A

We show here that independence of $n_{\mathrm{e}}(q, \dot{q})$ from $\dot{q}_{\mathrm{r}}$ in the dynamic model (12)–(14) is equivalent to assumption (ii) of Theorem 1. For this, the definition of Christoffel symbols for the Coriolis and centrifugal terms will be used.

In equations (12)–(13), let

$$B(q) = \begin{bmatrix} B_{\mathrm{rr}}(q) & 0 \\ 0 & B_{\mathrm{ee}}(q) \end{bmatrix} = [b_1(q) \; \cdots \; b_n(q)]$$

and

$$n(q,\dot{q}) = \begin{bmatrix} n_r(q,\dot{q}) \\ n_e(q,\dot{q}) \end{bmatrix} = \begin{bmatrix} c_r(q,\dot{q}) \\ c_e(q,\dot{q}) \end{bmatrix} + \begin{bmatrix} g_r(q) \\ g_e(q) \end{bmatrix} = c(q,\dot{q}) + g(q) \tag{30}$$

Dependence on generalized velocities $\dot{q}$ arises only within the components of vector $c(q,\dot{q})$ of Coriolis and centrifugal terms, which are computed from the columns and elements of the inertia matrix $B(q)$ as

$$c_i(q,\dot{q}) = \tfrac{1}{2}\dot{q}^{\mathrm{T}} C_i(q)\dot{q}$$

$$C_i(q) = \left(\frac{\partial b_i(q)}{\partial q}\right) + \left(\frac{\partial b_i(q)}{\partial q}\right)^{\mathrm{T}} - \left(\frac{\partial B(q)}{\partial q_i}\right), \quad i = 1, \ldots, n$$

For $i \in \{r+1, \ldots, n\} \equiv e$, independence of $c_i(q,\dot{q})$ from $\dot{q}_r$ implies

$$c_i = \tfrac{1}{2}[\dot{q}_r^{\mathrm{T}} \quad \dot{q}_e^{\mathrm{T}}] \begin{bmatrix} C_{i,rr} & C_{i,er}^{\mathrm{T}} \\ C_{i,er} & C_{i,ee} \end{bmatrix} \begin{bmatrix} \dot{q}_r \\ \dot{q}_e \end{bmatrix} = \tfrac{1}{2}\dot{q}_e^{\mathrm{T}} C_{i,ee}\dot{q}_e \tag{31}$$

On the other hand, from $b_i = [0^{\mathrm{T}} \quad b_{ee,i}^{\mathrm{T}}]^{\mathrm{T}}$ we have

$$C_i = \begin{bmatrix} 0 & \left(\dfrac{\partial b_{ee,i}}{\partial q_r}\right)^{\mathrm{T}} \\ \dfrac{\partial b_{ee,i}}{\partial q_r} & \dfrac{\partial b_{ee,i}}{\partial q_e} + \left(\dfrac{\partial b_{ee,i}}{\partial q_r}\right)^{\mathrm{T}} \end{bmatrix} - \begin{bmatrix} \dfrac{\partial B_{rr}}{\partial q_i} & 0 \\ 0 & \dfrac{\partial B_{ee}}{\partial q_i} \end{bmatrix} \tag{32}$$

so that

$$\frac{\partial b_{ee,i}}{\partial q_r} = 0 \Leftrightarrow B_{ee} = B_{ee}(q_e)$$

$$\frac{\partial B_{rr}}{\partial q_i} = 0 \ (i \in e) \Leftrightarrow B_{rr} = B_{rr}(q_r)$$

From this equivalence, using similar arguments, it follows also that $n_r = n_r(q, \dot{q}_r)$.

In addition, under the assumptions of Theorem 1, the dependence of Coriolis and centrifugal terms on configuration variables is also particularized. From equations (31) and (32), we have, in fact, $c_e = c_e(q_e, \dot{q}_e)$. Similarly, $c_r = c_r(q_r, \dot{q}_r)$. Nonetheless, in the expressions of the control law (15)–(16) we have left a general dependence on $q = (q_r, q_e)$ in both n_r and n_e, because there is no restriction on the contribution of the gravitational terms $g(q)$ to $n(q,\dot{q})$ (see equation (30)).

APPENDIX B

A simple example of a robot with mixed rigid/elastic joints violating condition (i) of Theorem 1 is given, whose nonlinear dynamics can still be linearized via static state feedback.

Consider a PR robot moving on a horizontal plane, with the first (prismatic) joint elastic and the second (rotational) joint rigid. Let θ_1 be the motor position of the first joint, q_1 the first-link position, and q_2 the second-link position. With the general notation used in the paper, we have $q_r = q_2$, $q_e = q_1$, and $\theta_e = \theta_1$ (all scalar). From the Euler–Lagrange equations, the dynamic model organized in the form (4)–(5) is

$$\begin{bmatrix} b_1 & -b_2 \sin q_2 \\ -b_2 \sin q_2 & b_3 \end{bmatrix} \begin{bmatrix} \ddot{q}_2 \\ \ddot{q}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ -b_2 \cos q_2 \dot{q}_2^2 \end{bmatrix} + \begin{bmatrix} 0 \\ k_1(q_1 - \theta_1) \end{bmatrix} = \begin{bmatrix} u_2 \\ 0 \end{bmatrix}$$

$$J_1 \ddot{\theta}_1 + k_1(\theta_1 - q_1) = u_1$$

with

$$b_1 = J_2 + I_2 + m_2 d_2^2, \quad b_2 = m_2 d_2, \quad b_3 = m_1 + m_2 + m_{m2}$$

where m_i is the mass of the ith link, J_i is the effective inertia of the rotor of the ith motor, m_{m2} is the mass of the second motor, I_2 is the baricentral inertia of the second link, $d_2 > 0$ is the distance of the center of mass of the second link from the second joint axis, and k_1 is the finite stiffness of the first joint. The above dynamic model is also the *complete* one, since the second motor does not rotate due to the linear motion of the first prismatic joint; thus, no simplifying assumptions are introduced in this case.

Let

$$x = [q_1 \;\; q_2 \;\; \theta_1 \;\; \dot{q}_1 \;\; \dot{q}_2 \;\; \dot{\theta}_1]^{\mathrm{T}} \in \mathbb{R}^6, \quad u = [u_1 \quad u_2]^{\mathrm{T}} \in \mathbb{R}^2$$

The state-space representation of the system is

$$\dot{x} = f(x) + \sum_{i=1}^{2} g_i(x) u_i$$

with

$$f(x) = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \\ b_1\delta(x_2)[b_2c_2x_5^2 + k_1(x_3 - x_1)] \\ b_2s_2\delta(x_2)[b_2c_2x_5^2 + k_1(x_3 - x_1)] \\ j_1k_1(x_1 - x_3) \end{bmatrix} \quad g_1(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ j_i \end{bmatrix} \quad g_2(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ b_2s_2\delta(x_2) \\ b_3\delta(x_2) \\ 0 \end{bmatrix}$$

where we defined $s_2 = \sin x_2$, $c_2 = \cos x_2$, $j_1 = 1/J_1$, and $\delta(x_2) = 1/(b_1b_3 - b_2^2s_2^2) > 0$.

It is lengthy but straightforward to verify that the conditions for feedback linearization via static state feedback (see, e.g., [22, Theorem 5.2.3, p. 233]) are satisfied. For, we need to compute a sequence of distributions G_i $(i = 0, 1, \ldots, 4)$ and check for constant dimension and involutivity at x^0. Denoting by $\mathrm{ad}_f g = [f, g]$ the Lie Bracket of two vector fields f and g, with $\mathrm{ad}_f^{k+1} g = [f, \mathrm{ad}_f^k g]$ $(k = 1, 2, \ldots)$, we have:

$$\dim(G_0) = \dim(\mathrm{span}\{g_1, g_2\}) = 2, \qquad G_0 \text{ involutive}, \quad \forall x,$$

$$\dim(G_1) = \dim(\mathrm{span}\{g_1, g_2, \mathrm{ad}_f g_2\}) = 3, \quad G_1 \text{ involutive}, \quad \forall x$$

Moreover, it can be shown that

$$\dim(G_i) = \dim(\mathrm{span}\,\{g_1, g_2, \mathrm{ad}_f g_2, \ldots, \mathrm{ad}_f^i g_2\}) = i + 2, \quad i = 2, 3, 4,$$

and G_2, G_3, and G_4 are involutive in the neighborhood of a generic point x^0, being

$$\mathrm{ad}_f g_2 = \begin{bmatrix} -b_2s_2\delta(x_2) \\ -b_3\delta(x_2) \\ 0 \\ x_5\phi_4(x_2) \\ x_5\phi_5(x_2) \\ 0 \end{bmatrix} \quad \mathrm{ad}_f^2 g_2 = \begin{bmatrix} -x_5\psi_1(x_2) \\ -x_5\psi_2(x_2) \\ 0 \\ \psi_4(x_1, x_2, x_3, x_5) \\ \psi_5(x_1, x_2, x_3, x_5) \\ j_1k_1b_2s_2\delta(x_2) \end{bmatrix}$$

$$\mathrm{ad}_f^3 g_2 = \begin{bmatrix} \sigma_1(x_1, x_2, x_3, x_5) \\ \sigma_2(x_1, x_2, x_3, x_5) \\ -j_1k_1b_2s_2\delta(x_2) \\ \sigma_4(x) \\ \sigma_5(x) \\ j_1k_1x_5[\psi_1(x_2) + b_2\partial(s_2\delta(x_2))/\partial x_2] \end{bmatrix} \quad \mathrm{ad}_f^4 g_2 = \begin{bmatrix} \omega_1(x) \\ \omega_2(x) \\ -j_1k_1x_5[\psi_1(x_2) + 2b_2\partial(s_2\delta(x_2))/\partial x_2] \\ \omega_4(x) \\ \omega_5(x) \\ \omega_6(x_1, x_2, x_3, x_5) \end{bmatrix}$$

with suitable functions ϕ_i, ψ_i, σ_i, and ω_i.