

Steering a Class of Redundant Mechanisms Through End-Effector Generalized Forces

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Abstract—A particular class of underactuated systems is obtained by considering kinematically redundant manipulators for which all joints are passive and the only available inputs are forces/torques acting on the end-effector. Under the assumption that the degree of redundancy is provided by prismatic joints located at the base, we address the problem of steering the robot between two arbitrary equilibrium configurations. By performing a preliminary partial feedback linearization, the dynamic equations take a convenient triangular form, which is further simplified under additional hypotheses. We give sufficient conditions for controllability of this kind of mechanisms. With a PPR robot as a case study, an algorithm is proposed for computing end-effector commands that produce the desired reconfiguration in finite time. Simulation results and a discussion on possible generalizations are given.

Index Terms—Holonomy angle, nonholonomic constraints, open-loop steering, redundant robots, underactuated mechanisms.

I. INTRODUCTION

In this paper, the control problem is addressed for a special class of underactuated mechanical systems with n generalized coordinates and $m < n$ control inputs. In particular, we are interested in controlling the configuration of manipulators that are kinematically redundant with respect to the task of positioning/orienting their end-effector, under the assumption that all joints are passive and the only available control inputs are forces/torques to be applied on the manipulator tip. Note that for conventional (nonredundant) robots such problem would be trivial, because there is a one-to-one mapping between end-effector and joint commands, provided that kinematic singularities are avoided.

The above problem may be of interest in multirobot cooperative systems, when one of the robots undergoes an energy failure thereby becoming a passive mechanism, or in robot assisted surgery, where the surgeon drives the instrumented end-effector of a passive or semi-active manipulator. Another example is given by superarticulated structures without intrinsic actuation, for which it is convenient to explore the possibility of steering the redundant arm to a desired configuration using only an external device (e.g., another robot).

In general, underactuated mechanical systems may be controllable via either kinematic or dynamic coupling. Typical examples of the first category are provided by first-order nonholonomic systems, such as wheeled mobile robots under pure rolling constraints [1]. The equations of these systems are nonlinear and driftless when generalized velocities are taken as control inputs. As a consequence, controllability of the linear approximation is lost and smooth time-invariant stabilization is not possible [2], so that the use of standard nonlinear control techniques is ruled out. Open-loop controls have been synthesized using special canonical forms (e.g., chained forms [1]), while the feedback stabilization problem has been solved using

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time-varying [3] and/or discontinuous feedback [4], [5]. In particular, the kinematic version of our problem (namely, the reconfiguration of a redundant arm driven by end-effector velocity commands under a particular inverse kinematic map) has been solved in [6]. Related work includes [7], which addresses the problem of arbitrarily positioning an object in the plane by pushing along a limited set of directions, under quasistatic assumptions.

The mechanisms considered in this paper belong to the second category, in which the inertial couplings among the various degrees of freedom play an essential role for control. In general, such systems inherit the limitations of underactuated kinematic systems, and in particular the lack of linear controllability. Notable exceptions are overhead cranes [8], gymnast robots (e.g., the Acrobot and the Pendubot [9], [10]), and manipulators with flexible elements [11]. These systems are in fact linearly controllable by virtue of gravity and/or elastic forces. This valuable property is lost when the drift term in the system equations vanishes as the generalized velocities go to zero.

The transposition to the dynamic case of control techniques devised for underactuated kinematic systems is made difficult by the presence of a nontrivial drift term in the model. As a result, there does not exist any well-established solution approach for our reconfiguration problem. Another way to appreciate the complexity of the control problem for underactuated dynamic mechanisms is to consider that these are subject to second-order differential constraints which are not integrable.

In order to perform a complete analysis and provide a solution, we restrict our attention to a specific kind of redundant mechanisms. In particular, we assume that the first $n - m$ joints of the structure are prismatic. This assumption is satisfied for robotic manipulators mounted on a track or on an omnidirectional mobile base, because the mobility of the base can be modeled by prismatic joints. Moreover, also many devices for robot assisted surgery have additional prismatic joints at the base in order to enlarge their workspace (see [12], [13], and the references therein). A further, although simpler, example is given by the planar motion of a rigid body subject to two noncolinear control forces applied at a point different from the center of mass [14].

The paper is organized as follows. In Section II, a partial feedback linearization procedure is presented which allows to obtain a triangular structure for the system equations. Based on this, in Section III, we give general sufficient conditions for controllability, together with some examples of application. In Section IV, it is shown that—in the absence of gravity—the mechanisms under study are neither controllable in the linear approximation nor smoothly stabilizable. An algorithm is then proposed for computing open-loop end-effector commands that achieve a desired reconfiguration in finite time. This is illustrated in Section V by using a planar PPR robot driven by two cartesian forces as a simulation case study. In the concluding section, we briefly outline possible extensions of the proposed approach.

II. PARTIAL FEEDBACK LINEARIZATION

Consider a robotic manipulator with n joints, whose end-effector pose is described by m variables, being $n - m > 0$ the degree of kinematic redundancy. Throughout the paper, we assume that the $n - m$ proximal joints (i.e., those located at the base) are prismatic, while the m distal joints may be prismatic or revolute. For convenience, reorder and partition the joint vector $q \in \mathbb{R}^n$ as (q_a, q_b) , where $q_b \in \mathbb{R}^{n-m}$ are the variables of the proximal joints and $q_a \in \mathbb{R}^m$ are those of the distal joints.

Following the Lagrangian approach, the dynamic model of the system can be written as

$$B(q_a)\ddot{q} + c(q_a, \dot{q}_a) + e(q_a) = J^T(q_a)F \quad (1)$$

where B is the $n \times n$ inertia matrix, c and e are, respectively, the n -vectors of centrifugal/Coriolis terms and of gravitational terms, J is the $m \times n$ geometric Jacobian of the robot, and F is the m -vector of generalized forces acting on the end-effector. Note that, in view of the considered mechanical structure, all the above terms do not depend on q_b and \dot{q}_b .

According to the partition of the joint vector, the dynamic model (1) can be rewritten as

$$\begin{bmatrix} B_{aa} & B_{ab} \\ B_{ab}^T & B_{bb} \end{bmatrix} \begin{bmatrix} \ddot{q}_a \\ \ddot{q}_b \end{bmatrix} + \begin{bmatrix} h_a \\ h_b \end{bmatrix} = \begin{bmatrix} J_a^T \\ J_b^T \end{bmatrix} F \quad (2)$$

having set $h(q_a, \dot{q}_a) = c(q_a, \dot{q}_a) + e(q_a)$. Left-multiplying (2) by the nonsingular matrix

$$T = \begin{bmatrix} I_m & -B_{ab}B_{bb}^{-1} \\ -B_{ab}^TB_{aa}^{-1} & I_{n-m} \end{bmatrix}$$

where I_p is the $p \times p$ identity matrix, one obtains

$$\begin{bmatrix} \tilde{B}_{aa} & O \\ O^T & \tilde{B}_{bb} \end{bmatrix} \begin{bmatrix} \ddot{q}_a \\ \ddot{q}_b \end{bmatrix} + \begin{bmatrix} \tilde{h}_a \\ \tilde{h}_b \end{bmatrix} = \begin{bmatrix} \tilde{J}_a^T \\ \tilde{J}_b^T \end{bmatrix} F$$

being O an $m \times (n - m)$ matrix with zero entries.

Assume now that

$$\text{rank} \begin{bmatrix} J_a^T & B_{ab} \\ J_b^T & B_{bb} \end{bmatrix} = n. \quad (3)$$

This condition turns out to be quite weak and generically satisfied, due to the mixing of kinematic and dynamic parameters.

Equation (3) implies that \tilde{J}_a is always nonsingular. In fact, being

$$\tilde{J}_a^T = [I_m \quad -B_{ab}B_{bb}^{-1}]J^T = (J_a^T - B_{ab}B_{bb}^{-1}J_b^T)$$

one has [15, p. 650]

$$\det \tilde{J}_a = \det \begin{bmatrix} J_a^T & B_{ab} \\ J_b^T & B_{bb} \end{bmatrix} \cdot \det B_{bb}^{-1} \neq 0.$$

If the end-effector generalized forces F are chosen as the partially linearizing feedback law

$$F = \tilde{J}_a^{-T}(\tilde{B}_{aa}u + \tilde{h}_a) \quad (4)$$

with $u \in \mathbb{R}^m$ an auxiliary input vector, the dynamic equations take the form

$$\ddot{q}_a = u, \quad (5)$$

$$\begin{aligned} \ddot{q}_b &= \tilde{B}_{bb}^{-1}(\tilde{J}_b^T \tilde{J}_a^{-T} \tilde{h}_a - \tilde{h}_b) + \tilde{B}_{bb}^{-1} \tilde{J}_b^T \tilde{J}_a^{-T} \tilde{B}_{aa} u \\ &= \tilde{f}(q_a, \dot{q}_a) + \tilde{G}(q_a)u = \tilde{f}(q_a, \dot{q}_a) + \sum_{i=1}^m \tilde{g}_i(q_a)u_i. \end{aligned} \quad (6)$$

Note the *triangular* structure of these equations: the acceleration of the joint variables q_b is not influenced by the values of q_b and \dot{q}_b . We shall refer to vector field \tilde{f} as the *acceleration drift*.

A further simplification of the triangular form (5) and (6) occurs

when the acceleration drift is zero. In this case, the dynamic equations become

$$\ddot{q}_a = u, \quad (7)$$

$$\ddot{q}_b = \sum_{i=1}^m \tilde{g}_i(q_a)u_i. \quad (8)$$

Henceforth, this system is called a *second-order Čaplygin form*, extending the definition of [16]. The acceleration \ddot{q}_b depends on q_a and u , but neither on \dot{q}_a nor—as already in (6)—on q_b , \dot{q}_b . We have the following result.

Proposition 1: The dynamic equations of the system take the second-order Čaplygin form (7) and (8) if and only if

$$c \in \mathcal{R}(J^T) \quad \text{and} \quad e \in \mathcal{R}(J^T) \quad (9)$$

where $\mathcal{R}(\cdot)$ denotes the range space of a matrix.

Proof: The sufficiency is shown first. Using (6), the acceleration drift is rewritten as

$$\tilde{f} = \tilde{B}_{bb}^{-1}[\tilde{J}_b^T \tilde{J}_a^{-T} \quad -I_{n-m}] \tilde{h} = \tilde{B}_{bb}^{-1}R(P - I_n)(c + e)$$

having set

$$\begin{aligned} R &= [-B_{ab}^TB_{aa}^{-1} \quad I_{n-m}], \\ P &= J^T \left([I_m \quad -B_{ab}B_{bb}^{-1}]J^T \right)^{-1} [I_m \quad -B_{ab}B_{bb}^{-1}]. \end{aligned}$$

It may be easily verified that matrix P is idempotent and of rank m . Moreover, the null space of $(P - I_n)$ coincides with the range space of J^T , since

$$\begin{aligned} v \in \mathcal{N}(P - I_n) &\iff v \perp \mathcal{N}(P^T) \iff v \perp \mathcal{N}(J) \\ &\iff v \in \mathcal{R}(J^T) \end{aligned}$$

where $\mathcal{N}(\cdot)$ denotes the null space of a matrix and we have used the properties of matrix P . Hence, $c + e \in \mathcal{R}(J^T)$ implies $\tilde{f} = 0$.

As for the necessity, for \tilde{f} to be zero in (6) one must have

$$c + e \in \mathcal{N}[R(P - I_n)].$$

It is easy to see that the null space of $R(P - I_n)$ coincides with the null space of $(P - I_n)$, i.e., with the range space of J^T , and therefore $c + e \in \mathcal{R}(J^T)$. Since c depends on the joint velocities \dot{q} , while e is a configuration-dependent vector, they must separately belong to $\mathcal{R}(J^T)$.

We conclude this section with two illustrative examples.

A. Example 1: PRR Robot

Consider the PRR robot of Fig. 1, with one prismatic and two revolute joints, that moves on the horizontal plane [$e(q) = 0$]. This manipulator is redundant for the task of positioning the end-effector in the plane ($n - m = 1$). The control input of the system is the two-dimensional vector F of cartesian forces acting on the end-effector.

Let m_i be the mass of the i th link ($i = 1, 2, 3$). For the j th link ($j = 2, 3$), let ℓ_j , d_j and \mathcal{I}_j be, respectively, its length, the distance between its center of mass and the j th joint axis, and its central moment of inertia. The dynamic model of this robot under

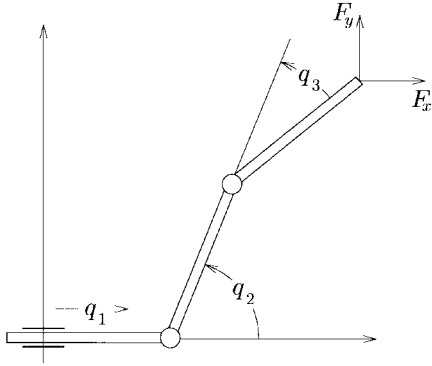


Fig. 1. A planar PRR robot.

end-effector commands is

$$B(q_2, q_3)\ddot{q} + c(q_2, q_3, \dot{q}_2, \dot{q}_3) = J^T(q_2, q_3)F$$

where

$$B(q_2, q_3) = \begin{bmatrix} a_1 & -a_5 s_2 - a_4 s_{23} & -a_4 s_{23} \\ -a_5 s_2 - a_4 s_{23} & a_2 + a_3 + 2a_4 \ell_2 c_3 & a_3 + a_4 \ell_2 c_3 \\ -a_4 s_{23} & a_3 + a_4 \ell_2 c_3 & a_3 \end{bmatrix}$$

$$c(q_2, q_3, \dot{q}_2, \dot{q}_3) = \begin{bmatrix} -(a_5 c_2 + a_4 c_{23})\dot{q}_2^2 - a_4 c_{23}\dot{q}_3(\dot{q}_2 + \dot{q}_3) \\ -a_4 \ell_2 s_3 \dot{q}_3(2\dot{q}_2 + \dot{q}_3) \\ a_4 \ell_2 s_3 \dot{q}_2^2 \end{bmatrix}$$

$$J(q_2, q_3) = \begin{bmatrix} 1 & -\ell_2 s_2 - \ell_3 s_{23} & -\ell_3 s_{23} \\ 0 & \ell_2 c_2 + \ell_3 c_{23} & \ell_3 c_{23} \end{bmatrix}$$

with the notation $s_2 = \sin q_2$, $c_2 = \cos q_2$, $s_{23} = \sin(q_2 + q_3)$, $c_{23} = \cos(q_2 + q_3)$, and

$$\begin{aligned} a_1 &= m_1 + m_2 + m_3 \\ a_2 &= \mathcal{I}_2 + m_2 d_2^2 + m_3 \ell_2^2 \\ a_3 &= \mathcal{I}_3 + m_3 d_3^2 \\ a_4 &= m_3 d_3 \\ a_5 &= m_2 d_2 + m_3 \ell_2. \end{aligned}$$

Both the inertia and the Jacobian matrices do not depend on q_1 , i.e., the prismatic joint variable. By partitioning the configuration vector into $q_a = (q_2, q_3)$ and $q_b = q_1$, it may be verified that the rank condition (3) holds almost everywhere, with the exception of the one-dimensional surface in the q_a -space defined by

$$a_1 \ell_2 \ell_3 s_3 + a_5 \ell_3 s_2 c_{23} - a_4 \ell_2 c_2 s_{23} = 0.$$

Since vector c does not belong to $\mathcal{R}(J^T)$, Prop. 1 indicates that, by using the feedback control (4), the dynamic equations of the robot will assume the form (5) and (6) with a nonzero acceleration drift \tilde{f} . For example, assuming $\ell_2 = \ell_3 = 1$ m, $m_i = 1$ kg ($i = 1, 2, 3$), uniform mass distribution for the links, and link shapes such that $\mathcal{I}_2 = \mathcal{I}_3 = 1$ kg·m², the following model of local validity is obtained after partial feedback linearization:

$$\begin{aligned} \ddot{q}_a &= \begin{bmatrix} \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ \ddot{q}_b &= \ddot{q}_1 = \tilde{f}(q_2, q_3, \dot{q}_2, \dot{q}_3) + \tilde{G}(q_2, q_3)u \\ &= \frac{2s_3(2c_2\dot{q}_2 - c_{23}\dot{q}_3)\dot{q}_2}{4s_3 + \sin(2q_2 + q_3)} + \frac{4c_2c_3}{4s_3 + \sin(2q_2 + q_3)}u_1, \end{aligned} \quad (10)$$

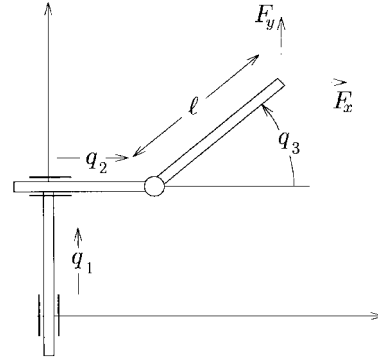


Fig. 2. A planar PPR robot.

B. Example 2: PPR Robot

Consider a PPR robot, having two prismatic and one revolute joint, moving on a horizontal plane (see Fig. 2). Like the PRR robot, this manipulator is redundant for the task of positioning the end-effector in the plane ($n - m = 1$).

Let m_i be the mass of the i th link ($i = 1, 2, 3$), d the distance between the center of mass of the third link and the third joint axis, \mathcal{I} the central moment of inertia of the third link, and ℓ its length.

The dynamic model of this robot under end-effector cartesian forces is

$$B(q_3)\ddot{q} + c(q_3, \dot{q}_3) = J^T(q_3)F \quad (11)$$

with

$$B(q_3) = \begin{bmatrix} a_1 & 0 & \neq a_4 c_3 \\ 0 & a_2 & -a_4 s_3 \\ \neq a_4 c_3 & -a_4 s_3 & a_3 \end{bmatrix}$$

$$c(q_3, \dot{q}_3) = -a_4 \dot{q}_3^2 \begin{bmatrix} s_3 \\ c_3 \\ 0 \end{bmatrix}$$

$$J(q_3) = \begin{bmatrix} 0 & 1 & -\ell s_3 \\ 1 & 0 & \ell c_3 \end{bmatrix}$$

where

$$\begin{aligned} a_1 &= m_1 + m_2 + m_3 \\ a_2 &= m_2 + m_3 \\ a_3 &= \mathcal{I} + m_3 d^2 \\ a_4 &= m_3 d. \end{aligned}$$

By partitioning the configuration vector into $q_a = (q_2, q_3)$ and $q_b = q_1$, the rank condition (3) is satisfied provided that $q_3 \neq \pi/2 + k\pi$. These values correspond to singularities for the partially linearizing and decoupling feedback (4).

For this manipulator one has $c \in \mathcal{R}(J^T)$ and $e = 0$, so that, according to Prop. 1, the dynamic equations will take a second-order Čaplygin form after partial feedback linearization. In fact, we obtain

$$\begin{aligned} \ddot{q}_a &= \begin{bmatrix} \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ \ddot{q}_b &= \ddot{q}_1 = \tilde{g}_1(q_3)u_1 + \tilde{g}_2(q_3)u_2 \\ &= \alpha_1 \tan q_3 u_1 + \beta_1 \sec q_3 u_2 \end{aligned} \quad (12)$$

where

$$\alpha_1 = \frac{a_4 - a_2 \ell}{a_4 - a_1 \ell}, \quad \beta_1 = \frac{a_4 \ell - a_3}{a_4 - a_1 \ell}.$$

Note that, by setting $m_1 = m_2 = 0$, $m_3 = m$, and $d = 0$, (11) collapses into the dynamic model of a planar rigid body of mass m and inertia \mathcal{I} subject to two orthogonal linear forces acting on a point at distance ℓ from the center of mass.

III. CONTROLLABILITY ANALYSIS

In analyzing the control problem for redundant robots under end-effector commands, one should first ascertain whether, for any choice of two robot states $x^0 = (q^0, \dot{q}^0)$ and $x^d = (q^d, \dot{q}^d)$, there exist a finite time T and an input u [related to the Cartesian commands F through (4)] such that $x(T, x^0, u) = x^d$, i.e., x^d is the state attained at time T starting from x^0 and applying the input u . However, necessary and sufficient conditions for this kind of controllability do not exist.

For nonlinear systems with drift, a useful concept is *small-time local controllability* (STLC) [17]. Roughly speaking, a STLC system can reach any point near x^0 in arbitrarily small time with trajectories remaining arbitrarily close to x^0 . Since STLC implies the natural form of controllability defined above, establishing such property for our mechanism would guarantee that the steering problem admits a solution.

Define the state vector as $x = (q_a, q_b, \dot{q}_a, \dot{q}_b)$. The state-space model corresponding to the partially linearized equations (5) and (6) is

$$\dot{x} = f(q_a, \dot{q}_a, \dot{q}_b) + \sum_{i=1}^m g_i(q_a) u_i \quad (13)$$

with

$$f(q_a, \dot{q}_a, \dot{q}_b) = \begin{bmatrix} \dot{q}_a \\ \dot{q}_b \\ \hat{f}(q_a, \dot{q}_a) \end{bmatrix}, \quad g_i(q_a) = \begin{bmatrix} 0_m \\ 0_{n-m} \\ \hat{g}_i(q_a) \end{bmatrix} \quad (14)$$

where the n -vectors \hat{f}_i, \hat{g}_i ($i = 1, \dots, m$) are defined as

$$\hat{f}(q_a, \dot{q}_a) = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ \tilde{f}(q_a, \dot{q}_a) \end{bmatrix} \quad (15)$$

$$\hat{g}_i(q_a) = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 1 \\ \vdots \\ 0 \\ \tilde{g}_i(q_a) \end{bmatrix}, \quad \leftarrow i\text{th entry}$$

with \tilde{f} and \tilde{g}_i vectors of dimension $n - m$.

The following result characterizes the controllability properties of system (13)–(15) at an equilibrium point x^e . For the proof, which is based on the computation of repeated Lie brackets of the vector fields f, g_1, \dots, g_m (see [18]), we refer to [19] and [20].

Proposition 2: Consider a system in the form (13)–(15). If at an equilibrium point $x^e = (q_a^e, q_b^e, 0, 0)$

$$\exists j, k_1, \dots, k_{n-m} : \begin{cases} 1. [g_j, [f, g_{k_1}]](x^e), \dots, \\ [g_j, [f, g_{k_{n-m}}]](x^e) \\ \quad \text{are linearly independent} \\ 2. [g_j, [f, g_j]](x^e) = 0 \end{cases} \quad (16)$$

then the system is small-time locally controllable at x^e .

It should be noted that the above proposition is applicable to any underactuated system which can be partially linearized via feedback. In particular, it is not necessary that the system equations are in triangular form. Below, we apply this result to the PRR and PPR robots.

A. Example 1: Controllability of PRR Robot

In the case of systems in triangular form with $n - m = 1$, one can show that the sufficient condition (16) for small-time local controllability becomes

$$\exists j, k : \begin{cases} 1. \left[\frac{\partial \tilde{g}_k}{\partial q_{a_j}} + \frac{\partial \tilde{g}_j}{\partial q_{a_k}} - \frac{\partial^2 \tilde{f}}{\partial \dot{q}_{a_j} \partial \dot{q}_{a_k}} \right] (x^e) \neq 0 \\ 2. \left[2 \frac{\partial \tilde{g}_j}{\partial q_{a_j}} - \frac{\partial^2 \tilde{f}}{\partial \dot{q}_{a_j}^2} \right] (x^e) = 0. \end{cases} \quad (17)$$

In order to apply condition (17) to the partially linearized model (10) of the PRR robot, we compute

$$\left[\frac{\partial \tilde{g}_1}{\partial q_3} + \frac{\partial \tilde{g}_2}{\partial q_2} - \frac{\partial^2 \tilde{f}}{\partial \dot{q}_2 \partial \dot{q}_3} \right] = \frac{-4c_2s_3 - 2s_3c_2s_3}{4s_3 + \sin(2q_2 + q_3)} - \frac{4c_2c_3[4c_3 + \cos(2q_2 + q_3)]}{[4s_3 + \sin(2q_2 + q_3)]^2}$$

which is generically nonzero, and

$$\left[2 \frac{\partial \tilde{g}_2}{\partial q_3} - \frac{\partial^2 \tilde{f}}{\partial \dot{q}_3^2} \right] \equiv 0.$$

Since with the chosen partition $q_2 = q_{a_1}, q_3 = q_{a_2}$, (17) holds by taking the indices $j = 2$ and $k = 1$. Thus, the PRR robot is small-time locally controllable by means of end-effector commands.

B. Example 2: Controllability of PPR Robot

In the particular case of systems in second-order Čaplygin form with $n - m = 1$, (16) simplifies to

$$\exists j, k : \begin{cases} 1. \left[\frac{\partial \tilde{g}_k}{\partial q_{a_j}} + \frac{\partial \tilde{g}_j}{\partial q_{a_k}} \right] (x^e) \neq 0 \\ 2. \left[\frac{\partial \tilde{g}_j}{\partial q_{a_j}} \right] (x^e) = 0. \end{cases} \quad (18)$$

For the partially linearized model (12) of the PPR robot, (18) is satisfied with $j = 1$ and $k = 2$, because we have

$$\frac{\partial \tilde{g}_2}{\partial q_2} + \frac{\partial \tilde{g}_1}{\partial q_3} = \frac{\alpha_1}{\cos^2 q_3} \neq 0 \quad \text{and} \quad \frac{\partial \tilde{g}_1}{\partial q_2} \equiv 0$$

wherever model (12) is defined. Hence, we have established small-time locally controllability under end-effector commands also for the PPR robot.

IV. POINT-TO-POINT STEERING

Under the assumption that the redundant robot is small-time locally controllable from the end-effector, we consider the problem of determining a sequence of input commands that steers the system from an initial equilibrium point $x^0 = (q^0, 0) = (q_a^0, q_b^0, 0)$ to a desired equilibrium point $x^d = (q^d, 0) = (q_a^d, q_b^d, 0)$. Such a sequence certainly exists in view of the STLC property, and may be computed either in an open-loop fashion or in a feedback mode. However, it is necessary to take into account the following result on feedback control.

Proposition 3: For a system in the form (13)–(15), there exists a smooth time-invariant feedback that stabilizes an equilibrium point $x^e = (q^e, 0)$ only if the mapping $\tilde{f}(q, 0)$ is surjective around the origin.

Proof: According to Brockett's necessary conditions for smooth stabilizability [2], the image of the map $\varphi: (x, u) \mapsto \dot{x}$ defined by (13)–(15) should contain a neighborhood of the origin. This is true

if and only if the system

$$\varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} + \begin{bmatrix} 0_n \\ \hat{g}_1(q) \end{bmatrix} u_1 + \cdots + \begin{bmatrix} 0_n \\ \hat{g}_m(q) \end{bmatrix} u_m$$

is solvable in (q, \dot{q}) for any ε near the origin. Due to the structure (15) of the vector fields in (13), letting $\varepsilon = (0_n, 0_m, \tilde{\varepsilon}_2)$ implies $\dot{q} = 0$ and $u = 0$, so that the above system reduces to

$$\tilde{\varepsilon}_2 = \tilde{f}(q, 0)$$

which must be solvable in q for any $\tilde{\varepsilon}_2$ near the origin. ■

Proposition 3 immediately leads to a negative result in two significant instances of our problem.

Corollary 1: A redundant robot moving in the horizontal plane is not smoothly stabilizable at an equilibrium point x^e via time-invariant feedback commands on the end-effector.

Corollary 2: A redundant robot that can be converted in the second-order Čaplygin form (7), (8) is not smoothly stabilizable at an equilibrium point x^e via time-invariant feedback commands on the end-effector.

Note that, although the presence of gravitational terms may imply the existence of smooth stabilizing control laws, it also restricts the region of equilibrium points for the closed-loop system. A similar situation occurs for robots with some unactuated joints [21], an example of which is provided by the Acrobot [9].

Since standard nonlinear control techniques produce smooth stabilizing laws [18], the above result indicates that in the absence of gravity there is no “simple” way to design end-effector commands in a feedback mode. In general, one must resort to time-varying and/or discontinuous feedback controllers. For example, a recent result by Coron [22] states that systems satisfying the STL condition are locally asymptotically stabilizable by means of a continuous time-varying feedback law. However, while systematic approaches to the design of time-varying feedback exist for controllable driftless (i.e., kinematic) systems, the case of systems with drift has received much less attention. In view of this, we present below an open-loop controller that generalizes the *holonomy angle method* [16], a technique for steering controllable driftless systems widely used in the nonholonomic motion planning context.

The proposed strategy for point-to-point motion prescribes the execution of two phases:

- 1) Drive in finite time T_1 the joint variables q_a to their desired values q_a^d by a proper choice of u . At the end of this phase we obtain $q_a(T_1) = q_a^d$ and $\dot{q}_a(T_1) = 0$. Correspondingly, we have $q_b(T_1) = q_b^I$ and $\dot{q}_b(T_1) = \dot{q}_b^I$, being in general $q_b^I \neq q_b^d$ and $\dot{q}_b^I \neq 0$.
- 2) Perform a *cyclic* motion of duration T_2 on the q_a variables [i.e., such that $q_a(T_1 + T_2) = q_a(T_1)$ and $\dot{q}_a(T_1 + T_2) = 0$] so as to obtain the desired value q_b^d for q_b , with zero final velocity.

The first phase can be performed using standard discontinuous feedback control for the decoupled chains of two integrators represented by (5). For example, one may set

$$u_i = -\gamma_i \text{sign}(q_{a_i} - q_{a_i}^d + 2\gamma_i \dot{q}_{a_i} |\dot{q}_{a_i}|), \quad i = 1, \dots, m \quad (19)$$

where γ_i is an arbitrary positive constant [16]. The final time T_1 will depend on γ_i as well as on the initial conditions for q_a .

For the second phase, which is inherently open-loop, it is convenient to select the cyclic control input u within a (sufficiently rich) parameterized class of commands. The computation of u is further simplified if the system equations can be put in second-order Čaplygin form. In the next section we shall work out a detailed case study to illustrate how to design a suitable cyclic control input.

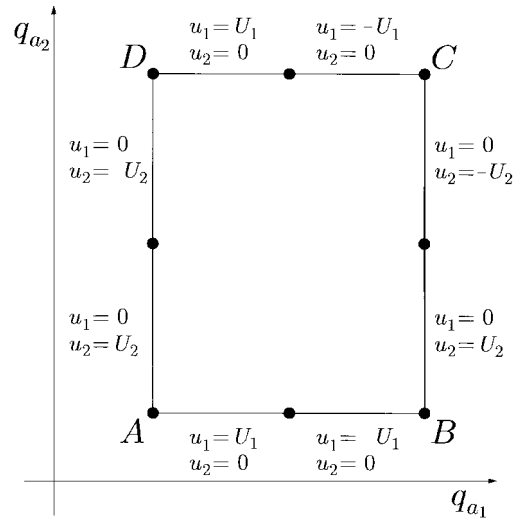


Fig. 3. A rectangular trajectory in the q_a variables.

V. A CASE STUDY: THE PPR ROBOT

Consider the PPR robot model (12) after partial feedback linearization. Although its small-time local controllability has been established in Section III-B, this system is not controllable in the first approximation due to the absence of gravity. The two-phase strategy outlined in the previous section can be applied for steering this robot to a desired joint configuration q^d via cartesian force commands.

The first phase can be performed by using the discontinuous feedback law (19). For the second phase, a convenient choice is to use rectangles as cyclic paths in the q_a variables, with bang-bang accelerations on each side and traveling time $T_2 = 4\delta$ (see Fig. 3). The corresponding class \mathcal{U} of piecewise-constant control inputs is parameterized by the magnitudes U_1 and U_2 of the accelerations along the sides of the rectangle. The generic input in this class is expressed as

$$u(U_1, U_2, t) = \begin{cases} u_1(t) = U_1, & u_2(t) = 0, & t \in [0, \delta/2) \\ u_1(t) = -U_1, & u_2(t) = 0, & t \in [\delta/2, \delta) \\ u_1(t) = 0, & u_2(t) = U_2, & t \in [\delta, 3\delta/2) \\ u_1(t) = 0, & u_2(t) = -U_2, & t \in [3\delta/2, 2\delta) \\ u_1(t) = -U_1, & u_2(t) = 0, & t \in [2\delta, 5\delta/2) \\ u_1(t) = U_1, & u_2(t) = 0, & t \in [5\delta/2, 3\delta) \\ u_1(t) = 0, & u_2(t) = -U_2, & t \in [3\delta, 7\delta/2) \\ u_1(t) = 0, & u_2(t) = U_2, & t \in [7\delta/2, 4\delta) \end{cases}$$

This choice yields a simple form for the q_b evolution. In fact, on each side of the rectangle, only one of the two inputs is active, while the other is zero, thus keeping the corresponding component of q_a constant. In particular, since $\dot{q}_3(T_1) = 0$, from (12) we have along sides AB and CD

$$u_2 = 0 \implies q_{a_2} = q_{a_3} = \text{constant}$$

so that

$$\ddot{q}_b = \ddot{q}_1 = \tilde{g}_1 u_1, \quad \text{with } \tilde{g}_1 = \alpha_1 \tan q_3 = \text{constant} \quad (20)$$

which shows that also the acceleration \ddot{q}_b is bang-bang along these sides. As a consequence, we have that

- 1) closed-form expression for $q_b(t)$ is available along AB and CD ;
- 2) the velocity \dot{q}_b is equal at the vertices of each of these two sides.

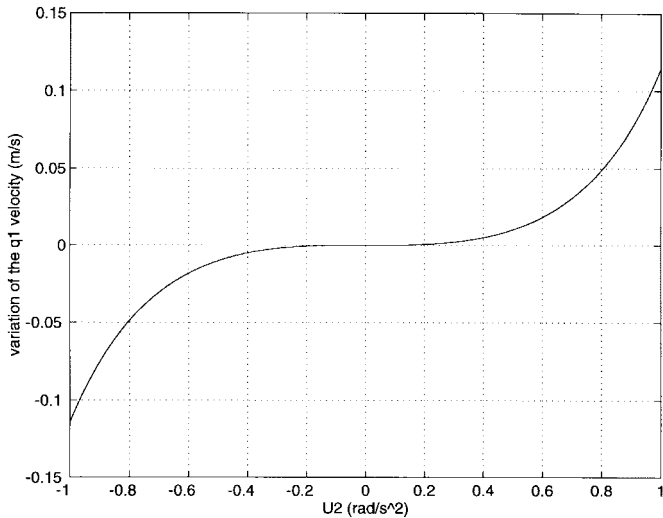


Fig. 4. Variation of $\dot{q}_b = \dot{q}_1$ after one cycle as a function of U_2 .

On the other hand, along sides BC and DA

$$u_1 = 0 \implies q_{a1} = q_2 = \text{constant}$$

so that

$$\ddot{q}_b = \ddot{q}_1 = \tilde{g}_2(t)u_2, \quad \text{with } \tilde{g}_2(t) = \beta_1 \sec q_3(t). \quad (21)$$

No closed-form is available for the solution of (21), and the variation of \dot{q}_b along BC and DA as a function of U_2 must be computed numerically. For illustration, Fig. 4 shows the relationship between U_2 and the variation of $\dot{q}_b = \dot{q}_1$ with the dynamic parameters specified in the following Section V-A.

Based on these considerations, we can determine the parameter values U_1^* , U_2^* that yield the desired reconfiguration according to the following procedure.

- 1) With the aid of Fig. 4, select U_2^* so as to obtain the desired variation $-\dot{q}_b^I$ for \dot{q}_b along BC and DA (and hence along the cycle). At this point, compute the corresponding variation Δq_b for q_b along BC and DA via forward integration of (21).
- 2) By using the closed-form expression for $q_b(t)$ along AB and CD [i.e., the solution of (20)], determine U_1^* so that the variation of q_b along AB and CD equals $q_b^d - q_b^I - \Delta q_b$. In this way, q_b will attain the desired value q_b^d at the end of the cycle.

To complete the analysis, we note the following points.

- a) If no variation of \dot{q}_b is required (i.e., if $\dot{q}_b^I = 0$), Fig. 4 would suggest to set $U_2 = 0$. This choice, however, is not feasible, because any value of U_1 would then give no variation for q_b at the end of the cycle. Therefore, in this case it is necessary to perform two cycles in the second phase giving velocity variations of equal magnitude and opposite sign.
- b) Assume that there exists an upper bound on the magnitudes U_1 and U_2 of the acceleration, e.g., $U_1^{\max} = 1 \text{ m/s}^2$, $U_2^{\max} = 1 \text{ rad/s}^2$. From Fig. 4, it appears that the maximum attainable variation for \dot{q}_b would be about 0.12 m/s. If a larger variation is needed, i.e., if $|\dot{q}_b^I| > 0.12 \text{ m/s}$, one must perform multiple cycles.
- c) Realistic bounds on U_1 and U_2 depend—through (4)—on the maximum applicable cartesian forces F . In particular, as the system approaches the singularities of the partial feedback linearizing control law, these bounds become smaller, and a larger number of cycles will be required in order to achieve the desired reconfiguration. In view of this, it is advisable to

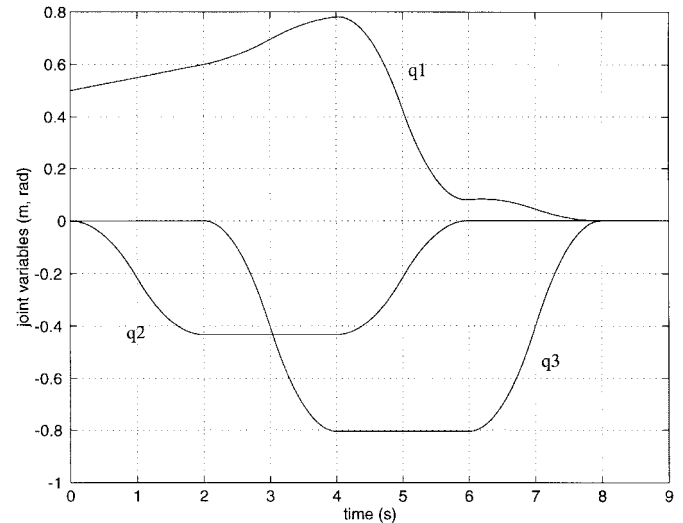


Fig. 5. Evolution of the joint variables along the cycle.

choose in advance the size of the cycles in such a way that the singularity regions are avoided.

- d) The choice of δ (and hence, of T_2) is related to the bounds on U_1 and U_2 . In particular, a longer δ will allow a larger reconfiguration within a single cycle.

A. Simulation Results

The proposed approach has been simulated for a PPR robot having all links of unit mass and a uniform thin rod of length 2 m as third link. We present only the results of the second phase, which is the most interesting. Suppose that, at the end of the first phase, the joint configuration is $q^I = (0.5, 0, 0)$ [m, m, rad] with velocity $\dot{q}^I = (0.05, 0, 0)$ [m/s, m/s, rad/s]. The final desired state at time $4\delta = 8$ s is $q^d = \dot{q}^d = (0, 0, 0)$.

From Fig. 4, it follows that the required variation of -0.05 m/s for \dot{q}_1 is obtained for $U_2^* = -0.80 \text{ rad/s}^2$. This introduces a net variation Δq_1 for q_1 along sides BC and DA equal to 0.07 m. As a result, the total variation needed for q_1 is -0.57 m . The desired value of U_1 is then computed as $U_1^* = -0.43 \text{ m/s}^2$.

The trajectories of the joint variables along the rectangular cycle are shown in Fig. 5, while the corresponding cartesian forces F acting on the end-effector are given in Fig. 6. A stroboscopic representation of the arm motion is shown in Fig. 7, together with the end-effector trajectory corresponding to the rectangle in the q_a space. Points A' , B' , C' , D' , and E' are the Cartesian-space images of the corners A , B , C , D , and A again, respectively. As expected, the closed rectangular trajectory in the q_a space does not correspond to a closed path in the cartesian space. Note also that we have introduced an offset (2, 1.5) on the prismatic joints for the sake of clarity.

VI. CONCLUSION

The configuration control problem has been addressed for a special class of underactuated mechanical systems. In particular, we have considered the case of kinematically redundant robots driven only by end-effector generalized forces, under the assumption that there are as many prismatic joints at the base as the degree of redundancy. After partial feedback linearization, the equations of such systems assume a convenient triangular structure, which may be further simplified under appropriate hypotheses. These model formats have been used to derive controllability conditions as well as to design an end-effector steering algorithm that achieves a desired joint reconfiguration in

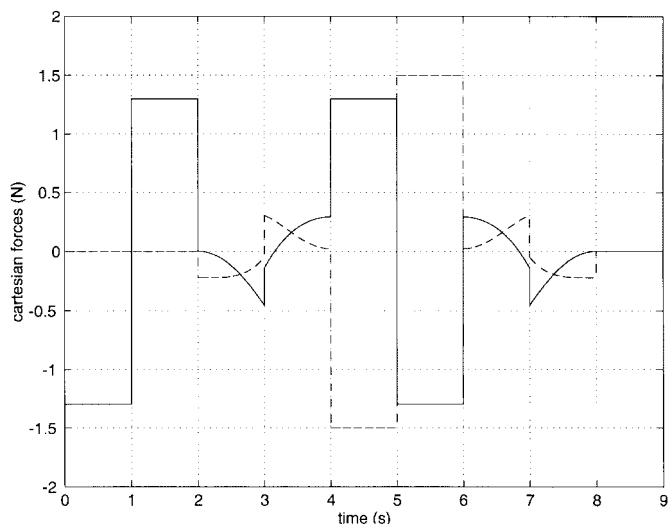


Fig. 6. Cartesian forces acting on the end-effector along the cycle: F_x (solid), F_y (dashed).

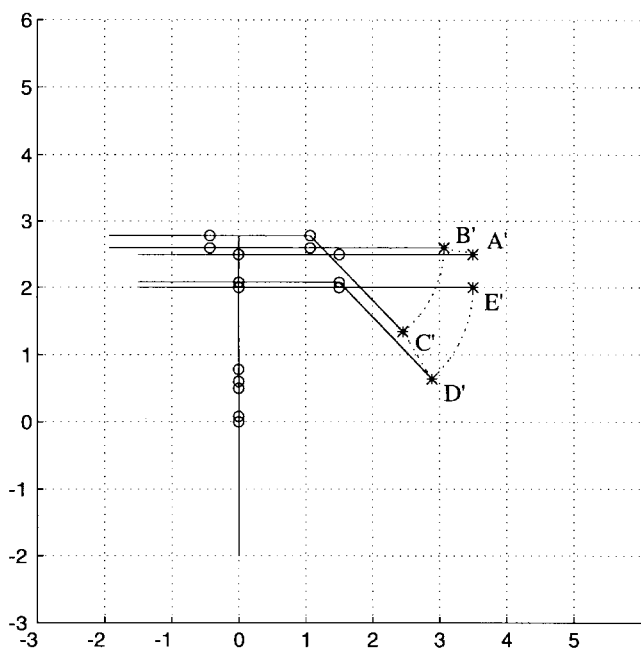


Fig. 7. Arm motion along the cycle.

finite time. The PPR planar robot has been used as a case study to illustrate the proposed approach.

We are currently considering the design of feedback controllers in order to perform the reconfiguration in a more robust fashion, as well as the application of our technique to more complex redundant robots. Furthermore, it would be desirable to gain more insight into the structure of the controllability conditions, in order to relate them to the dynamic properties of the mechanism.

The tools introduced in this paper might prove useful also in more general cases of underactuation. In particular:

- 1) the assumption that the first $n - m$ joints are prismatic may be replaced with the existence of $n - m$ cyclic joint variables [23], i.e., generalized coordinates whose value does not affect the system Lagrangian;
- 2) the nonlinear controllability analysis and the reconfiguration algorithm can be extended to underactuated manipulators, in which some joints are passive [24].

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