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Dynamic Mobility of Redundant Robots using End-Effector Commands*

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Abstract

We analyze the dynamic mobility of a kinematically redundant robot driven by forces/torques imposed on the end-effector, an interesting example of underactuated system. Under suitable assumptions, the system can be put via feedback in two special forms, namely the second-order triangular and Caplygin forms. Nonlinear controllability tools are used to derive conditions under which the robot can be steered between two given configurations using end-effector commands. With a PPR robot as a case study, a steering algorithm is proposed that achieves reconfiguration in finite time.

1 Introduction

The dynamic mobility problem is considered for a kinematically redundant robot controlled only through forces/torques imposed at the end-effector level. In particular, we analyze the conditions under which the robot can be arbitrarily reconfigured, and we propose a steering algorithm that achieves the reconfiguration in finite time.

From an applicative point of view, this problem and its solution may be of interest in the manipulation with multifingered hands or in cooperating tasks with multiple robot arms [1]. In both cases, while the natural input commands are defined at the task level (i.e., in terms of forces/torques at the tip of each open kinematic chain), one is also interested in the internal configurations assumed by each robotic subsystem.

For conventional (non-redundant) robots, the mobility problem is trivial because there is a one-to-one mapping between end-effector and joint commands. Instead, for a kinematically redundant robot with n joints, only $m < n$ end-effector commands are available. Therefore, we are dealing with a class of *underactuated* mechanical systems, namely with strictly less control inputs than generalized coordinates.

Underactuated systems often arise in the presence of *nonholonomic* constraints, e.g., in wheeled mobile robots [2], in dextrous manipulation [3], and in satellite-mounted manipulators [4]. The presence of non-integrable differential constraints introduces a fundamental difference between local and global mobility of these systems. In fact, while feasible instantaneous motions at each configuration are restricted, accessibility of the whole configuration space is still possible by appropriate maneuvers. From a control point of view, it is known that nonholonomic systems have a structural obstruction to the existence of smooth time-invariant stabilizing feedback laws [5]. This has motivated the use of open-loop controllers [6] and of time-varying [7] or discontinuous feedback [8, 9].

While nonholonomy is most of the times intrinsic to the nature of the problem, there are instances where enforcing a nonholonomic behavior may present advantages. Recently, a planar nonholonomic manipulator has been designed [10] so as to allow configuration control of its n joints using only two velocity input commands at the robot base. In the same spirit, in [11] we have determined conditions for choosing one (of the many) inverse kinematic maps of a redundant manipulator so that full accessibility of the configuration space is guaranteed by using only $m < n$ task velocity commands. Finally, [12] addresses the problem of arbitrarily positioning an object in the plane by pushing it along a limited set of directions.

In all the above cases, both the system analysis and the control synthesis are performed at a first-order kinematic level, assuming the direct applicability of generalized velocity inputs. The underlying differential constraints on the system are in the *first-order (Pfaffian) form* $A(q)\dot{q} = 0$, q being the system generalized coordinates [13]. Second-order dynamic

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models of nonholonomic systems have been considered in [14, 15], with generalized forces as inputs, but still in the presence of first-order nonholonomic constraints.

However, there are many control problems for underactuated systems where the underlying differential constraints appear directly in a *second-order form*

$$R(q)\ddot{q} + s(q, \dot{q}) = 0. \tag{1}$$

For example, Arai and Tachi have considered a robot with one passive joint and on/off brakes [16]. Hauser and Murray [17] and later Spong [18] have developed control laws for the Acrobot, a 2R robot with unactuated shoulder joint moving in the vertical plane.

For this class of mechanical systems, inclusion of the dynamics in the analysis is mandatory. As in the first-order case, constraint (1) *may* or *may not* be integrable. In the first case, one may distinguish between *partial integrability* to a velocity-dependent constraint and *total integrability* to a purely configuration-dependent constraint. When constraint (1) is not integrable, the system is nonholonomic and there is no limitation to the accessibility of the robot state space. Oriolo and Nakamura [19] have performed a detailed analysis of eq. (1) for underactuated manipulators, giving conditions for partial or total integrability.

The case of redundant robots driven only by end-effector forces/torques falls in this class of problems, and in fact it is possible to write a set of dynamic constraints of the form (1). Instead of checking the total or partial integrability of the second-order differential constraint, we will perform a controllability test in the proper nonlinear setting of the problem (see [20, 21]), often a more systematic procedure. If this test is verified, it is possible to apply an end-effector steering algorithm for reconfiguring the redundant arm between two equilibrium points. Such an algorithm can be inspired in principle to the literature on nonholonomic motion planning. However, the presence of a second-order differential (and dynamic) constraint brings forth a drift term in the system equations, requiring special caution in the extension.

The paper is organized as follows. In the next section, we show that a partial feedback linearization allows to put the robot dynamic equations in a simpler format useful for analysis and control. The existence of two special forms is pointed out in Sect. 3. A detailed controllability analysis is performed in Sect. 4, and in Sect. 5, we describe a point-to-point steering algorithm applicable to redundant robots that admit a second-order triangular or Caplygin form. The algorithm is illustrated using a PPR planar robot as a simulation case study.

2 Partial feedback linearization

Consider a manipulator with n joints whose end-effector pose is described by m variables, being $n - m > 0$ the degree of kinematic redundancy. Denote by $q \in I\!R^n$ the joint coordinates vector, and by $J(q)$ the $m \times n$ standard Jacobian of the robot. We shall assume that kinematic singularities are avoided.

Following the Lagrangian approach, the dynamic model of the system can be written as

$$B(q)\ddot{q} + h(q, \dot{q}) = J^T(q)F, \tag{2}$$

where $B(q)$ is the $n \times n$ inertia matrix, $h(q, \dot{q}) = c(q, \dot{q}) + e(q)$ collects the vector $c(q, \dot{q})$ of centrifugal and Coriolis terms and the vector $e(q)$ of gravitational terms, while F is the m-vector of generalized forces acting on the end-effector.

Partition the joint vector as $q = (q_a, q_b)$, with $q_a \in I\!R^m$ and $q_b \in I\!R^{n-m}$. Correspondingly, the dynamic model (2) may be written as

$$\begin{bmatrix} B_{aa} & B_{ab} \\ B_{ab}^T & B_{bb} \end{bmatrix} \begin{bmatrix} \ddot{q}_a \\ \ddot{q}_b \end{bmatrix} + \begin{bmatrix} h_a \\ h_b \end{bmatrix} = \begin{bmatrix} J_a^T \\ J_b^T \end{bmatrix} F. \tag{3}$$

Assume that, with the given partition,

$$\text{rank} \begin{bmatrix} J_a^T & B_{ab} \\ J_b^T & B_{bb} \end{bmatrix} = n. \tag{4}$$

Due to the full row rank assumption for $J(q)$, this can always be achieved, after possibly reordering the joint variables. Model (3) can be left-multiplied by the following nonsingular matrix $T(q)$

$$T = \begin{bmatrix} I_m & -B_{ab}B_{bb}^{-1} \\ -B_{ab}^T B_{aa}^{-1} & I_{n-m} \end{bmatrix},$$

where I_m is the $m \times m$ identity matrix, so as to obtain

$$\begin{bmatrix} \tilde{B}_{aa} & O \\ O^T & \tilde{B}_{bb} \end{bmatrix} \begin{bmatrix} \ddot{q}_a \\ \ddot{q}_b \end{bmatrix} + \begin{bmatrix} \tilde{h}_a \\ \tilde{h}_b \end{bmatrix} = \begin{bmatrix} \tilde{J}_a^T \\ \tilde{J}_b^T \end{bmatrix} F,$$

where O is an $m \times (n - m)$ matrix with zero entries, and $\tilde{J}_a$ is always nonsingular by virtue of property (4).

At this point, the end-effector generalized forces F can be chosen as a partially linearizing and decoupling feedback control

$$F = \tilde{J}_a^{-T} \left(\tilde{B}_{aa} u + \tilde{h}_a \right), \tag{5}$$

with $u \in I\!R^m$ an auxiliary input vector, so that the dynamic equations take the form

$$\ddot{q}_a = u, \tag{6}$$

$$\begin{aligned} \ddot{q}_b &= \tilde{B}_{bb}^{-1} \left(\tilde{J}_b^T \tilde{J}_a^{-T} \tilde{h}_a - \tilde{h}_b \right) + \tilde{B}_{bb}^{-1} \tilde{J}_b^T \tilde{J}_a^{-T} \tilde{B}_{aa} u \\ &= \tilde{f}(q, \dot{q}) + \tilde{G}(q) u. \end{aligned} \tag{7}$$

Interestingly, we can derive from eqs. (6–7) the *second-order differential constraint*

$$\tilde{G}(q)\ddot{q}_a - \ddot{q}_b + \tilde{f}(q,\dot{q}) = 0, \tag{8}$$

that is always satisfied by the robot during its motion. If this constraint could be integrated twice to a purely configuration-dependent constraint, the robot would not have complete mobility in the configuration space. We will come back on this at the end of Sect. 4.

3 Special second-order forms

While the system can *always* be put in the form (6–7), simplifications are obtained under suitable hypotheses. When the vector field $\tilde{f}$ and the matrix $\tilde{G}$ in eqs. (6–7) depend only on $q_a, \dot{q}_a$ and on q_a, respectively, the dynamic equations become

$$\ddot{q}_a = u, \tag{9}$$

$$\ddot{q}_b = \tilde{f}(q_a,\dot{q}_a) + \tilde{G}(q_a)u. \tag{10}$$

We call this a *second-order triangular form.* Below, we give conditions under which this form can be obtained.

Assumption 1. The robot Lagrangian $\mathcal{L}$ and the Jacobian J do not depend on the joint variables q_b.◁

This property is indeed restrictive, but can be achieved in many interesting cases. One possibility is to exploit the existence of *cyclic* variables [22], i.e., generalized coordinates whose value does not affect the Lagrangian. In the proper joint coordinates, such variables do not appear neither in the inertia matrix B nor in the gravitational vector e.

As a consequence of Assumption 1, both vectors c and e do not depend on q_b. Let

$$c(q_a,\dot{q}_a,\dot{q}_b) = c'(q_a,\dot{q}_a) + c''(q_a,\dot{q}_a,\dot{q}_b),$$

where $c'(q_a,\dot{q}_a)$ includes the velocity terms involving only the $\dot{q}_{a_i}$'s, while $c''(q_a,\dot{q}_a,\dot{q}_b)$ collects the velocity terms in which at least one $\dot{q}_{b_j}$ appears. We have the following result, whose proof is given in [23].

Proposition 1. *Under Assumption 1, the dynamic equations (6–7) of the system take the second-order triangular form (9–10) if and only if*

$$c''(q_a,\dot{q}_a,\dot{q}_b) \in \mathcal{R}(J^T),$$

where $\mathcal{R}(J^T)$ denotes the range space of matrix J^T.

A further simplification of the form (9–10) occurs when the acceleration drift $\tilde{f}$ term in eq. (10) is zero. In this case, the dynamic equations become

$$\ddot{q}_a = u, \tag{11}$$

$$\ddot{q}_b = \tilde{g}_1(q_a)u_1 + \ldots + \tilde{g}_m(q_a)u_m. \tag{12}$$

We refer to this system as a *second-order Čaplygin form*, extending the definition of [14].

Proposition 2. *Under Assumption 1, the dynamic equations (6–7) of the system take the second-order Čaplygin form (11–12) if and only if*

$$c \in \mathcal{R}(J^T) \quad \textit{and} \quad e \in \mathcal{R}(J^T).$$

Again, we refer to [23] for the proof. We shall see that the existence of a triangular form or, even better, of a Čaplygin form has consequences on the controllability analysis as well as on the synthesis of a control law.

4 Controllability analysis

In investigating the dynamic mobility of redundant robots under end-effector commands, we are basically interested in determining whether, for any choice of two robot states $x^0 = (q^0,\dot{q}^0)$ and $x^d = (q^d,\dot{q}^d)$, there exists a finite time T and an input u (related to the generalized forces F through eq. (5)) such that $x(T,x^0,u) = x^d$. This amounts to testing the controllability of the corresponding dynamical system

$$\dot{x} = f(x) + g_1(x)u_1 + \ldots + g_m(x)u_m. \tag{13}$$

While general criteria for verifying this kind of controllability do not exist, it is known that the latter is implied by *small-time local controllability* (STLC) [21]. Thus, establishing the STLC property would guarantee that our dynamic mobility problem is solvable. Below, we recall a sufficient condition for STLC.

Assume that the control input $u = (u_1,\ldots,u_m)$ of system (13) takes values in the limited region $\Omega = \{u \in I\!R^m : |u_i| \le \rho_i, i = 1,\ldots,m\}$. Define the *accessibility distribution* $\Delta_{\mathcal{C}}$ as the distribution generated by the smallest Lie algebra $\mathcal{C}$ containing $f, g_1,\ldots,g_m$. Given a Lie bracket $v \in \mathcal{C}$, denote by $\delta^0(v),\delta^1(v),\ldots,\delta^m(v)$ the number of occurrences of $f, g_1,\ldots,g_m$, respectively, in v. Any vector of nonnegative integers $\theta = (\theta_0,\theta_1,\ldots,\theta_m)$ such that $\theta_i \ge \theta_0$, $\forall i = 1,\ldots,m$, is called an *admissible weight vector*, and the *θ-degree* of v is defined as $\sum_{i=0}^m \theta_i\delta^i(v)$.

Theorem ([24]). *Assume $\dim\Delta_{\mathcal{C}}(x^0) = n$ and that, for any Lie bracket $v \in \mathcal{C}$ such that $\delta^0(v)$ is odd and $\delta^1(v),\ldots,\delta^m(v)$ are even, there exists an admissible weight vector θ such that v is θ-*neutralized, *i.e., it can be written as a linear combination of brackets of lower θ-degree. Then, system (13) is STLC from x^0.*

In the following, we shall simply call 'bad' the Lie brackets with δ^0 odd and $\delta^1,\ldots,\delta^m$ even. Thus, to establish the STLC property for system (13), one needs *(i)* to exhibit a basis of the state space (a $2n$-dimensional manifold) whose elements are chosen among $f, g_1, \ldots, g_m$ and their 'good' Lie brackets, and *(ii)* to show that there exists an admissible weight vector θ such that all bad Lie Brackets are θ-neutralized.

Since controllability properties are invariant under invertible state feedback, we may conveniently analyze our system by using the state-space form (13) corresponding to the partially linearized eqs. (6–7), in which the state vector is $x = (q_a, q_b, \dot{q}_a, \dot{q}_b)$ and

$$f(q,\dot{q}) = \begin{bmatrix} \dot{q}_a \\ \dot{q}_b \\ \hat{f}(q,\dot{q}) \end{bmatrix}, \qquad g_i(q) = \begin{bmatrix} 0_m \\ 0_{n-m} \\ \hat{g}_i(q) \end{bmatrix}, \quad (14)$$

with the n-vectors $\hat{f}$, $\hat{g}_i$ defined as

$$\hat{f}(q,\dot{q}) = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ \tilde{f}(q,\dot{q}) \end{bmatrix}, \quad \hat{g}_i(q) = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ \tilde{g}_i(q) \end{bmatrix} \begin{matrix} \\ \\ \longleftarrow i\text{-th row} \\ \\ \\ \\ \end{matrix} \quad (15)$$

We shall characterize the controllability properties of system (13–15) at an equilibrium point x^e in the case of a single degree of redundancy. However, a similar discussion can be repeated for the general case.

Proposition 3. *Consider the system (13–15) with $n - m = 1$. If at $x^e = (q_a^e, q_b^e, 0, 0)$*

$$\exists\, j,k \in \{1,\ldots,m\} : \begin{cases} [g_j,[f,g_k]](x^e) \neq 0, \\ [g_j,[f,g_j]](x^e) = 0, \end{cases} \quad (16)$$

then the system is STLC at x^e.

Proof. Consider the set of $2n$ good vector fields

$$\begin{aligned} \mathcal{B} \;=\; & \{g_1,\ldots,g_{n-1},[f,g_1],\ldots,[f,g_{n-1}], \\ & [g_j,[f,g_k]],[f,[g_j,[f,g_k]]]\}. \end{aligned}$$

When $\dot{q} = 0$, the structure of the elements of $\mathcal{B}$ is:

$$\begin{aligned} g_i &= \begin{bmatrix} 0_n \\ \hat{g}_i \end{bmatrix}, \quad i = 1,\ldots,n-1 \\ [f,g_i] &= \begin{bmatrix} -\hat{g}_i \\ 0_n \end{bmatrix}, \quad i = 1,\ldots,n-1 \\ [g_j,[f,g_k]] &= \begin{bmatrix} 0_n \\ \hat{\psi}_{jk} \end{bmatrix}, \\ [f,[g_j,[f,g_k]]] &= \begin{bmatrix} -\hat{\psi}_{jk} \\ 0_n \end{bmatrix}, \end{aligned}$$

with

$$\hat{\psi}_{jk}(q) = \begin{bmatrix} 0_{n-1} \\ \tilde{\psi}_{jk}(q) \end{bmatrix},$$

and

$$\tilde{\psi}_{jk}(q) = \frac{\partial \tilde{g}_k}{\partial q_{a_j}} + \frac{\partial \tilde{g}_j}{\partial q_{a_k}} + \frac{\partial \tilde{g}_k}{\partial q_b}\tilde{g}_j + \frac{\partial \tilde{g}_j}{\partial q_b}\tilde{g}_k - \frac{\partial}{\partial \dot{q}}\left(\frac{\partial \tilde{f}}{\partial \dot{q}}\hat{g}_k\right)\hat{g}_j, \quad (17)$$

where q_{a_i} is the i-th component of the subvector q_a. If $[g_j,[f,g_k]](x^e) \neq 0$ holds, its only nontrivial scalar entry $\tilde{\psi}_{jk}(q)$ is nonzero at x_e, and the vectors in $\mathcal{B}$ span a $2n$-dimensional space at x^e.

We have to show now that all bad Lie brackets are θ-neutralized. The first group of bad brackets to be considered is $[g_i,[f,g_i]]$, for $i = 1,\ldots,m$. Since

$$[g_i,[f,g_i]] = \begin{bmatrix} 0_n \\ \hat{\psi}_{ii} \end{bmatrix}, \quad \text{with } \hat{\psi}_{ii} = \begin{bmatrix} 0_{n-1} \\ \tilde{\psi}_{ii} \end{bmatrix},$$

the vector field $[g_i,[f,g_i]]$ is aligned with $[g_j,[f,g_k]]$. Consider the admissible weight vector θ defined by

$$\theta_0 = \theta_j = 1 \quad \text{and} \quad \theta_i = 2,\; i \neq 0, i \neq j.$$

With this choice of the weight vector, the θ-degree of $[g_j,[f,g_k]]$ is equal to 4, while the θ-degree of $[g_i,[f,g_i]]$ is 5 for $i \neq j$, and 3 for $i = j$. However, by hypothesis the Lie bracket $[g_j,[f,g_j]]$ is zero at x^e. Hence, nonzero bad brackets of the first group can be written as linear combinations of brackets of lower θ-degree (namely, of $[g_j,[f,g_k]]$ alone). The proof may be completed by verifying that all other bad brackets are θ-neutralized in a similar way. ∎

Note that, in order to test condition (16), one simply needs to compute the scalar functions $\tilde{\psi}_{hl}$ at x^e.

In the particular case of systems in the second-order triangular form (9–10), the sufficient condition (16) for small-time local controllability becomes

$$\exists\, j,k \in \{1,\ldots,m\} : \begin{cases} \left[\dfrac{\partial \tilde{g}_k}{\partial q_{a_j}} + \dfrac{\partial \tilde{g}_j}{\partial q_{a_k}} - \dfrac{\partial^2 \tilde{f}}{\partial \dot{q}_{a_j}\partial \dot{q}_{a_k}}\right](x^e) \neq 0, \\ \left[2\dfrac{\partial \tilde{g}_j}{\partial q_{a_j}} - \dfrac{\partial^2 \tilde{f}}{\partial \dot{q}_{a_j}^2}\right](x^e) = 0, \end{cases} \quad (18)$$

where expression (17) has been used.

Finally, for systems in the second-order Čaplygin form (11–12), condition (16) simplifies to

$$\exists\, j,k \in \{1,\ldots,m\} : \begin{cases} \left[\dfrac{\partial \tilde{g}_k}{\partial q_{a_j}} + \dfrac{\partial \tilde{g}_j}{\partial q_{a_k}}\right](x^e) \neq 0, \\ \left[\dfrac{\partial \tilde{g}_j}{\partial q_{a_j}}\right](x^e) = 0. \end{cases} \quad (19)$$

We briefly point out the close relationship between the controllability of system (13–15) and the non-integrability of the second-order differential constraint (8). If the robot is controllable via end-effector commands, then we have accessibility to any point in its state space. Hence, there does not exist any *geometric* restriction on the robot attainable configurations, implying that the differential constraint (8) is not integrable. In other words, eq. (8) represents a second-order *nonholonomic* constraint for our system.

On the other hand, the loss of controllability is equivalent to the integrability of eq. (8), under suitable regularity assumptions [23]. However, since condition (16) is only sufficient for STLC, and STLC is sufficient for controllability, its violation does not necessarily mean that constraint (8) is integrable.

5 Point-to-point steering

Assume our redundant robot is STLC from the end-effector. We now address the problem of determining a proper sequence of input commands so as to transfer the system from an initial equilibrium point $x^0 = (q^0, 0) = (q_a^0, q_b^0, 0)$ to a desired equilibrium point $x^d = (q^d, 0) = (q_a^d, q_b^d, 0)$.

In principle, one may either use an *open-loop* control law, or a *feedback* control law that renders x^d asymptotically stable. The design of feedback stabilizing laws is more difficult, but they are preferable for real-time motion control under uncertain or perturbed conditions. However, it is necessary to take into account the following result [23].

Proposition 4. *A redundant robot moving in the horizontal plane is not smoothly stabilizable at an equilibrium point x^e via time-invariant feedback end-effector commands. In particular, this is true for redundant robots that can be put in second-order Čaplygin form.*

Since standard nonlinear control techniques typically produce smooth stabilizing laws [20], the above corollaries indicate that there is no 'simple' way to design end-effector commands in a feedback mode. In view of this, we present below an open-loop controller that generalizes the *holonomy angle method* [14], a technique for steering controllable driftless systems widely used in the nonholonomic motion planning context.

The proposed strategy for point-to-point motion prescribes the execution of two phases:

1. Drive in finite time T_1 the joint variables q_a to their desired values q_a^d with zero velocity, by a proper choice of u. Correspondingly, we have $q_b(T_1) = q_b^I$ and $\dot{q}_b(T_1) = \dot{q}_b^I \neq 0$ in general.
2. Perform a *cyclic* motion of duration T_2 on the q_a variables (i.e., a motion such that $q_a(T_1 + T_2) = q_a(T_1)$ and $\dot{q}_a(T_1 + T_2) = 0$) so as to obtain the desired value q_b^d for q_b with zero final velocity.

The first phase can be performed using standard discontinuous feedback control for the decoupled chains of two integrators represented by eq. (6).

In the second phase, which is inherently open-loop, it is convenient to select the cyclic control input u within a *parameterized class*, in order to simplify the computation of the required command. The chosen class of inputs should be sufficiently rich to contain a solution for the problem [23]. This procedure is greatly simplified if the system equations can be put in second-order triangular or Čaplygin form.

The main difference of the proposed technique with respect to the ancestor method in [14] lies in the structure of the second phase, due to the nonholonomic constraint (8) being expressed at the acceleration level. As a consequence, the variations of q_b and $\dot{q}_b$ along the cycle depend on the *trajectory* of the q_a variables, i.e., not only on the geometric path but also on the time history. Thus, it is not possible to implement the second phase as a sequence of feedback stabilization steps, like in [14].

In the next section, we present a case study to illustrate how to design a suitable class of input trajectories for a robot that admits a Čaplygin form.

6 A case study: The PPR robot

Consider a PPR robot, having two prismatic and one revolute joint, moving on a horizontal plane (see Fig. 1). This manipulator is redundant for the task of positioning the end-effector in the plane ($n - m = 1$).

The dynamic model of this robot under end-effector cartesian forces is

$$B(q_3)\ddot{q} + c(q_3, \dot{q}_3) = J^T(q_3)F.$$

The expression of the various terms can be found in [23]. Note that the inertia matrix and the Jacobian matrix depend only on q_3, i.e., the revolute joint position, so that Assumption 1 holds choosing either $q_b = q_1$ or $q_b = q_2$. It can be verified that, in both cases, the rank property (4) is generically achieved; in particular, it must be $q_3 \neq \pi/2 + k\pi$ for the first case and $q_3 \neq k\pi$ for the second case. These values correspond to singularities for the feedback law (5). However, at any point of the state space, at least one of the two feedback laws is well-defined.

For this manipulator, it is $c \in \mathcal{R}(J^T)$ and $e = 0$, so that Prop. 2 applies. This means that, by using the

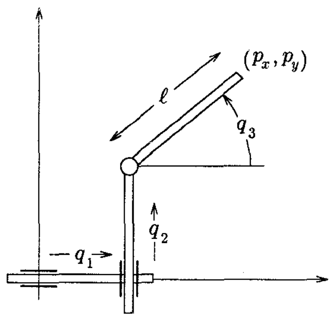

Figure 1 – A planar PPR robot

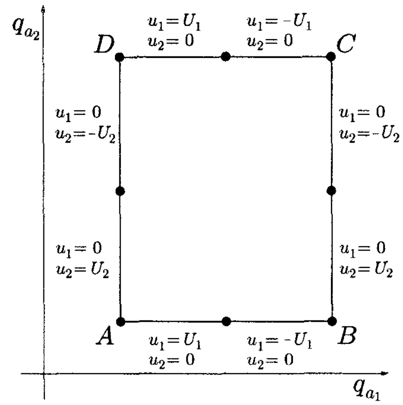

Figure 2 – A rectangular trajectory in the q_a variables

feedback control (5), the dynamic equations will take a second-order Čaplygin form. As a matter of fact, depending on the choice of q_b, two alternative forms are obtained. For example, we obtain

$$\begin{aligned} \ddot{q}_a &= \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \\ \ddot{q}_b &= \quad \ddot{q}_2 \quad = \alpha_1 \tan q_3 \, u_1 + \beta_1 \sec q_3 \, u_2, \end{aligned} \tag{20}$$

where α_1 and β_1 are constant coefficients.

Being $\tilde{g}_1 = \alpha_1 \tan q_3$ and $\tilde{g}_2 = \beta_1 \sec q_3$, the simplified controllability condition (19) is satisfied with $j = 1$ and $k = 2$, since

$$\frac{\partial \tilde{g}_2}{\partial q_1} + \frac{\partial \tilde{g}_1}{\partial q_3} = \frac{\alpha_1}{\cos^2 q_3} \neq 0 \quad \text{and} \quad \frac{\partial \tilde{g}_1}{\partial q_1} \equiv 0.$$

Hence, complete dynamic mobility under end-effector commands is guaranteed. Below, we apply the two-phase strategy of the previous section in order to steer the PPR robot to a desired configuration q_d.

The first phase can be performed by using the discontinuous feedback law

$$u_i = -\gamma_i \, \mathrm{sign}(q_{a_i} - q^d_{a_i} + 2\gamma_i \dot{q}_{a_i} |\dot{q}_{a_i}|), \quad i = 1, 2,$$

where γ_i is an arbitrary positive constant [14].

For the second phase, a convenient choice is to use rectangles as cyclic paths in the q_a variables, with bang-bang accelerations on each side and traveling time $T_2 = 4\delta$ (see Fig. 2). The generic input in this class is expressed as

$$u(U_1, U_2, t) = \begin{cases} u_1(t) = U_1, \ u_2(t) = 0, & t \in [0, \delta/2), \\ u_1(t) = -U_1, \ u_2(t) = 0, & t \in [\delta/2, \delta), \\ u_1(t) = 0, \ u_2(t) = U_2, & t \in [\delta, 3\delta/2), \\ u_1(t) = 0, \ u_2(t) = -U_2, & t \in [3\delta/2, 2\delta), \\ u_1(t) = -U_1, \ u_2(t) = 0, & t \in [2\delta, 5\delta/2), \\ u_1(t) = U_1, \ u_2(t) = 0, & t \in [5\delta/2, 3\delta), \\ u_1(t) = 0, \ u_2(t) = -U_2, & t \in [3\delta, 7\delta/2), \\ u_1(t) = 0, \ u_2(t) = U_2, & t \in [7\delta/2, 4\delta). \end{cases}$$

This choice yields a simple form for the q_b evolution: in fact, on each side of the rectangle, only one input is active, while the other is zero, keeping the corresponding component of q_a constant. In particular, along sides AB and CD we have $u_2 = 0$, implying $q_{a_2} = q_3 =$ constant and $\ddot{q}_b = \ddot{q}_2 = \tilde{g}_1 u_1$, with $\tilde{g}_1 = \tilde{g}_1(q_3) = \alpha_1 \tan q_3 =$ constant. This shows that also the acceleration $\ddot{q}_2$ is bang-bang. Wrapping up, we have that *(i)* a closed-form expression for $q_2(t)$ is available along AB and CD, and *(ii)* the velocity $\dot{q}_2$ is equal at the vertices of each of these two sides.

On the other hand, along BC and DA it is $u_1 = 0$ so that $q_{a_1} = q_1 =$ constant. Hence, $\ddot{q}_b = \ddot{q}_2 = \tilde{g}_2(t) u_2$, with $\tilde{g}_2(t) = \tilde{g}_2(q_3(t)) = \beta_1 \sec q_3$. No closed-form is available for the solution of the latter equation, and the variation of $\dot{q}_b = \dot{q}_2$ along BC and DA as a function of U_2 must be computed numerically. For illustration, Fig. 3 shows the relationship between U_2 and the variation of $\dot{q}_2$ obtained for model (20), with the dynamic parameters given in Sect. 6.1.

Based on these considerations, we can determine

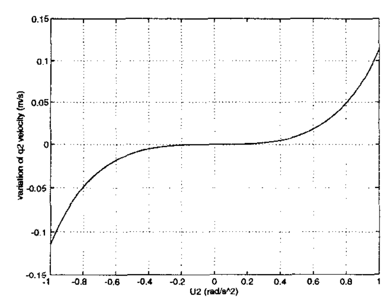

Figure 3 – Variation of $\dot{q}_b = \dot{q}_2$ after one cycle vs. U_2

the parameter values U_1^*, U_2^* that yield the desired reconfiguration according to the following procedure:

1. Using Fig. 3, select U_2^* so as to obtain the desired variation $-\dot{q}_2^I$ for $\dot{q}_2$ along BC and DA (hence, along the cycle). Compute the corresponding variation $\bar{q}_2$ of q_2 via forward integration.

2. By using the closed-form expression for $q_2(t)$ along AB and CD, determine U_1^* so that the variation of q_2 along AB and CD equals $q_2^d - q_2^I - \bar{q}_2$.

Note that, if no variation of $\dot{q}_b$ is required (i.e., if $\dot{q}_b^I = 0$), Fig. 3 would suggest to set $U_2 = 0$. This choice, however, is not feasible, because any value of U_1 would give no variation for q_b at the end of the cycle. Therefore, in this case it is necessary to perform two cycles giving velocity variations of equal magnitude and opposite sign.

6.1 Simulation results

The proposed approach has been simulated for a PPR robot having all links of unit mass and a uniform thin rod of length 2 m as third link. We present only the results of the second phase, which is the most interesting. Suppose that at the end of the first phase, the joint configuration is $q^I = (0, 0.5, 0)$ [m,m,rad] with velocity $\dot{q}^I = (0, 0.05, 0)$. The final desired state at time $4\delta = 8$ sec is $q^d = \dot{q}^d = (0, 0, 0)$. The desired joint reconfiguration corresponds to an end-effector displacement from (3.5,2.5) to (3.5,2).

A careful examination of Fig. 3 shows that the required variation of -0.05 m/sec for $\dot{q}_2$ is obtained for $U_2^* \approx -0.80$ rad/sec^2. This introduces a net variation $\bar{q}_2$ for q_2 along sides BC and DA approximately equal to -0.07 m. As a result, the total variation needed for q_2 is -0.43 m/sec^2 The desired value of U_1 is then easily computed as $U_1^* = -0.43$ m/sec^2.

The trajectories of the joint variables along the rectangular cycle, the corresponding cartesian forces F acting on the end-effector and a stroboscopic representation of the arm motion are shown in Fig. 4. Points A', B', C', D' and E' are respectively the cartesian-space images of the corners A, B, C, D, and A again. As expected, the closed rectangular trajectory in the q_a space does not correspond to a closed path in the cartesian space.

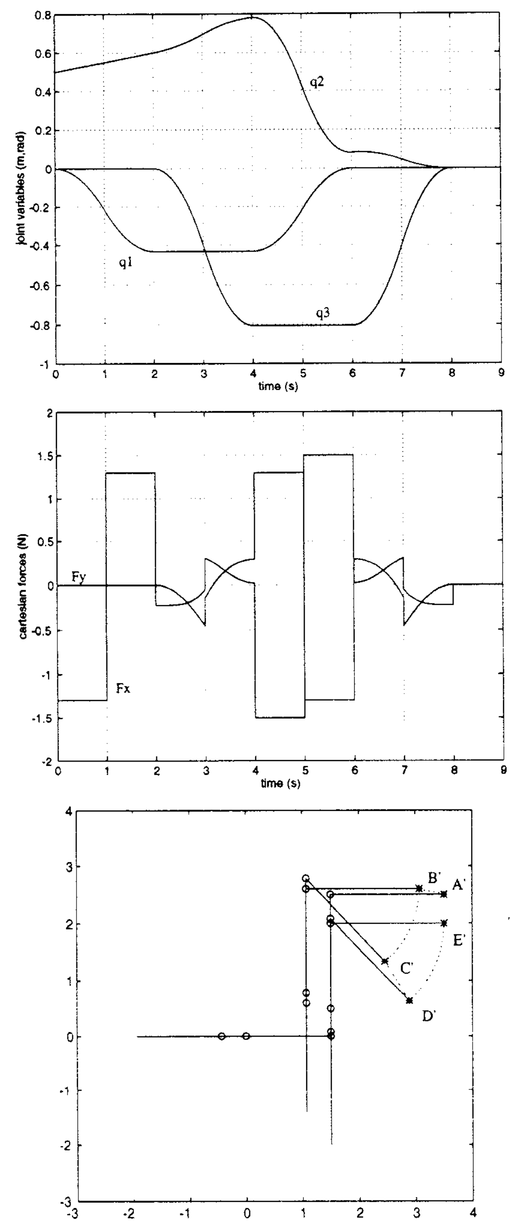

Figure 4 – Joint evolutions (top), cartesian forces on the end-effector (center) and arm motion (bottom) along the cycle

7 Conclusions

A mobility analysis of redundant robots driven by end-effector generalized forces has been performed by using tools from nonlinear controllability theory. We have identified conditions under which such systems may be cast into second-order triangular or Čaplygin forms, and we have exploited these particular structures in order to design an end-effector steering algorithm that achieves a desired joint reconfiguration in finite time. The PPR planar robot was used as a case study to illustrate the proposed approach.

We are currently considering the design of feedback controllers to perform the reconfiguration in a more robust fashion, as well as the application of our technique to more complex redundant robots. Finally, the tools introduced in this paper with reference to a special class of underactuated mechanical systems might prove beneficial also in more general cases. In particular, both the nonlinear controllability analysis and the reconfiguration algorithm are quite naturally applicable to underactuated robots.

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