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Decoupling and Feedback Linearization of Robots with Mixed Rigid/Elastic Joints*

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Abstract

We consider some theoretical aspects of the control problem for robots with rigid links, having some joints rigid and some with non-negligible elasticity. We start from the reduced model of robots with all elastic joints introduced by Spong, which is linearizable by static feedback (as for the rigid robot model). For the mixed situation, we give first structural necessary and sufficient conditions for input-output decoupling and full state linearization via static state feedback. These turn out to be very restrictive. However, when a robot fails to satisfy these conditions, we show that a dynamic state feedback always guarantees the same result. This implies that, for the mixed rigid/elastic joint case, the role of dynamic feedback is essential. The explicit forms of the needed nonlinear controllers are provided in terms of the dynamic model elements.

1 Introduction

Joint flexibility is present in many current industrial robots. When harmonic drives, belts or long shafts are used as motion transmission elements, a dynamic displacement is introduced between the position of the driving actuators and that of the driven links, which is the output to be controlled. This small deflection concentrated at the robot joints is a major source of oscillatory problems, when accurate trajectory tracking or high sensitivity to cartesian forces is required.

The experimental findings of [1] on the GE P-50 arm first showed that to overcome the above effects one should include joint elasticity in the model used for control design. An early study on the modeling of robots with joint elasticity can be found in [2]. A more detailed analysis of the model structure has been later presented in [3].

When considering the performance of nonlinear control techniques for trajectory tracking, such as feedback linearization or input-output decoupling, the first reported results were of negative flavor. In particular, it was shown in [4] that a 3R elbow-type arm with *all* joints being elastic fails to satisfy the necessary conditions for linearization or for decoupling via static state feedback.

Since rigid joints are the limit case of a flexible behavior, with stiffness going to infinity, the dynamic model of robots with elastic joints lend itself to a singularly perturbed format [4]. Several *approximate* nonlinear controllers have been proposed using the intrinsic two-time scale dynamics of the system [4, 5, 6]. Performance is satisfactory, provided that the joints are sufficiently stiff.

A solution to the problem of *exact* tracking of smooth trajectories for the robot links (i.e., beyond elasticity) can be obtained via *dynamic state feedback*. In this approach, there is no limitation on joint stiffness, which may be large or very small. The method has been proposed first in [7] for a 3R elbow-type arm, while a general approach to dynamic linearization and decoupling of robots with *all* joints being elastic is described in [8, 9]. Physically, dynamic feedback is introduced to compensate, at a given joint, for the small torque couplings arising from the acceleration of the actuators at other joints. Slowing down this low-energy effect allows to make use of higher authority commands. In fact, after dynamic compensation, the effective control action at a given joint is given by the local motor torque, which affects the associated link position through the fourth-order dynamics of the elastic joint.

Indeed, the design of a nonlinear dynamic state feedback control is quite complex and this leaves open the way to simpler approaches with a similar provably good performance. A *reduced model* for robots with arbitrary joint elasticity was introduced in [10], assuming that the kinetic energy of the electrical actuators is due only to their own rotor spinning. This assumption is very reasonable, especially when the reduction gear ratios are large. It was shown in [10] that this reduced model of robots with elastic joints satisfies the conditions for exact linearization and decoupling via static state feedback.

Both dynamic models —the reduced and the complete one— have been used extensively by researchers working on the control of robots with joint elasticity. A review of the modeling assumptions and of various proposed con-

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trol techniques can be found in [11]. We just note here that the distinction between the complete and the reduced model vanishes for some common robot kinematic configurations.

In this paper, we consider a different class of robot arms having *some* joints that can be considered completely rigid and *some other* where elasticity is relevant. This mixed picture is the common rule rather than the exception for industrial arms, e.g. in the Scara family, just because the actuator/transmission arrangements are mechanically different at each joint. A couple of kinematic configurations with mixed joints were already studied in [12], where it was found that *dynamic state feedback* was still needed for obtaining exact linear and decoupled closed-loop behavior. We generalize here the approach to the whole class of robot arms with mixed rigid/elastic joints and study the structural situations that arise from the point of view of applicability of nonlinear control methods. However, differently from [12], we assume the same simplifying hypothesis of [10] in the modeling phase. The purpose of this choice is two-fold: *i)* to obtain results based on a model that is as practical as possible; *ii)* to show that the need for dynamic compensation in this class of robots is *not* related to the use of a complete or of a reduced model.

A practical goal of our work is to find the minimum complexity of nonlinear feedback controllers enabling perfect trajectory tracking in nominal conditions. By avoiding the use of a state-space approach (i.e., working directly with the Euler-Lagrange dynamics), we are able to provide a complete answer to the question whether static or dynamic state feedback is needed. This answer is based on the structure of the dynamic model terms. In this way we easily recognize the physical role that dynamic feedback plays in robots with mixed rigid/elastic joints.

2 Dynamic Modeling and Preliminaries

Consider a serial robot arm with n rigid links and *all* joints being elastic. Let q be the n-vector of link positions, and θ represent the n-vector of actuator (viz., rotor of electrical motor) positions, as reflected through the transmission ratios. The difference $q_i - \theta_i$ is the ith joint deformation.

The rotors of the actuators are modeled as uniform bodies having their center of mass on the rotation axis, implying that both the inertia matrix and the gravity term in the dynamic model are independent from the actual internal position of the motors.

The *complete* dynamic model of a robot arm with *all elastic joints* can be written as [3]

$$\begin{bmatrix} B(q) & B_1(q) \\ B_1^T(q) & J \end{bmatrix} \begin{bmatrix} \ddot{q} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} C(q,\dot{q}) + C_1(q,\dot{\theta}) & C_2(q,\dot{q}) \\ C_3(q,\dot{q}) & 0 \end{bmatrix} \begin{bmatrix} \dot{q} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} g(q) \\ 0 \end{bmatrix} + \begin{bmatrix} K(q-\theta) \\ K(\theta - q) \end{bmatrix} = \begin{bmatrix} 0 \\ u \end{bmatrix}, \quad (1)$$

where $B(q)$ is the inertia matrix of the rigid arm, J is the constant diagonal inertia matrix of the actuators (reflected through the transmission ratios), $B_1(q)$ represents inertial couplings between actuators and links, the C's terms are Coriolis and centrifugal forces, $g(q)$ is the gravity vector of the rigid arm, K is the diagonal joint stiffness matrix, and u is the n-vector of torques provided by the actuators (performing work on θ).

Assuming that the angular part of the kinetic energy of each rotor is due only to its own spinning, a *reduced* dynamic model of a robot arm with *all elastic joints* has been introduced in [10]:

$$\begin{aligned} B(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) + K(q-\theta) &= 0 \quad (2) \\ J\ddot{\theta} + K(\theta - q) &= u, \quad (3) \end{aligned}$$

where the dynamic terms left over are the same appearing in (1).

In the following, we will use the shorthand notations $n(q,\dot{q}) = C(q,\dot{q})\dot{q} + g(q)$ and $q^{(i)} = d^i q/dt^i$.

It is known that model (2,3) can be transformed into a linear one through a static state feedback and a change of coordinates. Moreover, the same transformation will give a decoupled input-output behavior between new inputs v and outputs $y = q$ [10]. In fact, differentiating (2) with respect to time gives

$$B(q)q^{(3)} + \dot{B}(q)\ddot{q} + \dot{n}(q,\dot{q}) + K(\dot{q} - \dot{\theta}) = 0 \quad (4)$$

and

$$B(q)q^{(4)} + 2\dot{B}(q)q^{(3)} + \ddot{B}(q)\ddot{q} + \ddot{n}(q,\dot{q}) + K(\ddot{q} - \ddot{\theta}) = 0. \quad (5)$$

Setting $q^{(4)} = v$ and replacing $\ddot{\theta}$ from (3) in eq. (5), we can solve for u as

$$u = JK^{-1}(B(q)v + \alpha_s(q,\dot{q},\ddot{q},q^{(3)})) + K(\theta - q) \quad (6)$$

where

$$\alpha_s = (K + \ddot{B})\ddot{q} + 2\dot{B}q^{(3)} + \ddot{n}.$$

Equation (6) constitutes the *static* state feedback linearizing law, and depends, in general, on q and its first three time derivatives and on θ. However, the measurements available in principle for implementing any control law are the motor position θ and velocity $\dot{\theta}$, and the link position q and velocity $\dot{q}$, i.e., the full state of the system. The mapping from measurable quantities to $\ddot{q}$ and $q^{(3)}$ is obtained from eq. (2) and, respectively, eq. (4) as

$$\ddot{q} = -B^{-1}(q)\left[K(q-\theta) + g(q) + C(q,\dot{q})\dot{q}\right], \quad (7)$$

$$q^{(3)} = -B^{-1}(q)\left[\dot{B}(q)\ddot{q} + K(\dot{q} - \dot{\theta}) + \dot{n}(q,\dot{q})\right], \quad (8)$$

where eq. (7) should be used in eq. (8) to eliminate the dependence on $\ddot{q}$. The global linearizing coordinates are indeed the output $y = q$, the link velocity $\dot{y} = \dot{q}$ and the second and third time derivatives, $\ddot{y}$ and $y^{(3)}$, where eqs. (7,8) can be used.

Assume now that r out of the n robot joints are rigid while the remaining $n-r$ are elastic. By reordering the generalized coordinates we can write the *reduced* dynamic model of a robot arm with *mixed rigid/elastic joints* as follows:

$$\begin{bmatrix} B_{rr}(q) & B_{re}(q) \\ B_{er}(q) & B_{ee}(q) \end{bmatrix} \begin{bmatrix} \ddot{q}_r \\ \ddot{q}_e \end{bmatrix} + \begin{bmatrix} n_r(q,\dot{q}) \\ n_e(q,\dot{q}) \end{bmatrix} + \begin{bmatrix} 0 \\ K_e(q_e-\theta_e) \end{bmatrix} = \begin{bmatrix} u_r \\ 0 \end{bmatrix} \quad (9)$$

$$J_e\ddot{\theta}_e + K_e(\theta_e - q_e) = u_e. \quad (10)$$

Here, the r-vector q_r collects the generalized coordinates of the rigid joints, where the position of the driven link equals the position of the motor as reflected through the transmission ratio. The $(n-r)$-vector q_e contains the positions of the links driven by elastic joints, while the motor positions of these joints are collected in the $(n-r)$-vector θ_e. The inertia matrix, Coriolis, centrifugal, and gravity terms have been partitioned according to the dimensions of q_r and q_e. In particular, $B_{re}(q) = B_{er}^T(q)$. Being diagonal blocks of the symmetric inertia matrix, matrices $B_{rr}(q)$ and $B_{ee}(q)$ are invertible for any value of q. The torques produced by the motors at rigid and at elastic joints are labeled u_r and u_e, respectively. Finally, the constant diagonal matrices of joint stiffness K_e and actuator inertia J_e have now dimension $n-r$.

The model (9,10) is quite general and thus the dependence of the sub-blocks of the inertia matrix is in principle on the whole link position vector $q=(q_r,q_e)$. The particular arrangement of the arm kinematics may lead to special internal structures, in much the same way as in the model of a fully rigid arm.

3 Inverse Dynamics Computation

We show first that the system (9,10) is invertible from the input-output point of view, so that we can explicitly compute the nominal feedforward that enables exact reproduction of a desired trajectory. This is specified in terms of link motion, both for the links driven through rigid and through elastic joints.

From the second equation in (9) we get

$$\ddot{q}_e = -B_{ee}^{-1}(q)\left[K_e(q_e-\theta_e) + n_e(q,\dot{q}) + B_{er}(q)\ddot{q}_r\right]$$

that replaced into the first equation in (9) yields

$$\left[B_{rr}(q) - B_{re}(q)B_{ee}^{-1}(q)B_{er}(q)\right]\ddot{q}_r = u_r - n_r(q,\dot{q}) + B_{re}(q)B_{ee}^{-1}(q)\left[K_e(q_e-\theta_e)+n_e(q,\dot{q})\right].$$

This equation can be used to define the nominal torques to be applied at the *rigid* joints. Let $q^d(t) = (q_r^d(t), q_e^d(t))$ be the desired link trajectory, then

$$\begin{aligned} u_r^d(t) = {} & n_r(q^d(t),\dot{q}^d(t)) - B_{re}(q^d(t))B_{ee}^{-1}(q^d(t)) \cdot \\ & \left[K_e(q_e^d(t)-\theta_e^d(t)) + n_e(q^d(t),\dot{q}^d(t))\right] \quad (11) \\ & + \left[B_{rr}(q^d(t)) - B_{re}(q^d(t))B_{ee}^{-1}(q^d(t))B_{er}(q^d(t))\right]\ddot{q}_r^d(t). \end{aligned}$$

In eq. (11), the desired position $\theta_e^d(t)$ of the motors at the elastic joints is solved from the second equation in (9) as

$$\theta_e^d(t) = q_e^d(t) + K_e^{-1}\left[B_{er}(q^d(t))\ddot{q}_r^d(t) + B_{ee}(q^d(t))\ddot{q}_e^d(t) + n_e(q^d(t),\dot{q}^d(t))\right].$$

In order to compute the nominal torques to be applied at the *elastic* joints, we need to differentiate twice the second equation in (9) and then replace $\ddot{\theta}_e$ from (10). This gives

$$\begin{aligned} u_e^d(t) = {} & K_e(\theta_e^d(t) - q_e^d(t)) + J_eK_e^{-1}\left[B_{ee}(q^d(t))q_e^{(4),d}(t)\right. \\ & \left. + \alpha_e(q^d(t),\dot{q}^d(t),\ddot{q}^d(t),q^{(3),d}(t),q_r^{(4),d}(t))\right], \quad (12) \end{aligned}$$

where (dropping dependencies)

$$\begin{aligned} \alpha_e = {} & B_{er}q_r^{(4)} + 2\dot{B}_{er}q_r^{(3)} + \ddot{B}_{er}\ddot{q}_r \\ & + 2\dot{B}_{ee}q_e^{(3)} + \ddot{B}_{ee}\ddot{q}_e + \ddot{n}_e + K_e\ddot{q}_e. \end{aligned}$$

Equations (11) and (12) solve the inverse dynamics problem for robots with mixed rigid/elastic joints. We remark that:

- The reference link trajectories should be smooth enough to guarantee perfect tracking in nominal conditions, namely with matched initial state and perfectly known dynamic parameters. In particular, the link trajectories associated to *both* rigid and elastic joint types should be in general *four times* differentiable. The higher order requirement for $q_r^d(t)$, instead of an expected value of two, results from the inertial cross-couplings represented by matrix B_{er}.
- The previous formulas are useful in practice for deriving feedforward terms in a nonlinear regulator for trajectory tracking problems [13]. However, when *actual state* measurements are used in place of *nominal state* reference trajectories, this inverse dynamics computation does not allow to obtain input-output decoupling and linearization of the error dynamics.

4 Decoupling and Linearization via Static Feedback

We analyze next the input-output decoupling properties of the mixed rigid/elastic joint robot model. In doing so, we will find that, for a special class of robots, a *static* state feedback will be sufficient to this purpose. The decoupling static controller will then automatically yield also linearity of the closed-loop dynamics. This parallels the properties of both full rigid and full elastic joints robots.

Let the output be defined as

$$y = \begin{bmatrix} q_r \\ q_e \end{bmatrix}. \quad (13)$$

Taking twice the derivative of (13) gives

$$\ddot{y} = \begin{bmatrix} [B_{rr} - B_{re}B_{ee}^{-1}B_{er}]^{-1} & 0 \\ [B_{ee} - B_{er}B_{rr}^{-1}B_{re}]^{-1}B_{er}B_{rr}^{-1} & 0 \end{bmatrix} \begin{bmatrix} u_r \\ u_e \end{bmatrix} + \gamma(q, \dot{q}, \theta_e) = A(q)u + \gamma(q, \dot{q}, \theta_e).$$

Matrix $A(q)$ will be the so-called *decoupling matrix* of the system, provided that all its rows are non-zero [14]. The *relative degree* of each output $q_{r,i}$ associated to a *rigid* joint is $\rho_{r,i} = 2$ $(i = 1, \ldots, r)$. In fact, from the matrix identity

$$\begin{bmatrix} B_{rr} & B_{re} \\ B_{er} & B_{ee} \end{bmatrix} \cdot \begin{bmatrix} I & 0 \\ -B_{ee}^{-1}B_{er} & B_{ee}^{-1} \end{bmatrix} = \begin{bmatrix} B_{rr} - B_{re}B_{ee}^{-1}B_{er} & B_{er}B_{ee}^{-1} \\ 0 & I \end{bmatrix},$$

it follows that the first r rows of $A(q)$ are of full rank r and thus none of them is identically zero. By similar matrix algebra, we have for the last $(n - r)$ rows

$$\text{rank } [B_{ee} - B_{er}B_{rr}^{-1}B_{re}]^{-1}B_{er}B_{rr}^{-1} = \text{rank } B_{er} \leq \min\{r, n - r\}. \quad (14)$$

From this result, it is immediate to see that matrix $A(q)$ will coincide in part or totally with the decoupling matrix, depending on the structure of matrix B_{er}. In turn, the actual decoupling matrix will always be *singular*, unless $B_{er} = 0$. Thus, when $B_{er} \neq 0$ the necessary condition for input-output decoupling by static state feedback is violated.

Consider then the case $B_{er} = 0$. The relative degrees of the *elastic* joint outputs will be $\rho_{e,j} > 2$ $(j = 1, \ldots, n-r)$ and no conclusion can be drawn at this stage about the decoupling property. However, we show next that a static state feedback law will both decouple and linearize the closed-loop dynamics, provided that some additional conditions are satisfied by the robot dynamic model. Indeed, the dynamic equations (9,10) become in this case

$$\begin{aligned} B_{rr}(q)\ddot{q}_r + n_r(q, \dot{q}) &= u_r \\ B_{ee}(q)\ddot{q}_e + n_e(q, \dot{q}) + K_e(q_e - \theta_e) &= 0 \\ J_e\ddot{\theta}_e + K_e(\theta_e - q_e) &= u_e. \end{aligned} \quad (15)$$

As a stronger assumption we require the term $n_e(q, \dot{q})$ to be independent from $\dot{q}_r$, i.e., $n_e = n_e(q, \dot{q}_e)$. Using the definition of Christoffel symbols for the Coriolis and centrifugal terms, it can be shown that this happens iff the diagonal blocks of the inertia matrix in eqs. (15) have only a local dependence of the form $B_{rr} = B_{rr}(q_r)$ and $B_{ee} = B_{ee}(q_e)$. This, in turn, implies also that $n_r = n_r(q, \dot{q}_e)$. The above assumption guarantees that all outputs associated to elastic joints (i.e., the last $n - r$ components of y) will have relative degree *equal to four*.

Define then the static state feedback control law

$$u_r = B_{rr}(q_r)v_r + n_r(q, \dot{q}_r) \quad (16)$$

$$u_e = J_eK_e^{-1}(B_{ee}(q_e)v_e + \alpha'_s(q, \dot{q}, \ddot{q}_e, q_e^{(3)})) + K_e(\theta_e - q_e) \quad (17)$$

with v_r and v_e as new inputs, and where (dropping dependencies)

$$\alpha'_s = (K_e + \ddot{B}_{ee})\ddot{q}_e + 2\dot{B}_{ee}q_e^{(3)} + \ddot{n}_e$$

in which

$$\begin{aligned} \ddot{q}_e &= -B_{ee}^{-1}(q_e)\,[K_e(q_e - \theta_e) + n_e(q, \dot{q}_e)] \\ q_e^{(3)} &= -B_{ee}^{-1}(q_e)\left[\dot{B}_{ee}(q_e)\ddot{q}_e + K_e(\dot{q}_e - \dot{\theta}_e) + \dot{n}_e(q, \dot{q}_e)\right]. \end{aligned}$$

The assumption on B_{rr} and B_{ee} ensures that the last term in $q_e^{(3)}$ will not depend on $\ddot{q}_r$, a fact that would lead again to a singular decoupling matrix thus invalidating the static feedback approach.

Using the control (16,17), we finally obtain the linear and input-output decoupled closed-loop system

$$\begin{aligned} \ddot{q}_r &= v_r \\ q_e^{(4)} &= v_e. \end{aligned}$$

This result is summarized in the following

Theorem 1 *The dynamic model (9,10) of robots with mixed rigid/elastic joints, with output (13), can be input-output decoupled and fully linearized by static state feedback if and only if:*

i) $B_{er} = 0$;

ii) $B_{rr} = B_{rr}(q_r)$ *and* $B_{ee} = B_{ee}(q_e)$.

Whenever it applies, the required static state feedback law is given by (16,17).

We note that the decoupling and linearizing control law (16,17) can be implemented using only the measures of the original state $(q_r, \dot{q}_r, q_e, \dot{q}_e, \theta_e, \dot{\theta}_e)$. Moreover, in this case the computed torque technique for rigid robots and the linearizing control of [10] are recovered in the two limit cases of $r = n$ and $r = 0$, respectively.

It is interesting to check how Theorem 1 applies to some examples of robot arms.

Example 1. (*Cylindrical (PRP) arm*) The robot inertia matrix takes on the diagonal structure

$$B(q) = \begin{bmatrix} b_{11} & 0 & 0 \\ 0 & b_{22}(q_3) & 0 \\ 0 & 0 & b_{33} \end{bmatrix}.$$

When the second and third joints are of the same kind (rigid or elastic), the system is completely linearizable by static feedback law. Note that when *all* joints are elastic (so that B_{rr} vanishes), this robot is feedback linearizable by static state feedback [15]. In fact, the complete and reduced dynamic models (i.e., eqs. (1) and (2,3)) coincide for this arm.

Example 2. (*3R elbow-type manipulator*) The inertia matrix has the structure

$$B(q) = \begin{bmatrix} b_{11}(q_2, q_3) & 0 & 0 \\ 0 & b_{22}(q_3) & b_{23}(q_3) \\ 0 & b_{23}(q_3) & b_{33} \end{bmatrix}.$$

If the base joint is rigid and the shoulder and elbow joints are elastic (or vice versa), this robot satisfies only condition *i)* of Theorem 1, but not condition *ii)*. If the shoulder and the elbow joints are of a different kind, both conditions are violated. Static feedback will then always fail for linearization and decoupling purposes.

5 Decoupling and Linearization via Dynamic Feedback

Consider now the general case in which one or both conditions of Theorem 1 are violated. In these cases the decoupling matrix will always be singular. Input-output decoupling and full state linearization may still be obtained, but we need to pursue a more general strategy based on *dynamic* state feedback. We will show this in a constructive way.

We start by adding *two integrators* in series on each input channel associated to the rigid joints. The state of each pair of integrators will be conveniently denoted as $\theta_{r,i}$ and $\dot{\theta}_{r,i}$, leading to the *dynamic extension*

$$u_r = \theta_r, \qquad \ddot{\theta}_r = u_{re}, \tag{18}$$

where u_{re} is an r-vector of new inputs. These added integrators will become the state of the dynamic compensator. The combination of (9,10) and (18) is rewritten as

$$\begin{bmatrix} B_{rr}(q) & B_{re}(q) \\ B_{er}(q) & B_{ee}(q) \end{bmatrix} \begin{bmatrix} \ddot{q}_r \\ \ddot{q}_e \end{bmatrix} + \begin{bmatrix} n_r(q,\dot{q}) \\ n_e(q,\dot{q}) \end{bmatrix} + \begin{bmatrix} -\theta_r \\ K_e(q_e-\theta_e) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{19}$$

and

$$\begin{bmatrix} I & 0 \\ 0 & J_e \end{bmatrix} \begin{bmatrix} \ddot{\theta}_r \\ \ddot{\theta}_e \end{bmatrix} + \begin{bmatrix} 0 \\ K_e(\theta_e-q_e) \end{bmatrix} = \begin{bmatrix} u_{re} \\ u_e \end{bmatrix}. \tag{20}$$

Defining the system outputs as in (13), and denoting by $B(q)$ and $n(q,\dot{q})$ the overall robot arm inertia matrix and the nonlinear dynamic terms in (19), we have

$$\begin{aligned} \dot{y} &= \begin{bmatrix} \dot{q}_r \\ \dot{q}_e \end{bmatrix} \\ \ddot{y} &= B^{-1}(q)\left(\begin{bmatrix} \theta_r \\ K_e(\theta_e-q_e) \end{bmatrix} - n(q,\dot{q})\right) \qquad (21) \\ y^{(3)} &= -B^{-1}(q)\left(\dot{B}(q)B^{-1}(q)\left(\begin{bmatrix} \theta_r \\ K_e(\theta_e-q_e) \end{bmatrix} + n(q,\dot{q})\right)\right. \\ &\qquad \left. - \begin{bmatrix} \dot{\theta}_r \\ K_e(\dot{\theta}_e-\dot{q}_e) \end{bmatrix} + \dot{n}(q,\dot{q})\right). \end{aligned}$$

Since no input appears explicitly at this differential stage, the relative degree is *larger than three* for all outputs. Finally, taking twice the time derivative of (19) and using (20) yields

$$B(q)q^{(4)} + 2\dot{B}(q)q^{(3)} + \ddot{n}(q,\dot{q}) + \begin{bmatrix} 0 \\ K_e\ddot{q}_e \end{bmatrix}$$

$$= \begin{bmatrix} u_{re} \\ K_eJ_e^{-1}[u_e + K_e(q_e-\theta_e)] \end{bmatrix}.$$

The resulting decoupling matrix of the *extended* robot system

$$A(q) = B^{-1}(q)\begin{bmatrix} I & 0 \\ 0 & K_eJ_e^{-1} \end{bmatrix}$$

is *nonsingular* for all q (and θ), while the relative degrees are $\rho_i = 4$, $i = 1, \ldots, n$. Since the sum of the relative degrees equals the dimension of the extended state space, namely $[2r + 4(n-r)] + 2r = 4n$, the input-output decoupling law

$$\begin{bmatrix} u_{re} \\ u_e \end{bmatrix} = A^{-1}(q)v + \alpha(q,\dot{q},\ddot{q},q^{(3)}), \tag{22}$$

with

$$\alpha = \begin{bmatrix} I & 0 \\ 0 & K_eJ_e^{-1} \end{bmatrix} \cdot \left(2\dot{B}(q)\ddot{q} + \ddot{n}(q,\dot{q}) + \begin{bmatrix} 0 \\ K_e\ddot{q}_e + K_eJ_e^{-1}K_e(\theta_e-q_e) \end{bmatrix}\right),$$

will also completely linearize the closed-loop system. The linearizing coordinates are defined by (13) and (21). The combination of the dynamic extension (18) with the static feedback law (22) from the extended state gives the required *dynamic state feedback law*.

This result is summarized in the following

Theorem 2 *When the conditions of Theorem 1 are violated, the dynamic model (9,10) of robots with mixed rigid/elastic joints, with output (13), can be input-output decoupled and fully linearized by dynamic state feedback. The required dynamic state feedback law is given by (18,22).*

We note explicitly that the dependence of α on $\ddot{q}$ and $q^{(3)}$ can be transformed into a dependence on q, $\dot{q}$, θ_e, and $\dot{\theta}_e$ (the original state of the robot) and on θ_r and $\dot{\theta}_r$ (the state of the dynamic compensator) by means of the right hand sides of (21), used in the proper sequence. The closed-loop system is described in the linearizing coordinates by n chains of four input-output integrators, each from v_i to q_i, i.e.,

$$q^{(4)} = v.$$

Stabilization and trajectory tracking can then be performed in the usual way, on the linear side of the problem.

6 Conclusions

We have analyzed the theoretical aspects of the problem of input-output decoupling and exact linearization by state feedback for robots having some joints rigid and some elastic. It was proved that static state feedback works successfully if and only if *no* coupling is present in the robot inertia matrix between the variables of the rigid joints and those of the elastic joints. This is a rather strict condition, hardly ever satisfied by robot kinematic/dynamic arrangements. However, when this condition is not met, dynamic state feedback will guarantee a linear and decoupled closed-loop dynamics. In these cases, the dimension of the dynamic compensator is always equal to $2r$, where r (with $0 < r < n$) is the number of rigid joints.

The obtained results may be somewhat surprising. While the two extreme cases of robots with all rigid joints and with all elastic joints are both linearizable and decouplable by static feedback, dynamic feedback is required in most cases for the intermediate situation if we desire the same control properties. As a matter of fact, the role of dynamic feedback is to slow down the effects of torques applied at the rigid joints with respect to those acting on the motor side of the elastic joints.

We were able to derive closed-form expressions for both the static and the dynamic feedback controllers. These may serve as a reference basis for designing and implementing robust and/or adaptive controllers.

All results were obtained using, for the elastic joint modeling part, the reduced model of Spong [10] which neglects small inertial terms. Indeed, a similar analysis can be performed starting from the complete dynamic model of robots with mixed joints.

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