Proceedings of the 2002 IEEE
International Conference on Robotics & Automation
Washington, DC • May 2002

A Simple STLC Test for Mechanical Systems Underactuated by One Control*

Alessandro De Luca Stefano Iannitti

Dipartimento di Informatica e Sistemistica
Università degli Studi di Roma "La Sapienza"
Via Eudossiana 18, 00184 Roma, Italy
deluca@dis.uniroma1.it
iannitti@labrob1.ing.uniroma1.it

Abstract

We consider the controllability problem, i.e., the existence of a suitable control input that achieves a desired reconfiguration, for underactuated mechanical systems. Since there is no general analytic tool for investigating this natural controllability property in nonlinear systems, one possibility is to study small-time local controllability (STLC), a property which is sufficient for stating controllability. The available STLC conditions require the computation of Lie brackets on the classical state-space form of the dynamic model equations. In this paper, we provide a simple sufficient condition for testing STLC in underactuated mechanical systems with n degrees of freedom and $n-1$ control inputs, directly based on the terms of the system inertia matrix. As an application, we analyze the STLC of planar robots with n rotational joints, one of which is passive.

1 Introduction

Underactuated mechanical systems are defined as systems having fewer control inputs than generalized coordinates. This class of dynamic systems is characterized by the presence of second-order differential constraints, i.e., involving generalized accelerations, and includes for example robot manipulators with passive joints and robots with elastic joints or flexible links [1]. The existence of such differential constraints on the motion may destroy controllability, namely the possibility of transfering the system between two given states by means of a suitable input. The latter, when it exists, may be defined as an open-loop command (motion planning) or in a closed-loop mode (feedback control). Certainly, controllability is lost when the second-order differential constraints are totally integrable, a situation that can be tested using the necessary and sufficient conditions given in [2]. As a result, since the interesting underactuated mechanical systems have non-integrable differential constraints, they are sometimes referred to as second-order nonholonomic systems.

An analytic tool for checking the above natural controllability of an underactuated mechanical system is not available. The linear approximation around an equilibrium can be used as a preliminary step for assessing local controllability. Unfortunately, many underactuated systems (e.g., robots with passive joints moving in the absence of gravity) are not controllable in the first approximation. For nonlinear systems with (non-trivial) drift, a necessary (but not sufficient) condition for controllability is the so-called local accessibility. The associated Lie algebraic test (LARC, see [3]) is always passed by underactuated mechanical systems with non-integrable second-order differential constraints [4].

On the other hand, controllability is implied by the so-called small-time local controllability (STLC). A system is said to be small-time local controllable from an equilibrium state x^e if, for every time $T > 0$ and for every neighborhood V of x^e, the set of states reachable from x^e within T while remaining in V is open and contains x^e in its interior. Only sufficient conditions are available for testing STLC (see [5] and [6]). Roughly speaking, the STLC property is of interest because it implies that only local maneuvers are needed to steer the system between two sufficiently close states. One example of successful application of the STLC sufficient conditions of [5] for an underactuated mechanical system is the planar 3R robot with passive last joint reported in [7].

We note that all the above conditions are defined in terms of the state of a generic nonlinear system. For mechanical systems, the state is usually represented by its configuration variables and associated general-

*Work supported by MURST within the *MISTRAL* project.

ized velocities, while most of the time one is interested in motion planning and control only between equilibrium configurations (thus, with zero velocity). The sufficient conditions for STLC given in [6] have been specialized to mechanical systems underactuated by one control in [8] and with higher degree of underactuation in [9] and [4]. However, these conditions still require the transformation of the Lagrangian second-order system dynamics into state-space equations.

In this paper, we improve the result of [8], deriving sufficient STLC conditions for mechanical systems with n degrees of freedom and $n-1$ control inputs (i.e., one degree of underactuation) to be tested directly on the elements of the system inertia matrix. We shall then use these conditions to prove STLC for a class of underactuated planar robots. Our result is simpler but along the same lines of the configuration controllability (and small-time local configuration controllability) analysis introduced in [10], where properties are studied for mechanical systems starting at an equilibrium within a more formal setting. A similar reduction of the controllability of second-order mechanical systems to the analysis of configuration-dependent terms alone has been proposed also in [11], through the introduction of the kinematic controllability concept. It should be noted that kinematic controllability is neither implied by nor implies small-time local controllability.

The paper is organized as follows. Preliminary material is reported in Sect. 2, while our main theoretical result is given in Sect. 3. Example applications of the proposed STLC test are presented in Sect. 4 for planar robots with $n \geq 3$ rotational joints underactuated by one control.

2 Preliminaries

For a mechanical systems with n degrees of freedom and $m = n-1$ control inputs, denote by $q \in \mathbb{R}^n$ the generalized coordinates and by $\tau \in \mathbb{R}^m$ the control input. Following the Lagrangian approach, the dynamics is written as

$$B(q)\ddot{q} + h(q,\dot{q}) + g(q) = F(q)\,\tau \tag{1}$$

where $B > 0$ is the $n \times n$ symmetric inertia matrix, h is the centrifugal and Coriolis vector, $g = (\partial U/\partial q)^T$ is the vector of potential terms, and F is the $n \times m$ input matrix assumed of full rank. No dissipative effects are considered. The velocity-related vector can be expressed in the form

$$h(q,\dot{q}) = S(q,\dot{q})\,\dot{q},$$

where S is a $n \times n$ matrix such that $\dot{B} - 2S$ is skew-symmetric. The generic element s_{ij} of S can be written as

$$s_{ij} = \sum_{k=1}^{n} c_{ijk}\dot{q}_k,$$

using the Christoffel coefficients

$$c_{ijk} = c_{ikj} = \frac{1}{2}\left(\frac{\partial b_{ij}}{\partial q_k} + \frac{\partial b_{ik}}{\partial q_j} - \frac{\partial b_{jk}}{\partial q_i}\right)$$

where b_{ij} are the elements of the inertia matrix B.

Assume, without loss of generality, that the last q_n coordinate is not actuated[1] and partition the configuration vector as $q = (q_a, q_n)$, with $q_a \in \mathbb{R}^{n-1}$ and $q_n \in \mathbb{R}$. Correspondingly, eq. (1) can be rewritten as

$$\begin{bmatrix} B_{aa} & B_{an} \\ B_{an}^T & b_{nn} \end{bmatrix}\begin{bmatrix} \ddot{q}_a \\ \ddot{q}_n \end{bmatrix} + \begin{bmatrix} S_{aa} & S_{an} \\ S_{na} & s_{nn} \end{bmatrix}\begin{bmatrix} \dot{q}_a \\ \dot{q}_n \end{bmatrix} + \begin{bmatrix} g_a \\ g_n \end{bmatrix} = \begin{bmatrix} F_a \\ 0 \end{bmatrix}\tau, \tag{2}$$

where F_a is an $(n-1)\times(n-1)$ nonsingular matrix and we have dropped dependence on q and $\dot{q}$ for compactness. The second (scalar) equation in (2) represents a second-order differential constraint on system motion.

To simplify analysis, we perform a preliminary partial feedback linearization of the system. Introducing a new input vector $u \in \mathbb{R}^{n-1}$ and defining the static state feedback

$$\tau = F_a^{-1}(q)\left[\left(B_{aa}(q) - \frac{B_{an}(q)}{b_{nn}(q)}B_{an}^T(q)\right)\,u + C(q,\dot{q})\right], \tag{3}$$

where $C = (S_{aa} - \frac{B_{an}}{b_{nn}}S_{na})\dot{q}_a + (S_{an} - \frac{B_{an}}{b_{nn}}s_{nn})\dot{q}_n + g_a - \frac{B_{an}}{b_{nn}}g_n$, system (2) becomes

$$\ddot{q}_a = u \tag{4}$$

$$\ddot{q}_n = -\frac{1}{b_{nn}}\left[S_{na}\dot{q}_a + s_{nn}\dot{q}_n + g_n + B_{an}^T u\right]. \tag{5}$$

Equations (3-5) are globally defined since $b_{nn} > 0$ and $(B_{aa} - B_{an}B_{an}^T/b_{nn}) > 0$, for all $q \in \mathbb{R}^n$ (see [12]).

We want to derive sufficient STLC conditions for the underactuated mechanical system (2), or equivalently (4–5), expressed in terms of the elements of the inertia matrix $B(q)$ only. The classical approach used in the control literature is based on the state-space equations for system (4–5). By defining $x = (x_1, x_2, x_3, x_4) = (q_a, q_n, \dot{q}_a, \dot{q}_n) \in \mathbb{R}^{2n}$, we have

$$\dot{x} = f(x) + \sum_{i=1}^{n-1} g_i(x)u_i = \begin{bmatrix} x_3 \\ x_4 \\ 0 \\ \varphi(x) \end{bmatrix} + \sum_{i=1}^{n-1}\begin{bmatrix} 0_{n-1} \\ 0 \\ e_i \\ \gamma_i(x_1,x_2) \end{bmatrix} u_i, \tag{6}$$

[1] If this is not the case, one can always perform a change of coordinates leading to a system in the form of eq. (2).

where $e_i \in \mathbb{R}^{n-1}$ is the i-th standard basis vector in $\mathbb{R}^{n-1}$, $\varphi(x) = -(S_{na}x_3 + s_{nn}x_4 + g_n)/b_{nn}$, and $\gamma_i(x_1, x_2) = -b_{ni}/b_{nn}$.

We finally recall a result that will be used in the proof of our main result. Denote by $[f,g] = \frac{\partial g}{\partial x}f - \frac{\partial f}{\partial x}g$ the Lie bracket of two vector fields $f(x)$ and $g(x)$.

Lemma 1 [8] *Consider the underactuated mechanical system in the state-space form (6). If at an equilibrium state $x^e = (q_a^e, q_n^e, 0, 0)$*

$$\exists i,j \in \{1,\dots,n-1\}, i \neq j : \begin{cases} [g_i,[f,g_j]](x^e) \neq 0 \\ [g_i,[f,g_i]](x^e) = 0, \end{cases}$$

then system (6) is small-time locally controllable at x^e.

3 Main result

Our main result is given by the following

Proposition 1 *Consider the mechanical system underactuated by one control described by eq. (2). If there exist two indices $i,j \in \{1,\dots,n-1\}$, $i \neq j$, such that at an equilibrium configuration q^e*

$$\left[b_{ni}\frac{\partial b_{nj}}{\partial q_n} + b_{nj}\frac{\partial b_{ni}}{\partial q_n} - b_{nn}\frac{\partial b_{ij}}{\partial q_n}\right]_{q=q^e} \neq 0 \qquad (7)$$

$$\left[2b_{ni}\frac{\partial b_{ni}}{\partial q_n} - b_{nn}\frac{\partial b_{ii}}{\partial q_n}\right]_{q=q^e} = 0, \qquad (8)$$

then system (2) is small-time locally controllable at q^e.

Proof. Since controllability properties are preserved under feedback transformations, we first apply the static feedback (3) and reconduct the system to the state-space form (6). According to Lemma 1, the evaluation of the Lie brackets $[g_i,[f,g_j]]$ at an equilibrium point $x^e = (q^e, 0)$ yields

$$[g_i,[f,g_j]] = \begin{bmatrix} 0_{n-1} \\ 0 \\ 0_{n-1} \\ \alpha_{ij}(q), \end{bmatrix}$$

with

$$\alpha_{ij}(q) = \frac{\partial \gamma_j}{\partial q_i} + \frac{\partial \gamma_i}{\partial q_j} + \frac{\partial \gamma_j}{\partial q_n}\gamma_i + \frac{\partial \gamma_i}{\partial q_n}\gamma_j - \left[\frac{\partial^2\varphi}{\partial \dot{q}_n^2}\gamma_i\gamma_j + \frac{\partial^2\varphi}{\partial \dot{q}_i\partial \dot{q}_j} + \frac{\partial^2\varphi}{\partial \dot{q}_i\partial \dot{q}_n}\gamma_j + \frac{\partial^2\varphi}{\partial \dot{q}_j\partial \dot{q}_n}\gamma_i\right], \quad (9)$$

where we used directly the generalized coordinates q. Thus, in order to state STLC, one has to find two indices $i,j \in \{1,\dots,n-1\}$, $i \neq j$, such that $\alpha_{ij}(q^e) \neq 0$ and $\alpha_{ii}(q^e) = 0$.

Recalling that a diagonal element of the inertia matrix of a mechanical system is always independent from the associated generalized coordinate, i.e., $\partial b_{ii}/\partial q_i = 0$, $\forall i = 1,\dots,n$, we have $b_{nn}(q) = b_{nn}(q_a)$. Moreover, since the Christoffel coefficient $c_{nnn} = 0$, the $s_{nn}(q,\dot{q})$ term does not depend on $\dot{q}_n$, i.e., $s_{nn}(q,\dot{q}) = s_{nn}(q,\dot{q}_a)$. With this in mind, we evaluate explicitly each term in eq. (9):

$$\begin{aligned}
\frac{\partial \gamma_j}{\partial q_i} &= -\frac{1}{b_{nn}^2(q_a)}\left[\frac{\partial b_{nj}}{\partial q_i}b_{nn} - b_{nj}\frac{\partial b_{nn}}{\partial q_i}\right] \\
\frac{\partial \gamma_i}{\partial q_j} &= -\frac{1}{b_{nn}^2(q_a)}\left[\frac{\partial b_{ni}}{\partial q_j}b_{nn} - b_{ni}\frac{\partial b_{nn}}{\partial q_j}\right] \\
\frac{\partial \gamma_j}{\partial q_n}\gamma_i &= \frac{b_{ni}(q)}{b_{nn}^2(q_a)}\frac{\partial b_{nj}}{\partial q_n} \\
\frac{\partial \gamma_i}{\partial q_n}\gamma_j &= \frac{b_{nj}(q)}{b_{nn}^2(q_a)}\frac{\partial b_{ni}}{\partial q_n} \\
\frac{\partial^2\varphi}{\partial \dot{q}_i\dot{q}_j} &= -\frac{1}{b_{nn}(q_a)}\left(\frac{\partial b_{nj}}{\partial q_i} + \frac{\partial b_{ni}}{\partial q_j} - \frac{\partial b_{ji}}{\partial q_n}\right) \\
\frac{\partial^2\varphi}{\partial \dot{q}_j\dot{q}_n}\gamma_i &= \frac{b_{ni}(q)}{b_{nn}^2(q_a)}\left(\frac{\partial b_{nj}}{\partial q_n} + \frac{\partial b_{nn}}{\partial q_j} - \frac{\partial b_{jn}}{\partial q_n}\right) \\
\frac{\partial^2\varphi}{\partial \dot{q}_i\dot{q}_n}\gamma_j &= \frac{b_{nj}(q)}{b_{nn}^2(q_a)}\left(\frac{\partial b_{nn}}{\partial q_i} + \frac{\partial b_{ni}}{\partial q_n} - \frac{\partial b_{ni}}{\partial q_n}\right) \\
\frac{\partial^2\varphi}{\partial \dot{q}_n^2}\gamma_i\gamma_j &= -\frac{\partial}{\partial \dot{q}_n}\left[\frac{2}{b_{nn}^3(q_a)}b_{ni}(q)b_{nj}(q)\sum_{k=1}^{n-1} c_{nnk}\dot{q}_k\right] = 0.
\end{aligned}$$

Since $b_{nn}(q_a) > 0$, $\forall q_a \in \mathbb{R}^{n-1}$, we obtain

$$\alpha_{ij}\, b_{nn}^2 = b_{ni}\frac{\partial b_{nj}}{\partial q_n} + b_{nj}\frac{\partial b_{ni}}{\partial q_n} - b_{nn}\frac{\partial b_{ij}}{\partial q_n},$$

and therefore $\alpha_{ij}(q^e) \neq 0$ iff condition (7) holds and $\alpha_{ii}(q^e) = 0$ iff condition (8) holds, concluding the proof. ∎

The following remarks are in order.

- In deriving eqs. (7-8), no special assumption has been made on the internal structure of the underactuated mechanical system (e.g., triangular or Caplygin system, as in [8] and [9]) or about the dependence of matrices $B(q)$ and $S(q,\dot{q})$ on q_a and q_n. Equations (7-8) exploit indeed the structural properties of the Christoffel coefficients.
- The presence of a potential term $g(q)$ does not affect eqs. (7-8). As a consequence, if a robot with one unactuated joint satisfies at q^e these sufficient conditions for STLC in the absence of gravity, then the same robot is STLC at q^e in the presence of gravity, provided q^e is an equilibrium configuration (i.e., $g(q^e) = 0$). Note, however, that the presence of $g(q)$ strongly restricts the region of system equilibria.

- Although the conditions are written assuming the last coordinate as the unactuated one, it is easy to see that Proposition 1 holds for mechanical systems with a generic k-th unactuated coordinate, substituting index n with k in eqs. (7–8).

- The proposed sufficient test for STLC has two limitations. First, it can be used only for systems with $n \geq 3$ degrees of freedom, similarly to the conditions in [4, 8, 9]. For systems with $n = 2$ and $m = 1$, one should resort to the general sufficient conditions given in [5] or in [6]. Second, just as the seminal result of Sussmann [5], it is not coordinate independent and therefore one should in general look for a suitable choice of coordinates.

4 STLC of underactuated planar robots with one passive joint

We apply our main result to analyze STLC of planar robots with rotational joints, one of which is unactuated. We shall put an overline on the passive joint (i.e., $\overline{R}$) to denote actuator configuration. A recent result about the controllability of this class of manipulators can be found in [13]. Using the concept of time-reversed control, it has been shown that an input that steers the robot from an initial configuration to a desired one exists, provided that the first joint is actuated. Here, under the same hypothesis, we want to show that a planar robot with one unactuated joint is in addition small-time locally controllable. Thus, at least theoretically, large maneuvers are not needed to steer such an underactuated robot between two close equilibrium configurations.

The test will be applied to robots with $n \geq 3$ joints. The planar robot with $n = 2$ rotational joints has been already proven to be not controllable when the first joint is passive in [2], and controllable for reversed actuation in several works (see [14]–[16]). In particular, in [16] it is pointed out that, although the system does not satisfy the sufficient conditions of [6] and thus nothing can be stated about its STLC, the need for spinning maneuvers when steering the passive joint from certain configuration regions suggests the lack of this property for the planar $R\overline{R}$ robot.

We state the following result, which will be shown to hold through analysis of the possible alternatives for the passive joint positioning.

Proposition 2 *A planar manipulator with $n \geq 3$ rotational joints, one of which is passive, is small-time local controllable (STLC) provided that the first joint is actuated.*

A general dynamic model of a planar manipulator with n rotational joints can be found in [13]. However, a full model of the system is not necessary for our analysis. Moreover, simple considerations show that, in order to prove STLC of planar robots with $n \geq 3$ rotational joints and underactuated by one control, it is sufficient to reduce the analysis to the case $n = 3$. The following situations may arise for $n \geq 4$:

I. The first joint is passive. In this case, the general n-dof robot inherits the negative control properties of an $\overline{R}RR$ robot (see Sect. 4.3).

II. The second joint is passive. In this case, any motion of the last $n-3$ joints can be realized through their local actuators. These joints can be then blocked, and the last $n-2$ links considered as a single (third) rigid body. If the STLC property of a $R\overline{R}R$ robot holds, the same applies also to the general case.

III. One of the last $n-2$ joints is passive, say the k-th joint, $k \in (3, \dots, n)$. In this case, the proximal joints 3 to $k-1$ and the distal joints $k+1$ to n, all actuated, can be steered to their desired positions and then blocked, so as to reduce robot motion to that of three rigid bodies. The system reflects then the control properties of an $RR\overline{R}$ robot.

In the remaining analysis, carried out in the next sections for $n = 3$ and $m = 2$ actuated joints, we shall make use of two different sets of generalized coordinates (see Fig. 1): the set $\{q_1, q_2, q_3\}$ consists of the three relative link angles (with the first one measured w.r.t. an arbitrary reference x-axis), while the set $\{x, y, \theta_3\}$ expresses the cartesian location of the third joint and the absolute angle of the third link w.r.t. an arbitrary reference x-axis.

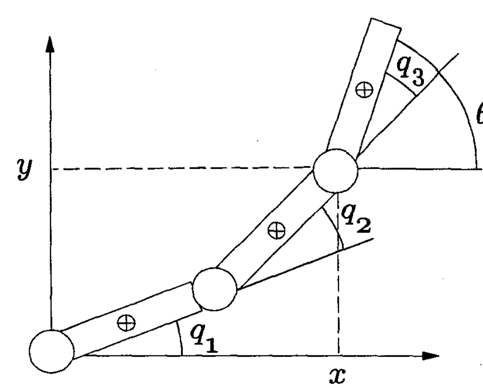

Figure 1: Coordinate variables for RRR planar robot

4.1 Actuator configuration: $RR\overline{R}$

This system has already been found to be STLC in [7]. We will check this result with our Proposition 1. Relying on the coordinate system $\{x, y, \theta_3\}$, the inertia matrix $B(q)$ is [17]

$$B(q) = \begin{bmatrix} b_{11}(x,y) & b_{12}(x,y) & -m_3 d_3 s_3 \\ b_{12}(x,y) & b_{22}(x,y) & m_3 d_3 c_3 \\ -m_3 d_3 s_3 & m_3 d_3 c_3 & I_3 + m_3 d_3^2 \end{bmatrix},$$

where $s_3 = \sin\theta_3$, $c_3 \doteq \cos\theta_3$, and I_3, m_3, and d_3 denote, respectively, the baricentral inertia, mass and distance of the center of mass from its base for the third link.

Since θ_3 can be defined arbitrarily, any equilibrium of the system may be translated to the equilibrium point $q^e = (x^e, y^e, \theta_3^e) = (x^e, y^e, \pi/2)$. Choosing $i = 1$, $j = 2$, and $n = 3$, conditions (7–8) become

$$\left[\frac{\partial b_{32}}{\partial \theta_3} b_{31} + \frac{\partial b_{31}}{\partial \theta_3} b_{32} - \frac{\partial b_{12}}{\partial \theta_3} b_{33}\right]_{\theta_3 = \frac{\pi}{2}} = -m_3^2 d_3^2 \cos 2\theta_3 \big|_{\theta_3 = \frac{\pi}{2}} \neq 0$$

$$\left[2 b_{31} \frac{\partial b_{31}}{\partial \theta_3} - b_{33} \frac{\partial b_{11}}{\partial \theta_3}\right]_{\theta_3 = \frac{\pi}{2}} = m_3^2 d_3^2 \sin 2\theta_3 \big|_{\theta_3 = \frac{\pi}{2}} = 0,$$

which proves the STLC property at any equilibrium.

4.2 Actuator configuration: $R\overline{R}R$

In this case, we rely on the coordinate system $\{q_1, q_2, q_3\}$. The inertia matrix $B(q)$ takes the form

$$B(q) = \begin{bmatrix} b_{11}(q_2,q_3) & b_{12}(q_2,q_3) & b_{13}(q_2,q_3) \\ b_{12}(q_2,q_3) & b_{22}(q_3) & b_{23}(q_3) \\ b_{13}(q_2,q_3) & b_{23}(q_3) & b_{33} \end{bmatrix}, \quad (10)$$

with

$$\begin{aligned} b_{11} &= A_1 + 2A_4 c_2 + 2A_5 c_3 + 2A_6 c_{23} \\ b_{12} &= A_2 + A_4 c_2 + 2A_5 c_3 + A_6 c_{23} \\ b_{13} &= A_3 + A_5 c_3 + A_6 c_{23} \\ b_{22} &= A_2 + 2A_5 c_3 \\ b_{23} &= A_3 + A_5 c_3 \\ b_{33} &= A_3, \end{aligned}$$

where $s_i = \sin q_i$, $c_i = \cos q_i$, $s_{ij} = \sin(q_i + q_j)$ and $c_{ij} = \cos(q_i + q_j)$, with $i, j = 2, 3$, and the constant coefficients are given by

$$\begin{aligned} A_1 &= I_1 + I_2 + I_3 + m_1 d_1^2 + m_2(l_1^2 + d_2^2) \\ &\quad + m_3(l_1^2 + l_2^2 + d_3^2) \\ A_2 &= I_2 + I_3 + m_2 d_2^2 + m_3(l_2^2 + d_3^2) \\ A_3 &= I_3 + m_3 d_3^2 \\ A_4 &= l_1(m_2 d_2 + m_3 l_2) \\ A_5 &= m_3 l_2 d_3 \\ A_6 &= m_3 l_1 d_3, \end{aligned}$$

being m_i, l_i, and I_i, respectively, the mass, length, and baricentral inertia of link i, and d_i the distance between the joint i and the center of mass of link i (with $i = 1, \ldots, 3$).

Choosing $j = 1$, $n = 2$, and $i = 3$, conditions (7–8) become

$$b_{23}\frac{\partial b_{12}}{\partial q_2} + b_{12}\frac{\partial b_{23}}{\partial q_2} - b_{22}\frac{\partial b_{13}}{\partial q_2} = (A_2 - A_3 + A_5 c_3) A_6 s_{23} - (A_3 + A_5 c_3) A_4 s_2$$

and

$$2 b_{23}\frac{\partial b_{23}}{\partial q_2} - b_{22}\frac{\partial b_{33}}{\partial q_2} \equiv 0.$$

Away from kinematic singularities (i.e., robot arm fully stretched or folded along a common line), the former is generically nonzero. This proves a generic STLC property of the $R\overline{R}R$ planar robot.

4.3 Actuator configuration: $\overline{R}RR$

The $\overline{R}RR$ configuration (as well as the $\overline{R}nR$ configuration, for any integer n) has been proven to be uncontrollable in [13], by means of an angular momentum analysis. In fact, since the first coordinate q_1 is cyclic (it never appears in the inertia matrix, given again by eq. 10) and the first joint is passive, the resulting second-order differential constraint is at least partially integrable [2]. We just show here that our simple test does not contradict the previous results. To this end, letting $n = 1$, $i = 2$, and $j = 3$, we compute the first condition (7) as

$$b_{12}\frac{\partial b_{13}}{\partial q_1} + b_{13}\frac{\partial b_{12}}{\partial q_1} - b_{11}\frac{\partial b_{23}}{\partial q_1} \equiv 0, \quad (11)$$

which shows that the sufficient STLC conditions are indeed never satisfied for any $q \in I\!R^3$.

5 Conclusions

A simple test for determining small-time local controllability (STLC) of mechanical systems underactuated by one control has been presented. The sufficient conditions are expressed directly in terms of the system inertia matrix elements. The test has been used to prove STLC of planar robots with $n \geq 3$ rotational joints and $n-1$ control inputs, provided that the first joint is actuated. When the number of unactuated degrees of freedom is larger than one, sufficient conditions for STLC depending only on configuration-dependent terms can still be stated [18], but they become much more involved as Christoffel coefficients and the inverse of the inertia matrix need to be computed. In any case, it would be interesting to provide a physical interpretation of the obtained conditions.

We conclude with a remark on the relevance of STLC for motion planning and control of underactuated mechanical systems. One of the most successful approaches is based on flatness (or, equivalently, dynamic feedback linearizability). In [19], sufficient conditions for a system underactuated by one control to be configuration flat and an associated planning method were provided. Motion planning techniques and trajectory tracking control based on dynamic feedback linearization were explored in [17] for a $RR\overline{R}$ planar robot. In all cases, the STLC property of the mechanical system was not explicitly used, so that rest-to-rest motion planning (which reduces to a simple interpolation problem) may typically generate large and swinging maneuvers even for close reconfiguration tasks. Therefore, inclusion of the STLC property at the level of planning should result in better motion. An example of this can be found in [20] for a nonholonomic kinematic system, but the problem is still open for underactuated dynamic systems.

References

[1] A. De Luca, S. Iannitti, R. Mattone, and G. Oriolo, "Control problems in underactuated manipulators," *2001 IEEE/ASME Int. Conf. on Advanced Mechatronics*, Como, I, pp. 855–861, 2001.

[2] G. Oriolo and Y. Nakamura, "Control of mechanical systems with second-order nonholonomic constraints: Underactuated manipulators," *30th IEEE Conf. on Decision and Control*, Brighton, UK, pp. 2398–2403, 1991.

[3] H. Nijmeijer and A. J. van der Schaft, *Nonlinear Dynamical Control Systems*, Springer Verlag, 1990.

[4] M. Reyhanoglu, A. J. van der Shaft, N. H. McClamroch, and I. Kolmanovsky, "Dynamics and control of a class of underactuated mechanical systems," *IEEE Trans. on Automatic Control*, vol. 44, no. 9, pp. 1663–1671, 1999.

[5] H. J. Sussmann, "A general theorem on local controllability," *SIAM J. on Control and Optimization*, vol. 25, no. 1, pp. 158–194, 1987.

[6] R. M. Bianchini and G. Stefani, "Controllability along a trajectory: A variational approach," *SIAM J. on Control and Optimization*, vol. 31, no. 4, pp. 900–927, 1993.

[7] H. Arai, K. Tanie, and N. Shiroma, "Nonholonomic control of a three-dof planar underactuated manipulator," *IEEE Trans. on Robotics and Automation*, vol. 14, no. 5, pp. 681–695, 1998.

[8] A. De Luca, R. Mattone and G. Oriolo, "Control of Redundant Robots under End-Effector Commands: A Case Study in Underactuated Robotic Systems," *J. of Applied Mathematics abd Computer Science*, vol. 7, no. 2, pp. 225–251, 1997.

[9] A. De Luca, R. Mattone, and G. Oriolo, "Steering a class of redundant mechanisms through end-effector generalized forces," *IEEE Trans. on Robotics and Automation*, vol. 14, no. 2, pp. 329–335, 1998.

[10] A. D. Lewis and R.M. Murray, "Configuration controllability of simple mechanical control systems," *SIAM J. on Control and Optimization*, vol. 35, no. 3, pp. 766–790, 1997.

[11] F. Bullo and K.M. Lynch, "Kinematic controllability and decoupled trajectory planning for underactuated mechanical systems," *2001 IEEE Int. Conf. on Robotics and Automation*, Seoul, KR, pp. 3300–3307, 2001.

[12] T. Kailath, *Linear Systems*, Prentice-Hall, 1980.

[13] K. Kobayashi and T. Yoshikawa, "Controllability of under-actuated planar manipulators with one unactuated joint," *2000 IEEE/RSJ Int. Conf. on Intelligent Robots and System*, Takamatsu, J, pp. 133–138, 2000.

[14] Y. Nakamura, T. Suzuki, and M. Koinuma "Nonlinear behavior and control of nonholonomic free-joint manipulator," *IEEE Trans. on Robotics and Automation*, vol. 13, no. 6, pp. 853–862, 1997.

[15] H. Arai, K. Tanie, and N. Shiroma, "Time-scaling control of an underactuated manipulator," *J. of Robotic Systems*, vol. 15, no. 9, pp. 525–536, 1998.

[16] A. De Luca, R. Mattone, and G. Oriolo, "Stabilization of an underactuated planar 2R manipulator," *Int. J. on Robust and Nonlinear Control*, vol. 10, no. 4, pp. 181–198, 2000.

[17] A. De Luca and G. Oriolo, "Motion planning and trajectory control of an underactuated three-link robot via dynamic feedback linearization," *2000 IEEE Int. Conf. on Robotics and Automation*, San Francisco, CA, pp. 2789–2795, 2000.

[18] S. Iannitti, *Motion Planning and Control of a Class of Underactuated Robots*, Ph.D. thesis, Dipartimento di Informatica e Sistemistica, Università di Roma "La Sapienza", Dec. 2001.

[19] M. Rathinam and R.M. Murray, "Configuration flatness of Lagrangian systems underactuated by one control," *SIAM J. on Control and Optimization*, vol. 36, no. 1, pp. 164–179, 1998.

[20] F. Lamiraux and J.-P. Laumond, "Smooth motion planning for car-like vehicles," *IEEE Trans. on Robotics and Automation*, vol. 17, no. 4, pp. 498–502, 2001.