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CONTROL OF ROBOT ARM WITH ELASTIC JOINTS VIA NONLINEAR DYNAMIC FEEDBACK

Alessandro De Luca, Alberto Isidori and Fernando Nicolò

Dipartimento di Informatica e Sistemistica, Università di Roma
"La Sapienza", Via Eudossiana 18, 00184 Roma, Italy.

Abstract

It is known that control of a rigid robot arm can easily be achieved via static state-feedback compensation of the nonlinearities. However, in many practical situations, the elasticity in gear boxes is not negligible. If this is the case, the use of such a control technique is not possible anymore because neither is the system feedback equivalent to a controllable linear one, nor its input-output behavior can be decoupled via static state-feedback.

The purpose of this paper is to show how dynamic state-feedback compensation may be used in order to obtain full state-space linearity, and to present an application to the model of a three link robot arm with elastic joints.

Introduction

The increasing interest for nonlinear control theory in the robotics literature is witnessed by a series of recent papers. Among the others we quote e.g. the works of Freund[1], Tarn and others[2], Singh and Schy[3], Marino and Nicosia[4]. A standard technique proposed for the control of rigid robots is the one based on input-output decoupling and nonlinearity compensation via static state-feedback. For robots with elastic transmission between actuators and arms, as belts or harmonic drives, this control strategy cannot be applied anymore since the associated model is such that the necessary conditions for the existence of the desired feedback fail to hold[4].

In a recent paper[5], the authors suggested the use of dynamic state-feedback and, applying the nonlinear model matching theory[6], solved the noninteracting control problem for the case of a two-link planar robot with elastic joints. The dynamic compensator thus found was such as to induce full linearity in suitable local coordinates for the resulting closed-loop system. This suggested further investigations addressed to the problem of getting full linearization via *dynamic* state-feedback. This control problem is apparently a new one, a natural generalization of that originally posed by Brockett and fully solved by Jacubczyk and Respondek[7] and independently by Hunt and others[8] by means of *static* state-feedback. As a matter of fact, a set of sufficient conditions for the solvability of this problem has been found, described in the first half of this paper. These conditions turn out to be satisfied for a three-link robot arm with elastic joints, which is considered as an example in the second part of the paper.

Exact Linearization via Dynamic State-Feedback

Consider a control system described by differential equations of the form:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \qquad (1)$$

with state x evolving on an open subset M of $\mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^\ell$. The vector f, the m columns of the matrix g, and the vector h are assumed throughout the paper to be analytic on M.

In what follows we will let the control u depend on the state x and on a reference variable v through equations of the form:

$$\begin{aligned} \dot{z} &= a(x,z) + b(x,z)v \\ u &= c(x,z) + d(x,z)v. \end{aligned} \qquad (2)$$

These equations characterize a dynamical system - a state-feedback compensator - whose state z evolves on an open subset N of $\mathbb{R}^\nu$. The vector a, the m columns of the matrix b, the vector c and the m columns of the matrix d are assumed to be analytic on an open subset of M × N.

The purpose of this section is to show how to design the compensator in such a way that the closed-loop system resulting from the composition of (1) and (2) becomes locally diffeomorphic to a linear controllable system.

In doing so we will make use to a large extent of some basic results from nonlinear differential geometric feedback control theory; some background material in this field is assumed to be known[9]. In particular, most of our results will rely upon certain properties of the so-called maximal controlled invariant distribution Algorithm[10].

We recall that with any system of the form (1), one may associate a sequence of codistribution defined in the following way:

$$\Omega_o(x) = \mathrm{span}\{dh_1(x),\ldots,dh_\ell(x)\}$$

$$\Omega_k(x) = \Omega_{k-1}(x) + (L_f(\Omega_{k-1} \cap G^\perp))(x) + \sum_{i=1}^{m} (L_{g_i}(\Omega_{k-1} \cap G^\perp))(x) \qquad (3)$$

where $G(x) = \mathrm{span}\{g_1(x),\ldots,g_m(x)\}$. This sequence is clearly increasing and, if $\Omega_{k^*} = \Omega_{k^*+1}$ for some k^*, then $\Omega_k = \Omega_{k^*}$ for all $k > k^*$.

For practical purpose, we shall henceforth assume that the codistributions involved in this Algorithm have constant dimension around the point of interest x°. This is precised in the following terms.

Definition. The point x° is a regular point for the Algorithm (3) if for all x in a neighborhood of x°.

(i) the dimension of $G(x)$ is constant
(ii) the dimension of $\Omega_k(x)$ is constant, for all $k \geq 0$
(iii) the dimension of $(\Omega_k \cap G^\perp)(x)$ is constant, for all $k \geq 0$. □

Note that if x° is a regular point for the Algorithm (3), then there exists an integer $k^* < n$ such that $\Omega_{k^*} = \Omega_{k^*+1}$ and this implies the convergence of the Algorithm, in a neighborhood of x°, in a finite number of stages. The codistribution Ω_{k^*} will be sometimes denoted by the simpler symbol Ω^* and its annihilator by

$$\Delta^* = \Omega_*^\perp$$

The Algorithm in question will be used in the sequel in order to compute the distribution Δ^*, to check some suitable structural conditions-stated in terms of properties of the codistributions Ω_k - and also in order to compute the so-called structure at infinity[11] of the system (1). We recall that the latter is defined in the following terms. Set

$$r_k = \dim \frac{\Omega_k}{\Omega_k \cap G^\perp} \quad , \qquad k \geq 0$$

and

$$\delta_1 = r_o \; , \; \delta_{i+1} = r_i - r_{i-1} \; , \; i \geq 1.$$

Then the system (1) is said to have δ_i (formal) zeros at infinity of multiplicity i.

The ingredients summarized so for enable us to give an answer to the problem of exact linearization via dynamic state feedback. The key tool in the procedure that follows is a nice canonical form under feedback-equivalence[7] which exists under the specific conditions stated hereafter. For the sake of notational simplicity we will restrict our considerations to the particular case of systems with three inputs and three outputs.

Theorem 1. Suppose $\ell = m = 3$ in (1). Moreover let the following assumptions be satisfied:

(A1) $\Delta^* = 0$

(A2) $\sum_{i=1}^{m} (L_{g_i}(\Omega_{k-1} \cap G^\perp))(x) \subset \Omega_{k-1}(x), \quad k \geq 1.$

Then system (1) has exactly $\ell = 3$ (formal) zeros at infinity, of multiplicity $\mu_1 \leq \mu_2 \leq \mu_3$, and

$$\mu_1 + \mu_2 + \mu_3 = n.$$

Moreover, there exists a feedback $u = \alpha(x) + \beta(x)w$, with α and β defined in a neighborhood of x^o, such that

$$\begin{aligned} \dot{x} &= (f+g\alpha)(x) + (g\beta)(x)w \\ y &= h(x) \end{aligned} \tag{4}$$

via the local diffeomorphism

$$\phi(x) = (\xi_1, \xi_2, \ldots, \xi_{\mu_1}, \eta_1, \eta_2, \ldots, \eta_{\mu_2}, \zeta_1, \zeta_2, \ldots, \zeta_{\mu_3})$$

where

$$\xi_i = L^{i-1}_{(f+g\alpha)} h_{j_1}$$

$$\eta_i = L^{i-1}_{(f+g\alpha)} h_{j_2}$$

$$\zeta_i = L^{i-1}_{(f+g\alpha)} h_{j_3}$$

and (j_1, j_2, j_3) is a permutation of (1,2,3), becomes

$$\dot{\xi}_1 = \xi_2$$
$$\ldots$$
$$\dot{\xi}_{\mu_1-1} = \xi_{\mu_1}$$
$$\dot{\xi}_{\mu_1} = w_1$$
$$\dot{\eta}_1 = \eta_2$$
$$\ldots$$
$$\dot{\eta}_{\mu_1-1} = \eta_{\mu_1}$$
$$\dot{\eta}_{\mu_1} = \eta_{\mu_1+1} + \gamma_{\mu_1}(\xi_1, \ldots, \xi_{\mu_1}, \eta_1, \ldots, \eta_{\mu_1}, \zeta_1, \ldots, \zeta_{\mu_1}) w_1$$
$$\dot{\eta}_{\mu_1+1} = \eta_{\mu_1+2} + \gamma_{\mu_1+1}(\xi_1, \ldots, \xi_{\mu_1}, \eta_1, \ldots, \eta_{\mu_1+1}, \zeta_1, \ldots, \zeta_{\mu_1+1}) w_1$$
$$\ldots$$
$$\dot{\eta}_{\mu_2-1} = \eta_{\mu_2} + \gamma_{\mu_2-1}(\xi_1, \ldots, \xi_{\mu_1}, \eta_1, \ldots, \eta_{\mu_2-1}, \zeta_1, \ldots, \zeta_{\mu_2-1}) w_1$$
$$\dot{\eta}_{\mu_2} = w_2$$
$$\dot{\zeta}_1 = \zeta_2$$
$$\ldots$$
$$\dot{\zeta}_{\mu_1-1} = \zeta_{\mu_1}$$
$$\dot{\zeta}_{\mu_1} = \zeta_{\mu_1+1} + \delta_{\mu_1}(\xi_1, \ldots, \xi_{\mu_1}, \eta_1, \ldots, \eta_{\mu_1}, \zeta_1, \ldots, \zeta_{\mu_1}) w_1$$
$$\dot{\zeta}_{\mu_1+1} = \zeta_{\mu_1+2} + \delta_{\mu_1+1}(\xi_1, \ldots, \xi_{\mu_1}, \eta_1, \ldots, \eta_{\mu_1+1}, \zeta_1, \ldots, \zeta_{\mu_1+1}) w_1$$
$$\ldots$$
$$\dot{\zeta}_{\mu_2-1} = \zeta_{\mu_2} + \delta_{\mu_2-1}(\xi_1, \ldots, \xi_{\mu_1}, \eta_1, \ldots, \eta_{\mu_2-1}, \zeta_1, \ldots, \zeta_{\mu_2-1}) w_1$$
$$\dot{\zeta}_{\mu_2} = \zeta_{\mu_2+1} + \delta_{\mu_2}(\xi_1, \ldots, \xi_{\mu_1}, \eta_1, \ldots, \eta_{\mu_2}, \zeta_1, \ldots, \zeta_{\mu_2}) w_1 + \varepsilon_{\mu_2}(\xi_1, \ldots, \xi_{\mu_1}, \eta_1, \ldots, \eta_{\mu_2}, \zeta_1, \ldots, \zeta_{\mu_2}) w_2$$
$$\dot{\zeta}_{\mu_2+1} = \zeta_{\mu_2+2} + \delta_{\mu_2+1}(\xi_1, \ldots, \xi_{\mu_1}, \eta_1, \ldots, \eta_{\mu_2}, \zeta_1, \ldots, \zeta_{\mu_2+1}) w_1 + \varepsilon_{\mu_2+1}(\xi_1, \ldots, \xi_{\mu_1}, \eta_1, \ldots, \eta_{\mu_2}, \zeta_1, \ldots, \zeta_{\mu_2+1}) w_2$$
$$\ldots$$
$$\dot{\zeta}_{\mu_3-1} = \zeta_{\mu_3} + \delta_{\mu_3-1}(\xi_1, \ldots, \xi_{\mu_1}, \eta_1, \ldots, \eta_{\mu_2}, \zeta_1, \ldots, \zeta_{\mu_3-1}) w_1 + \varepsilon_{\mu_3-1}(\xi_1, \ldots, \xi_{\mu_1}, \eta_1, \ldots, \eta_{\mu_2}, \zeta_1, \ldots, \zeta_{\mu_3-1}) w_2$$
$$\dot{\zeta}_{\mu_3} = w_3$$
$$y_{j_1} = \xi_1$$
$$y_{j_2} = \eta_1$$
$$y_{j_3} = \zeta_1. \tag{5}$$

□

The proof of this Theorem may be found elsewhere[12]. Anyway, the interested reader may recover the fundamental steps of this proof from the application to the robot equations discussed in the second half of the paper.

The possibility of having exact linearization via dynamic feedback is shown in the following Corollary, whose proof is an easy consequence of the existence of the canonical form (5).

Corollary. Suppose $\ell = m = 3$. Moreover, let the assumptions (A1),(A2) be satisfied. Consider the following dynamic extension of system (4)

$$\dot z_{11} = z_{12} \qquad \dot z_{21} = z_{22}$$
$$\cdots \qquad \cdots$$
$$\dot z_{1,\mu_3-\mu_1-1} = z_{1,\mu_3-\mu_1} \qquad \dot z_{2,\mu_3-\mu_2-1} = z_{2,\mu_3-\mu_2} \tag{6}$$
$$\dot z_{1,\mu_3-\mu_1} = \bar w_1 \qquad \dot z_{2,\mu_3-\mu_2} = \bar w_2$$
$$w_1 = z_{11}, \quad w_2 = z_{21}, \quad w_3 = \bar w_3.$$

Then the composition of (4) and (6) yields a dynamical system which is feedback-equivalent to a system of the form

$$\dot \xi_{i1} = \xi_{i2}$$
$$\cdots$$
$$\dot \xi_{i,\mu_3-1} = \xi_{i,\mu_3}, \quad i = 1,2,3, \tag{7}$$
$$\dot \xi_{i,\mu_3} = v_i$$
$$y_{j_i} = \xi_{i1}$$

Proof. Let $\bar x = (x,z)$ and

$$\dot{\bar x} = \bar f(\bar x) + \bar g(\bar x)\bar w$$
$$y = \bar h(\bar x) \tag{8}$$

denote the composition of (4) and (6). A direct computation based on the canonical form (5) shows that

$L_{\bar g} L_{\bar f}^k \bar h_i = 0$, $i = 1,2,3$; $k = 0,\ldots,\mu_3-2$.

and that the 3×3 matrix

$$\bar A(\bar x) = L_{\bar g} L_{\bar f}^{\mu_3-1} \bar h$$

is nonsingular. Then, there exist a feedback $\bar w = \bar\alpha(\bar x) + \bar\beta(\bar x)v$ which makes (8) input-output-wise linear and decoupled[10]. Moreover, since the dimension of $\bar x$ is $3\mu_3$, the mapping

$$\bar\Phi(\bar x) = \{\xi_{ij},\ j = 1,\ldots,\mu_3;\ i = 1,2,3\}$$

with $\xi_{ij} = L_{\bar f}^{j-1}\bar h_i(x)$, is a local diffeomorphism, which brings the system

$$\dot{\bar x} = (\bar f + \bar g\bar\alpha)(\bar x) + (\bar g\bar\beta)(\bar x)v$$
$$y = \bar h(\bar x)$$

to the form (7). □

A series of remarks are now in order.

Remark 1. The composition of the feedback $u = \alpha(x)+\beta(x)w$, the dynamic extension (6) and the feedback $\bar w = \bar\alpha(\bar x)+\bar\beta(\bar x)v$ characterizes a dynamic compensator of the form (2) which solves the exact linearization problem. The structure of this compensator is shown in Fig. 1. Note that system (1) has dimension $n = \mu_1+\mu_2+\mu_3$, the dynamic compensator has dimension $\nu = 2\mu_3-\mu_2-\mu_1$. The closed loop system has dimension $n+\nu = 3\mu_3$ and in suitable local coordinates appears as three decoupled chains of μ_3 integrators each.

Remark 2. It is well known[10] that a system in which the noninteracting control problem (via static state-feedback) is solvable, if $\Delta^* = 0$, is feedback-equivalent to a linear controllable system. As a matter of fact, the same feedback which yields noninteraction makes the system diffeomorphic to a linear controllable system. If $\Delta^* \neq 0$, the above feedback yields input-output linearity but a possibly nonlinear unobservable part is left. In the present case we keep the assumption $\Delta^* = 0$ (see (A1)) but we replace the condition needed for solvability of the noninteracting control problem by the *weaker* assumption (A2). We still get full linearity at the state-space level and noninteraction but using now a dynamic, rather than static, state-feedback.

Remark 3. Note that in the canonical form (5) the drift vector field is linear and all the nonlinearity is concentrated in the vector fields which multiply the inputs. The triangular structure of the latter and the specific dependencies of their entries from the local coordinates is a direct consequence of the structural assumption (A2). The most important feature of the canonical form (5) is that the addition of integrators to any input channel does not destroy the condition $\Delta^* = 0$ (this is not always the case for nonlinear systems[13]). This explains why the composition of (4) with the dynamic extension (6), having still $\Delta^* = 0$, and being such that the noninteracting control problem is solvable, is feedback-equivalent to a linear (and decoupled) system.

Remark 4. In the applications one might be interested in a further, now linear, feedback from the state variables ξ_{ij}, in order to place all the $n+\nu$ eigenvalues of the resulting closed loop system.

Remark 5. If two or three of the indexes μ_i are equal, the canonical form (5) particularizes in an obvious way. If, for instance, $\mu_1 = \mu_2$, not only the dynamics of the ξ_i's but also that of the η_i's is fully linear. It may be worth noting the relation between the μ_i's and the so-called characteristic numbers ρ_i's (the least integer such that $L_g L_f^{\rho_i} h_i \neq 0$). Assuming $\rho_1 \leq \rho_2 \leq \rho_3$ one has $\rho_1 = \mu_1-1$, $\rho_2 \leq \mu_2-1$, $\rho_3 \leq \mu_3-1$, equalities being true if and only if the noninteracting control problem is solvable via static state-feedback.

Exact Linearization of the Robot Arm with Elastic Joints

In this section we will apply the results described before to the control of a robot arm with elastic joints. The mathematical model of this kind of robot arm is briefly summarized hereafter[14].

Consider the mechanical structure of a robot as being constituted by an open chain of N+1 bodies (links) interconnected through N rotational/translational joints. The joints are activated by motors with transmission gears or belts; when the links and the transmissions are assumed to be rigid the dynamical behavior is that of a chain of N rigid bodies. In this case the Lagrangian formulation[15] leads to equations of motion in the form:

$$B(q)\ddot q + c(q,\dot q) + e(q) = m(t) \tag{9}$$

where q is the N-vector of joint variables giving the relative displacement between two adjacent links, $B(q)$ is the N×N nonsingular inertial matrix, $m(t)$ is the N-vector of generalized forces delivered by the motors, $e(q)$ is the N-vector of conservative forces and $c(q,\dot q)$ is the N-vector collecting centrifugal and Coriolis forces.

When the transmissions are not rigid the N actuating bodies of the motors are elastically coupled to the driven links. Therefore, the dynamical behavior is that of 2N rigid bodies, N of which are directly actuated while the other N include elasticity; this is the case of interest here. The equations of motion are still given by (9), but with the following peculiarities:

- the number of second order equations is 2N;
- q is a 2N-vector in which q_{2i} denotes the displacement of link i w.r.t. link i-1 and q_{2i-1} denotes the displacement of the driving body of joint i w.r.t. link i-1, for i = 1,...,N;
- B(q) is the 2N×2N inertial nonsingular matrix of the 2N rigid bodies;
- e(q) and $c(q,\dot{q})$ are 2N-vectors and e(q) includes the effects of elasticity;
- m(t) is a 2N-vector with the even components equal to zero.

Starting from mechanical parameters, the model (9) is given automatically by the DYMIR code both for rigid and elastic robots[16]; (9) may be rewritten in the standard form

$$\dot{x} = f(x) + g(x)u \qquad y = h(x) \tag{10}$$

with state $x \triangleq \begin{bmatrix} x_p^T & x_v^T \end{bmatrix}^T \triangleq \begin{bmatrix} q^T & \dot{q}^T \end{bmatrix}^T \in M \subset R^n$, input $u \in R^m$ and output $y \in R^\ell$. In the elastic case n = 4N; moreover, the input u collects only the nonzero components of m(t) while the output y may be defined as the vector of link displacements $x_{2i} = q_{2i}$ (i=1,...,N). Thus, $m = \ell = N$. The expressions for f and g are given by:

$$f(x) = \begin{bmatrix} x_v \\ -B(x_p)^{-1}[\,c(x_p,x_v) + e(x_p)] \end{bmatrix}, \qquad g(x) = \begin{bmatrix} 0 \\ B(x_p)^{-1}\mathrm{diag}\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\} \end{bmatrix}. \tag{11}$$

The equations of a PUMA-like three-link robot arm with elastic joints (see Fig. 2) are reported in Appendix 1.

It is well known[1,2,15] that the rigid robot can be decoupled and linearized via static state-feedback, whereas this is no longer the case whenever joint elasticity is not negligible[4]. In view of this we consider now the problem of achieving linearity via dynamic state-feedback. To this end the first thing to do is to perform the maximal invariant distribution Algorithm on the equations of the robot under consideration. All computations may be found with full details in Appendix 2.

As a result of these computations we find that assumptions (A1) and (A2) of Theorem 1 are satisfied. Moreover, since

$$r_o = 0,\ r_1 = 1,\ r_2 = 1,\ r_3 = 2,\ r_4 = 2,\ r_5 = r_{k^*} = 3$$

we have

$$\delta_1 = 0,\ \delta_2 = 1,\ \delta_3 = 0,\ \delta_4 = 1,\ \delta_5 = 0,\ \delta_6 = 1$$

and thus $\mu_1 = 2$, $\mu_2 = 4$, $\mu_3 = 6$. In addition we see that the set of functions

$$\xi_i = L^{i-1}_{(f+g\alpha)}h_2 \qquad i = 1,2;$$

$$\eta_i = L^{i-1}_{(f+g\alpha)}h_1 \qquad i = 1,\dots,4;$$

$$\zeta_i = L^{i-1}_{(f+g\alpha)}h_3 \qquad i = 1,\dots,6$$

qualifies a new set of local coordinates in the state space. The function α(x) is given by:

$$\alpha(x) = \begin{bmatrix} \left(\dfrac{\phi_1(x)f_{10}(x)}{g_{10,3}(x)} + \phi_2(x)\right)/\left(g_{71}\cdot\dfrac{\partial f_8(x)}{\partial x_1}\right) \\ 0 \\ -f_{10}(x)/g_{10,3}(x) \end{bmatrix} \tag{12}$$

where all terms involved may be found in either Appendices. The choice of this α(x) together with a β(x) given by:

$$\beta(x) = \begin{bmatrix} L_g L_{(f+g\alpha)} h_2 \\ L_g L^3_{(f+g\alpha)} h_1 \\ L_g L^5_{(f+g\alpha)} h_3 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & g_{10,3}(x) \\ g_{71}\dfrac{\partial f_8(x)}{\partial x_1} & 0 & \phi_1(x) \\ \phi_3(x) & \phi_4 & \phi_5(x) \end{bmatrix}^{-1} \tag{13}$$

in system (4) yields, in the local coordinates ξ_i, η_i, ζ_i, the canonical form (5). The dynamic extension (6) considered in the Corollary of Theorem 1 consists here of the addition of $\mu_3-\mu_1 = 4$ integrators on the input w_1 and of $\mu_3-\mu_2 = 2$ integrators on the input w_2 i.e.

$$\begin{aligned} &\dot{z}_{11} = z_{12},\ \dot{z}_{12} = z_{13},\ \dot{z}_{13} = z_{14},\ \dot{z}_{14} = \bar{w}_1 \\ &\dot{z}_{21} = z_{22},\ \dot{z}_{22} = \bar{w}_2 \\ &w_1 = z_{11},\ w_2 = z_{21},\ w_3 = \bar{w}_3 \end{aligned} \tag{14}$$

The robot model (10) subject to a feedback $u = \alpha(x)+\beta(x)w$, with α and β specified by (12) and (13), together with the dynamic extension (14) is now a system which can be decoupled and fully linearized by a static state-feedback of the form $\bar{w} = \bar{\alpha}(\bar{x}) + \bar{\beta}(\bar{x})v$. In the notation of the previous section (recall that (8) indicates the composition of (4) and (6)) the functions $\bar{\alpha}$ and $\bar{\beta}$ are now given by

$$\bar{\beta}(\bar{x}) = \left[L_{\bar{g}} L^5_{\bar{f}} \bar{h}(\bar{x}) \right]^{-1}$$

$$\bar{\alpha}(\bar{x}) = -\bar{\beta}(\bar{x}) \cdot L^6_{\bar{f}} \bar{h}(\bar{x}).$$

The resulting closed-loop system is locally diffeomorphic to three chains of $\mu_3 = 6$ integrators each.

Conclusions

In this paper we have shown how, under suitable assumptions, dynamic state-feedback can be used in order to make a given nonlinear system diffeomorphic to a linear controllable (and decoupled) one. The assumptions in question are indeed weaker than the ones which guarantee the achievement of the same result via static state-feedback. In particular, the assumption of nonsingularity of the so-called decoupling matrix has been replaced by the structural assumption (A2) which characterizes a specific property of the sequence of codistribution generated by means of the maximal controlled invariant distribution Algorithm. Intuitively speaking, the structural assumption (A2) simply means that, from the point of view of its formal structure at infinity, the system under consideration essentially behaves like a linear one.

The technique of dynamic extension used here in order to achieve decoupling is similar to the one proposed by Descusse and Moog[17]. The replacement of their conditions with the stronger assumptions (A1),(A2) provides the required state-space full linearization. Related results based on Hirschorn's inversion algorithm are due to Singh[18].

In the second part of the paper we applied our synthesis procedure to the case of a three-link robot arm with elastic joints. On the DYMIR-generated model[16] we checked the fulfillment of assumptions (A1),(A2) and showed how to compute all the relevant functions associated with the dynamic compensator. The complexity of the actual computations requires symbolic manipulation systems like MACSYMA or REDUCE. We considered as outputs the joint coordinates but the proposed approach is likewise successfull for task-oriented synthesis problems. Moreover, we conjecture that any robot model with joint elasticity satisfies the assumptions (A1) and (A2). The idea of using nonlinear feedback in order to compensate nonlinearities and to achieve noninteraction dates back to early works of Porter[19] and Singh and Rugh[20] ; similar techniques have been simultaneously and independently developed in the robotic field dealing with the case of rigid robots[21] . The solution of the same kind of problems for robots with joint elasticity can still be accomplished but now requires, as shown here, full exploitment of nonlinear differential geometric control techniques.

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Appendix 1

We report here the dynamic model of a three-link robot arm with joint elasticity (see Fig. 2). The state space representation has been obtained by means of a symbolic manipulation system (REDUCE) starting from the DYMIR code[16] which outputs the matrix and vector entries in (9). We have:

$$\dot{x} = f(x) + \sum_{i=1}^{3} g_i(x)u_i = f(x) + g(x)u,$$

$$y = h(x)$$

with

$$f(x) = [x_7\ x_8\ x_9\ x_{10}\ x_{11}\ x_{12} \,|\, f_7\ f_8\ f_9\ f_{10}\ f_{11}\ f_{12}]^T,$$

$$g(x) = \left[\begin{array}{c|cccccc} & g_{71} & 0 & 0 & 0 & 0 & 0 \\ O_{3\times 6} & 0 & 0 & g_{92} & 0 & 0 & 0 \\ & 0 & 0 & 0 & g_{10,3} & g_{11,3} & g_{12,3} \end{array}\right]^T,$$

$$h(x) = [x_2\ x_4\ x_6]^T$$

where

$$g_{71} = G_1$$

$$g_{92} = G_3$$

$$g_{10,3} = -G_5 H_8/\omega_1$$

$$g_{11,3}=[\,4H_8(H_3\cos x_6+H_7)-(H_3\cos x_6+2H_8)^2]\,/4\omega_1$$

$$g_{12,3} = G_5(H_3\cos x_6+2H_8)/2\omega_1$$

$$f_7 = (N_1x_2-x_1)K_1G_1/N_1^2$$

$$\begin{aligned}f_8=\{&N_1x_8[\,x_{10}(H_1\sin(2x_4)+H_2\sin(2x_4+2x_6)+H_3\sin(2x_4+x_6))\\&+x_{12}(H_2\sin(2x_4+2x_6)+H_3\cos x_4\sin(x_4+x_6))]\\&-K_1(N_1x_2-x_1)\}/N_1\omega_2\end{aligned}$$

$$f_9 = (N_2x_4-x_3)K_2G_3/N_2^2$$

$$\begin{aligned}f_{10}=G_5\{&N_2N_3^2[\,(H_3\cos x_6+2H_8)(x_8^2(H_2\sin(2x_4+2x_6)\\&+H_3\cos x_4\sin(x_4+x_6))+x_{10}^2H_3\sin x_6)\\&+2H_8(x_{12}(2x_{10}+x_{12})H_3\sin x_6\\&-x_8^2(H_1\sin(2x_4)+H_2\sin(2x_4+2x_6)+H_3\sin(2x_4+x_6))]\\&+2N_2[\,N_3(H_3\cos x_6+2H_8)(N_3\omega_3+K_3(N_3x_6-x_5))\\&-2H_8K_3(N_3x_6-x_5)]\\&-4N_3^2H_8[N_2H_5\cos x_4+N_2\omega_3+K_2(N_2x_4-x_3)]\,\}/4N_2N_3^2\omega_1\end{aligned}$$

$$\begin{aligned}f_{11}=-\{&G_5N_2N_3^2[\,(H_3\cos x_6+2H_8)(x_8^2(H_2\sin(2x_4+2x_6)\\&+H_3\cos x_4\sin(x_4+x_6))+x_{10}^2H_3\sin x_6)\\&+2H_8(x_{12}(2x_{10}+x_{12})H_3\sin x_6\\&-x_8^2(H_1\sin(2x_4)+H_2\sin(2x_4+2x_6)\\&+H_3\sin(2x_4+x_6))))]\\&-4G_5N_3^2H_8[\,N_2H_5\cos x_4+N_2\omega_3+K_2(N_2x_4-x_3)]\\&+N_2[\,2G_5N_3(H_3\cos x_6+2H_8)(N_3\omega_3+K_3(N_3x_6-x_5))\\&-K_3(N_3x_6-x_5)(4H_8(H_3\cos x_6+H_7)\\&-(H_3\cos x_6+2H_8)^2)]\,\}/4N_2N_3^2\omega_1\end{aligned}$$

$$\begin{aligned}f_{12}=G_5\{&-N_2N_3^2[\,2(H_3\cos x_6+H_7-G_5)(x_8^2(H_2\sin(2x_4+2x_6)\\&+H_3\cos x_4\sin(x_4+x_6))+x_{10}^2H_3\sin x_6)\\&+(H_3\cos x_6+2H_8)(x_{12}(2x_{10}+x_{12})H_3\sin x_6\\&-x_8^2(H_1\sin(2x_4)+H_2\sin(2x_4+2x_6)\\&+H_3\sin(2x_4+x_6))))]\\&+2N_3^2(H_3\cos x_6+2H_8)[\,N_2H_5\cos x_4+N_2\omega_3+K_2(N_2x_4-x_3)]\\&-2N_2[\,2N_3(H_3\cos x_6+H_7-G_5)(N_3\omega_3+K_3(N_3x_6-x_5))\\&-K_3(N_3x_6-x_5)(H_3\cos x_6+2H_8)]\,\}/4N_2N_3^2\omega_1.\end{aligned}$$

In the expressions above we defined for compactness the terms:

$$\omega_1(x_4) = H_9 + H_{10}\cos^2 x_6$$

$$\omega_2(x_4,x_6)=H_1\cos^2 x_4+H_2\cos^2(x_4+x_6)+H_3\cos x_4\cos(x_4+x_6)+H_4$$

$$\omega_3(x_6) = H_3\cos x_6+H_7.$$

The constants $H_1 \ldots H_{10}$ and G_1,G_3,G_5 depend on the robot data which include length, mass, inertia tensor and center of mass for each link, mass and inertia tensor for each rotor; furthermore at joint i, N_i is the reduction ratio of the gear box and K_i is its elastic constant.

We collect in this Appendix also the relevant terms which are computed during the application of the maximal controlled invariant distribution Algorithm to the robot arm under consideration (see Appendix 2 and formulas (12) and (13) in the text):

$$\tilde{g}_{11,3}=g_{11,3}/g_{10,3}=-[\,1+\omega_1/H_8G_5^2]$$

$$\tilde{g}_{12,3}=g_{12,3}/g_{10,3}=-[\,1+(H_3/2H_8)\cos x_6]$$

$$\begin{aligned}\tilde{f}_{11}&=f_{11}-f_{10}\tilde{g}_{11,3}\\&=-\{[\,x_8^2(H_2\sin(2x_4+2x_6)+H_3\cos x_4\sin(x_4+x_6))\\&\qquad+x_{10}^2H_3\sin x_6]\,/2H_8+[\,N_3\omega_3+K_3(N_3x_6-x_5)]\,/N_3H_8\}\end{aligned}$$

$$\begin{aligned}\tilde{f}_{12}=f_{12}-f_{10}\tilde{g}_{12,3}=\{&[\,(x_{10}+x_{12})^2H_3\sin x_6)\\&-x_8^2(H_1\sin(2x_4)+H_3\sin x_4\cos(x_4+x_6))]\,/2\\&+H_3\cos x_6[\,x_8^2(H_2\sin(2x_4+2x_6)+H_3\cos x_4\sin(x_4+x_6))\\&+x_{10}^2H_3\sin x_6+2\omega_3]\,/4H_8+(1+H_3\cos x_6/2H_8)(N_3x_6-x_5)K_3/N_3\\&-(N_2x_4-x_3)K_2/N_2-H_5\cos x_4\}/G_5\end{aligned}$$

$$\begin{aligned}\phi_1(x)=g_{10,3}\Big(&\frac{\partial f_8}{\partial x_4}+x_{10}\frac{\partial^2 f_8}{\partial x_4\partial x_{10}}+x_{12}\frac{\partial^2 f_8}{\partial x_6\partial x_{10}}+f_8\frac{\partial^2 f_8}{\partial x_8\partial_{10}}\\&+\frac{\partial f_8}{\partial x_8}\frac{\partial f_8}{\partial x_{10}}+\frac{\partial f_8}{\partial x_{12}}\frac{\partial \tilde{f}_{12}}{\partial x_{10}}\Big)+g_{12,3}\Big(\frac{\partial f_8}{\partial x_6}+x_{10}\frac{\partial^2 f_8}{\partial x_4 x_{12}}\\&+x_{12}\frac{\partial^2 f_8}{\partial x_6\partial x_{12}}+f_8\frac{\partial^2 f_8}{\partial x_8\partial x_{12}}+\frac{\partial f_8}{\partial x_8}\frac{\partial f_8}{\partial x_{12}}\Big)\end{aligned}$$

$$\begin{aligned}\phi_2(x)=-\{&x_7\Big(2x_{10}\frac{\partial^2 f_8}{\partial x_1\partial x_4}+2x_{12}\frac{\partial^2 f_8}{\partial x_1\partial x_6}+\frac{\partial f_8}{\partial x_1}\frac{\partial f_8}{\partial x_8}\Big)\\&+x_8\Big(2x_{10}\frac{\partial^2 f_8}{\partial x_2\partial x_4}+2x_{12}\frac{\partial^2 f_8}{\partial x_2\partial x_6}+\frac{\partial f_8}{\partial x_2}\frac{\partial f_8}{\partial x_8}\Big)\\&+x_{10}\Big(x_{10}\frac{\partial^2 f_8}{\partial x_4^2}+2x_{12}\frac{\partial^2 f_8}{\partial x_4\partial x_6}+\frac{\partial f_8}{\partial x_4}\frac{\partial f_8}{\partial x_8}+2f_8\frac{\partial^2 f_8}{\partial x_4\partial x_8}\\&+\tilde{f}_{12}\frac{\partial^2 f_8}{\partial x_4\partial x_{12}}+\frac{\partial f_8}{\partial x_{12}}\frac{\partial \tilde{f}_{12}}{\partial x_4}\Big)+x_{11}\Big(\frac{\partial f_8}{\partial x_{12}}\frac{\partial \tilde{f}_{12}}{\partial x_5}\Big)\\&+x_{12}\Big(x_{12}\frac{\partial^2 f_8}{\partial x_6^2}+\frac{\partial f_8}{\partial x_6}\frac{\partial f_8}{\partial x_8}+2f_8\frac{\partial^2 f_8}{\partial x_6\partial x_8}\\&+\tilde{f}_{12}\frac{\partial^2 f_8}{\partial x_6\partial x_{12}}+\frac{\partial f_8}{\partial x_{12}}\frac{\partial \tilde{f}_{12}}{\partial x_6}\Big)+f_7\Big(\frac{\partial f_8}{\partial x_1}\Big)\\&+f_8\Big(\frac{\partial f_8}{\partial x_2}+\Big(\frac{\partial f_8}{\partial x_8}\Big)^2+f_{12}\frac{\partial^2 f_8}{\partial x_8\partial x_{12}}+\frac{\partial f_8}{\partial x_{12}}+\frac{\partial \tilde{f}_{12}}{\partial x_8}\Big)\end{aligned}$$

$$+f_{10}\left(\frac{\partial f_8}{\partial x_4}+x_{10}\frac{\partial^2 f_8}{\partial x_4\partial x_{10}}+x_{12}\frac{\partial^2 f_8}{\partial x_6\partial x_{10}}+f_8\frac{\partial^2 f_8}{\partial x_8\partial x_{10}}+\frac{\partial f_8}{\partial x_8}\frac{\partial f_8}{\partial x_{10}}+\frac{\partial f_8}{\partial x_{12}}\frac{\partial \tilde f_{12}}{\partial x_{10}}\right)+f_{12}\left(\frac{\partial f_8}{\partial x_6}+x_{10}\frac{\partial^2 f_8}{\partial x_4\partial x_{12}}+x_{12}\frac{\partial^2 f_8}{\partial x_6\partial x_{12}}+f_8\frac{\partial^2 f_8}{\partial x_8\partial x_{12}}+\frac{\partial f_8}{\partial x_8}\frac{\partial f_8}{\partial x_{12}}\right)$$

$$\phi_3(x)=G_1\Big\{\frac{K_1}{N_1}\Big[x_{10}\frac{\partial}{\partial x_4}\Big(\frac{1}{\omega_2}\frac{\partial \tilde f_{12}}{\partial x_8}\Big)+x_{12}\frac{\partial}{\partial x_6}\Big(\frac{1}{\omega_2}\frac{\partial \tilde f_{12}}{\partial x_8}\Big)+\frac{f_8}{\omega_2}\frac{\partial^2 \tilde f_{12}}{\partial x_8^2}\Big]-\frac{K_1}{\omega_2}\Big[2x_{10}\frac{\partial^2 \tilde f_{12}}{\partial x_4\partial x_8}+2x_{12}\frac{\partial^2 \tilde f_{12}}{\partial x_6\partial x_8}+2f_8\frac{\partial^2 \tilde f_{12}}{\partial x_8^2}+\frac{\partial \tilde f_{12}}{\partial x_8}\frac{\partial f_8}{\partial x_8}\Big]+\frac{K_1}{\omega_2^2}\frac{\partial \tilde f_{12}}{\partial x_8}\Big[x_{10}\frac{\partial\omega_2}{\partial x_4}-\frac{x_{12}}{N_1}\frac{\partial\omega_2}{\partial x_6}\Big]\Big\}$$

$$\phi_4 = G_3K_2K_3/N_2N_3G_5H_8$$

$$\phi_5(x)=g_{10,3}\frac{\partial L_{\tilde f}^5 h_3}{\partial x_{10}}+g_{11,3}\frac{\partial L_{\tilde f}^5 h_3}{\partial x_{11}}+g_{12,3}\frac{\partial L_{\tilde f}^5 h_3}{\partial x_{12}}$$

$$\tilde g_{71} = g_{71}\tilde\beta_{11} = N_1\omega_2/K_1$$

$$\tilde g_{73} = g_{71}\tilde\beta_{13} = N_1\omega_1\omega_2\phi_1/K_1G_5H_8$$

$$\tilde f_7 = f_7+g_{71}\tilde\alpha_1 = f_7+\frac{N_1\omega_2}{K_1}\Big(\phi_2-\frac{\omega_1\phi_1}{G_5H_8}f_{10}\Big).$$

Appendix 2

In this Appendix we apply the maximal controlled invariant distribution Algorithm, in the form suggested by Krener[22], to the three-link robot arm with non negligible joint elasticity whose model is reported in Appendix 1; we will show that this model satisfies the assumptions (A1) and (A2) of Theorem 1. For the sake of completeness we report here the above Algorithm.

From the components $h_1,\dots,h_\ell$ of the map h one constructs first of all the (x-dependent) subspace (of row vectors)

$$\Omega_o(x) = \mathrm{span}\{dh_1(x),\dots,dh_\ell(x)\}$$

Suppose $\Omega_o(x)$ has dimension $s_o \le \ell$ in a neighborhood of a point x^o. Then there exists an $s_o\times 1$ column vector λ_o, whose entries $\lambda_{o1},\dots,\lambda_{os_o}$ are entries of h, with the property that the differentials $d\lambda_{o1},\dots,d\lambda_{os_o}$ are linearly independent at all x in a neighborhood of x^o. The Algorithm consists of a finite number of iterations, each one defined as follows.

Iteration (k). Consider the $s_k\times m$ matrix $A_k(x)$ whose (i,j)-entry is $d\lambda_{ki}(x)g_j(x)$. Suppose that in a neighborhood of x^o the rank of $A_k(x)$ is constant and equal to r_k. Then it is possible to find r_k rows of $A_k(x)$ which, for all x in a neighborhood of x^o, are linearly independent. Let $P_k^T=[P_{k1}^T \mid P_{k2}^T]$ be an $s_k\times s_k$ permutation matrix, such that the r_k rows of $P_{k1}A_k(x)$ are linearly independent. Let $B_k(x)$ be an s_k-vector whose i-th element is $d\lambda_{ki}(x)f(x)$. As a consequence of the assumptions on P_{k1}, the equations

$$\begin{aligned}P_{k1}A_k(x)\alpha(x) &= -P_{k1}B_k(x)\\ P_{k1}A_k(x)\beta(x) &= K\end{aligned}\tag{15}$$

(where K is a matrix of real numbers, of rank r_k) may be solved for α and β, an m-vector and an m×m invertible matrix whose entries are real-valued smooth functions defined in a neighborhood of x^o. Set $\tilde f=\tilde g_o = f+g\alpha$ and $\tilde g_i = (g\beta)_i$, $1\le i\le m$.

Consider the set of functions

$$\Lambda_k = \{\lambda = L_{\tilde g_i}\lambda_{kj} : 1\le j\le s_k,\ 0\le i\le m\}$$

and the two (x-dependent) subspaces (of row vectors)

$$\Omega_k(x) = \mathrm{span}\{d\lambda_{k1}(x),\dots,d\lambda_{ks_k}(x)\}$$

$$\Omega_k'(x) = \mathrm{span}\{d\lambda(x) : \lambda\in\Lambda_k\}$$

Set $\Omega_{k+1}(x) = \Omega_k(x) + \Omega_k'(x)$.

Suppose $\Omega_{k+1}(x)$ has constant dimension $s_{k+1}(\ge s_k)$ in a neighborhood of x^o. Let $\lambda_{k+1,1},\dots,\lambda_{k+1,s_{k+1}}$ be entries of λ_k and/or elements of Λ_k such that the differentials $d\lambda_{k+1,1},\dots,d\lambda_{k+1,s_{k+1}}$ are linearly independent at all x in a neighborhood of x^o. Define the s_{k+1}-vector λ_{k+1} whose i-th entry is the function $\lambda_{k+1,i}$. This concludes the k-th iteration. At each stage of the Algorithm two integers are considered

$$s_k = \dim\Omega_k(x),\quad r_k = \mathrm{rank}\,A_k(x).$$

Since $s_k\le s_{k+1}\le n$, a dimensionality argument shows that there exists an integer k^* such that $s_k=s_{k^*}$, $r_k=r_{k^*}$ for all $k\ge k^*$. The sequence $\{r_o,r_1,\dots\}$ provides the structure at the infinity associated with the triplet (f,g,h).

We perform next this Algorithm on the triplet (f,g,h) which describes the robot arm system dynamics. In the *initial step* we use the output functions h_1,h_2 and h_3 and we get

$$\Omega_o = sp\{dh_1dh_2dh_3\} = sp\left\{\begin{array}{cccccc|c}0&1&0&0&0&0&\\0&0&0&1&0&0&O_{3\times6}\\0&0&0&0&0&1&\end{array}\right\}$$

and thus $s_o=3$ and $\lambda_o=h$.

In the *0-th iteration* we have:

$$A_o = d\lambda_o\cdot g = L_g h = 0\quad,\quad r_o = 0$$

and hence $\Omega_o\cap G^\perp = \Omega_o$ so that it is easy to see that assumption (A2) holds for k = 1. Furthermore,

$$\Omega_1=\Omega_o\oplus sp\{dL_fh_1dL_fh_2dL_fh_3\}=\Omega_o\oplus sp\left\{\begin{array}{c|cccccc}&0&1&0&0&0&0\\O_{3\times6}&0&0&0&1&0&0\\&0&0&0&0&0&1\end{array}\right\}$$

giving $s_1 = 6$ and $\lambda_1 = [h^T \; L_f h^T]^T \equiv [x_2 \, x_4 \, x_6 \, x_8 \, x_{10} \, x_{12}]$. This way of "translating" dependencies from the first group of states (x_p) to the second one (x_v) reflects the Newtonian structure of the considered system.

In the 1-*st iteration*, the matrix

$$A_1 = d\lambda_1 \cdot g = \left[O_{3\times 3} \;\middle|\; \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & g_{10,3} & g_{12,3} \end{matrix} \right]^T$$

has rank $r_1 = 1$; thus, we have to compute a feedback pair (α,β) from equation (15). Choosing P_1 such that $P_{11} = [0\;0\;0\;0\;1\;0]$, since $B_1 = d\lambda_1 \cdot f = [x_8 \; x_{10} \; x_{12} \; f_8 \; f_{10} \; f_{12}]$ we have as a solution:

$$\alpha_1 = \alpha_2 = 0 \;, \; \alpha_3 = -f_{10}/g_{10,3} \;,$$

$$\beta_{11} = \beta_{22} = 1, \; \beta_{33} = 1/g_{10,3}, \; \beta_{ij} = 0 \; (i \neq j).$$

This gives:

$$\tilde{f} = f + g\alpha = [x_7 \; x_8 \; x_9 \; x_{10} \; x_{11} \; x_{12} \mid f_7 \; f_8 \; f_9 \; 0 \; \tilde{f}_{11} \; \tilde{f}_{12}]^T$$

$$\tilde{g} = g\beta = \left[O_{3\times 6} \;\middle|\; \begin{matrix} g_{71} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & g_{82} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \tilde{g}_{11,3} & \tilde{g}_{12,3} \end{matrix} \right]^T .$$

The complete expression of the new terms involved is reported in Appendix 1; notice only that the new vector fields $\tilde{f}, \tilde{g}_i$ have much simpler forms than the original ones. Furthermore since $L_f h = L_{\tilde{f}} h$, the set Λ_1 where we have to look for functions with linear independent differentials is the following:

$$\Lambda_1 = \{L^2_{\tilde{f}} h_j, \; L_{\tilde{g}_i} L_{\tilde{f}} h_j ; \; i,j = 1,2,3\}.$$

We get:

$$L_{\tilde{g}_1} L_{\tilde{f}} h = L_{\tilde{g}_2} L_{\tilde{f}} h = 0 \;,$$

$$L_{\tilde{g}_3} L_{\tilde{f}} h = [0 \;\; 1 \;\; \tilde{g}_{12,3}]^T \;,$$

$$L^2_{\tilde{f}} h = [f_8 \;\; 0 \;\; \tilde{f}_{12}].$$

From $\tilde{g}_{12,3} = \tilde{g}_{12,3}(x_4 x_6)$ we have at this step that $\sum_{i=1}^{3} L_{\tilde{g}_i}(\Omega_1 \cap G^{\perp}) \subset \Omega_1$ (assumption (A2) for k = 2) holds. Thus,

$$\Omega_2 = \Omega_1 \oplus \mathrm{sp}\{dL^2_{\tilde{f}} h_1 \; dL^2_{\tilde{f}} h_3\}$$

$$= \Omega_1 \oplus \mathrm{sp}\left\{ \begin{matrix} \partial f_8/\partial x_1 & 0 & * & 0 & 0 & * & | & 0 & * & 0 & * & 0 & * \\ 0 & 0 & 0 & * & \partial\tilde{f}_{12}/\partial x_5 & * & | & 0 & * & 0 & * & 0 & 0 \end{matrix} \right\}$$

where $*$ denotes non relevant terms and $\partial f_8/\partial x_1 = K_1/N_1 \omega_2 \neq 0$, $\partial \tilde{f}_{12}/\partial x_5 = K_3/N_3 H_8 \neq 0$ (a constant). Note that ω_2 is always nonzero being the second diagonal element of the inertia matrix $B(x_p)$ of the robot, which is positive definite for all x_p. So $s_2 = 8$ everywhere and $\lambda_2 = [h^T \; L_{\tilde{f}} h^T \; L^2_{\tilde{f}} h_1 \; L^2_{\tilde{f}} h_3]^T$. Moreover, the characteristic numbers for the second and third outputs are $\rho_2 = \rho_3 = 1$ while $\rho_1 > 1$.

In the 2-*nd iteration*,

$$A_2 = d\lambda_2 \cdot g = \left[O_{3\times 3} \;\middle|\; \begin{matrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & g_{10,3} & g_{12,3} & * & * \end{matrix} \right]^T, \; r_2 = 1.$$

As long as r_k remains constant we do not need to recompute a feedback pair (α,β). The functions in the set Λ_2 are the following:

$$L_{\tilde{g}} L^2_{\tilde{f}} h_1 = [0 \;\; 0 \;\; (\partial f_8/\partial x_{10} + \tilde{g}_{12,3} \partial f_8/\partial x_{12})]$$

$$L_{\tilde{g}} L^2_{\tilde{f}} h_3 = [0 \;\; 0 \;\; (\partial \tilde{f}_{12}/\partial x_{10})]$$

$$L^3_{\tilde{f}} h_1 = x_7(\partial f_8/\partial x_1) + \psi_1(x)$$

$$L^3_{\tilde{f}} h_3 = x_{11}(\partial \tilde{f}_{12}/\partial x_5) + \psi_2(x)$$

where $L_{\tilde{g}_3} L^2_{\tilde{f}} h_1$, $L_{\tilde{g}_3} L^2_{\tilde{f}} h_3$, ψ_1 and ψ_2 are all independent from $x_3 x_7 x_9 x_{11}$.

Again we have

$$\sum_{i=1}^{3} L_{\tilde{g}_i}(\Omega_2 \cap G^{\perp}) \subset \Omega_2$$

and

$$\Omega_3 = \Omega_2 \oplus \mathrm{sp}\{dL^3_{\tilde{f}} h_1 \;\; dL^3_{\tilde{f}} h_3\}$$

$$= \Omega_2 \oplus \mathrm{sp}\left\{ \begin{matrix} * & * & 0 & * & * & * & | & \partial f_8/\partial x_1 & * & 0 & * & 0 & * \\ * & * & 0 & * & 0 & * & | & 0 & * & 0 & * & \partial \tilde{f}_{12}/\partial x_5 & * \end{matrix} \right\}$$

giving $s_3 = 10$ everywhere and $\lambda_3 = [h^T \; L_{\tilde{f}} h^T \; L^2_{\tilde{f}} h_1 \; L^2_{\tilde{f}} h_3 \; L^3_{\tilde{f}} h_1 \; L^3_{\tilde{f}} h_3]^T$.

We can see that $\rho_1 = 2$; the rank of the decoupling matrix[10] $A(x)$ - which is a feedback invariant - is thus:

$$\mathrm{rank}\, A(x) = \mathrm{rank} \begin{bmatrix} L_{\tilde{g}} L^2_{\tilde{f}} h_1 \\ L_{\tilde{g}} L_{\tilde{f}} h_2 \\ L_{\tilde{g}} L_{\tilde{f}} h_3 \end{bmatrix} = \mathrm{rank} \begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & * \end{bmatrix} = 1$$

and we conclude that, as expected, the system is not decoupable by static state-feedback. It is also worth mentioning that this system does not satisfy the necessary and sufficient conditions[23] for the existence of a static state-feedback law which makes the input-dependent part of the response of the closed loop system linear in the input and independent from the initial state, as shown by Marino and Nicosia[4].

Coming back to the 3-*rd iteration* of the Algorithm we have:

$$A_3 = d\lambda_3 \cdot g = \left[O_{3\times 3} \;\middle|\; \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & g_{10,3} & g_{12,3} \end{matrix} \;\middle|\; \begin{matrix} 0 & 0 & (g_{71} \partial f_8/\partial x_1) & 0 \\ 0 & 0 & 0 & 0 \\ * & * & \phi_1(x) & * \end{matrix} \right]^T$$

and $r_3 = 2$. Choose the permutation matrix P_3 so that P_{31} picks up rows 5 and 9 from A_3. Then $-P_{31} B_3 = -P_{31} \cdot d\lambda_3 \cdot f = [-f_{10} \;\; \phi_2(x)]^T$ and a feedback pair $(\tilde{\alpha}, \tilde{\beta})$ is obtained solving the matrix equation (15) which gives:

$\tilde{\alpha}_1 = [(\phi_1(x) f_{10}/g_{10,3}) + \phi_2(x)] / (g_{71} \cdot \partial f_8/\partial x_1)$

$\tilde{\alpha}_2 = 0$, $\tilde{\alpha}_3 = -f_{10}/g_{10,3}$,

$\tilde{\beta}_{11} = 1/(g_{71} \cdot \partial f_8/\partial x_1)$, $\tilde{\beta}_{22} = 1$, $\tilde{\beta}_{33} = 1/g_{10,3}$,

$\tilde{\beta}_{13} = -\phi_1(x)/(g_{71} g_{10,3} \cdot \partial f_8/\partial x_1)$, $\tilde{\beta}_{ij} = 0$ (else).

The new vector fields $\tilde{f}, \tilde{g}$ are then:

$$\tilde{f} = f + g\tilde{\alpha} = [\, x_7 \; x_8 \; x_9 \; x_{10} \; x_{11} \; x_{12} \mid \tilde{f}_7 \; f_8 \; f_9 \; 0 \; \tilde{f}_{11} \; \tilde{f}_{12} \,]^T,$$

$$\tilde{g} = g\tilde{\beta} = \left[\begin{array}{c|cccccc} & \tilde{g}_{71} & 0 & 0 & 0 & 0 & 0 \\ 0_{3\times 6} & 0 & 0 & g_{92} & 0 & 0 & 0 \\ & \tilde{g}_{73} & 0 & 0 & 1 & \tilde{g}_{11,3} & \tilde{g}_{12,3} \end{array} \right]^T .$$

Again, the expression of the new terms involved are given in full in Appendix 1.

Compute next the functions in the set

$$\Lambda_3 = \{ L_{\tilde{f}} L^3_{\tilde{f}} h_j ,\; L_{\tilde{g}_i} L^3_{\tilde{f}} h_j \;;\; j = 1,3;\; i = 1,2,3 \}$$

which have the following expressions:

$$L_{\tilde{g}_1} L^3_{\tilde{f}} h_1 = 1, \; L_{\tilde{g}_1} L^3_{\tilde{f}} h_1 = L_{\tilde{g}_2} L^3_{\tilde{f}} h_1 = L_{\tilde{g}_2} L^3_{\tilde{f}} h_3 = 0,$$

$$L_{\tilde{g}_3} L^3_{\tilde{f}} h_1 = \psi_3(x), \; L_{\tilde{g}_3} L^3_{\tilde{f}} h_3 = \psi_4(x)$$

$$L_{\tilde{f}} L^3_{\tilde{f}} h_1 = \psi_5(x), \; L_{\tilde{f}} L^3_{\tilde{f}} h_3 = x_3 (\partial \tilde{f}_{11}/\partial x_3)(\partial \tilde{f}_{12}/\partial x_5) + \psi_6(x)$$

with $\partial \tilde{f}_{11}/\partial x_3 = K_2/N_2 G_5$ (constant) $\neq 0$ and $\psi_3(x), \ldots, \psi_6(x)$ all independent from x_3, x_9. Thus we still find $\sum_{i=1}^{3} L_{\tilde{g}_i}(\Omega_3 \cap G^\perp) \subset \Omega_3$ and

$$\Omega_4 = \Omega_3 \oplus sp\{d(L_{\tilde{f}} L^3_{\tilde{f}} h_1)\} = \Omega_3 \oplus sp\{* \, * \, (\frac{\partial \tilde{f}_{11}}{\partial x_3} \frac{\partial \tilde{f}_{12}}{\partial x_5}) \, * \, * \, * \mid * \, * \, 0 \, * \, 0 \, *\}$$

with $s_4 = 11$, globally; finally

$$\lambda_4 = [\, h^T \; L_{\tilde{f}} h^T \; L^2_{\tilde{f}} h_1 \; L^2_{\tilde{f}} h_3 \; L^3_{\tilde{f}} h_1 \; L^3_{\tilde{f}} h_3 \; L_{\tilde{f}} L^3_{\tilde{f}} h_3 \,]^T .$$

In the *4-th iteration* the rank r_4 of the matrix $A_4 = d\lambda_4 \cdot g$ is again 2; after similar computation as in the previous steps we can see that the only "new" function from the set Λ_4 is

$$L^2_{\tilde{f}} L^3_{\tilde{f}} h_3 = x_9 (\partial \tilde{f}_{11}/\partial x_3)(\partial \tilde{f}_{12}/\partial x_5) + \psi_7(x)$$

where $\psi_7(x)$ does not depend on x_9. The structural assumption (A2) is again satisfied at this step (and hence for all $k \geq 1$) and

$$\Omega_5 = \Omega_4 \oplus sp\{d L^2_{\tilde{f}} L^3_{\tilde{f}} h_3\} = \Omega_4 \oplus sp\{* \, * \, * \, * \, * \, * \mid * \, * \, (\frac{\partial \tilde{f}_{11}}{\partial x_3} \frac{\partial \tilde{f}_{12}}{\partial x_5}) \, * \, * \, *\}$$

giving $s_5 = 12 = \dim M$. Thus $k^* = 5$, $\Omega_* = T^* M$ for any x and $\Delta^* = \Omega_*^\perp = 0$, assumption (A1) of Theorem 1.

Last,

$$\lambda_5 = [\, h^T \; L_{\tilde{f}} h^T \; L^2_{\tilde{f}} h_1 \; L^2_{\tilde{f}} h_3 \; L^3_{\tilde{f}} h_1 \; L^3_{\tilde{f}} h_3 \; L_{\tilde{f}} L^3_{\tilde{f}} h_3 \; L^2_{\tilde{f}} L^3_{\tilde{f}} h_3 \,]^T$$

and we need to compute the matrix

$$A_5 = d\lambda_5 g = \left[\begin{array}{ccc|ccc|cc|cc|c|c} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{71} \partial f_8/\partial x_1 & 0 & * & \phi_3(x) \\ & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \phi_4 \\ 0 & 0 & 0 & 0 & g_{10,3} & * & * & * & \phi_1(x) & * & * & \phi_5(x) \end{array} \right]^T$$

which has rank $r_5 = 3$, being $\phi_4 = g_{92} (\partial \tilde{f}_{11}/\partial x_3)(\partial \tilde{f}_{12}/\partial x_5)$ a nonzero constant.

To conclude this Appendix, note that the following identities hold:

$$L_{\tilde{f}} h_i = L_{f} h_i = x_{6+2i} \; , \; i = 1,2,3;$$

$$L^2_{\tilde{f}} h_1 = L^2_{f} h_1 = f_8 \; , \; L^2_{\tilde{f}} h_3 = L^2_{f} h_3 = \tilde{f}_{12} \; ;$$

Furthermore $L^3_{\tilde{f}} h_1 = L^3_{f} h_1$ which is due to the fact that the vector fields $\tilde{f}$ and f differ only in the seventh component while f_8 is independent from x_7; for the same reason $L^3_{\tilde{f}} h_3 = L^3_{f} h_3$.

Thus Ω_* can also be spanned by the differentials of the following set of functions:

$$\xi_1 = h_2, \; \xi_2 = L_{\tilde{f}} h_2;$$

$$\eta_1 = h_1, \; \eta_2 = L_{\tilde{f}} h_1 \; , \; \eta_3 = L^2_{\tilde{f}} h_1 \; , \; \eta_4 = L^3_{\tilde{f}} h_1 \; ;$$

$$\zeta_1 = h_3, \; \zeta_2 = L_{\tilde{f}} h_3 \; , \; \zeta_3 = L^2_{\tilde{f}} h_3 \; , \; \zeta_4 = L^3_{\tilde{f}} h_3 \; , \; \zeta_5 = L^4_{\tilde{f}} h_3 \; ,$$

$$\zeta_6 = L^5_{\tilde{f}} h_3 \; .$$

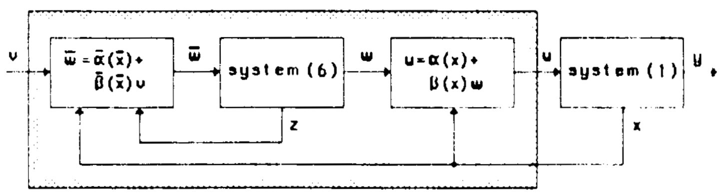

Fig. 1.

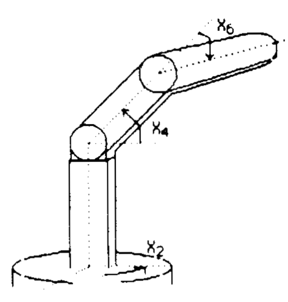

Fig. 2.