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A globally convergent steering algorithm for regular nonholonomic systems

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Abstract— **We present a steering algorithm for regular – i.e., without singularities – nonholonomic systems which are not required to possess special properties such as flatness or exact nilpotentizability. The method makes use of local steering laws, with suitable contraction properties, designed on the basis of a continuous approximation of the system. The algorithm is amenable to extension to systems with singularities.**

I. INTRODUCTION

Nonholonomic systems attract the attention of the scientific community for the theoretical challenges arising from the research on the control of these systems and for their relevance in applications. In particular, the problem of generating feasible trajectories joining two system configurations (referred to as nonholonomic path planning) has been solved for specific classes of driftless systems by effective techniques. These include a Lie-theoretical method for steering nilpotentizable systems [3], open-loop control (e.g., sinusoidal inputs [5]) for chained-form transformable systems and trajectory generation for flat systems [2].

However, there exist nonholonomic robots — also called *general* in this paper — whose kinematic model does not fall into any of the aforementioned classes. For example, mobile robots with more than one trailer cannot be transformed in chained form unless each trailer is hinged to the midpoint of the previous wheel axle — a particular arrangement, very unusual in real trailer vehicles, known as 'on-hooking'. Another such example are robotic systems that perform object manipulation by rolling contacts [6]: even the simplest mechanism in this category, the so-called plate-ball system, does not admit a chained-form transformation. More in general, for 2-input systems, as soon as the dimension of the state space reaches 5, exact nilpotentizability becomes the exception rather than the rule (whereas all systems up to dimension 4 possess this property [4]).

Techniques for steering general nonholonomic systems include the iterative method of [3], the generic loop method of [7] and the continuation method of [10]. However, the practical applicability of these methods is limited. In fact, the first two essentially require an a priori estimate of some "critical distance" which is generally unknown[1], while the third imposes strong assumptions on the system.

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[1]In [7], this is "masked" by the fact that an optimization problem is solved at each iteration.

On a similar line, we mention the iterative approach in [6], where stabilization (i.e., arbitrary boundedness of trajectories within the iterations) is seeked, so that appropriate continuity conditions must be satisfied by the steering control law.

In this paper, assuming that the considered system is regular (i.e., it has no singularities), we first give a local steering algorithm based on continuous approximations and steering laws characterized by a suitable contraction property. Then, we use the local method as an inspiration to devise a global steering algorithm. Our work is closely related to a suggestion in [9], where an iterative scheme of the type implemented in this paper is envisaged to overcome the necessity of knowing a priori the aforementioned critical distance.

The paper is organized as follows. In Section II we fix the notation and recall the basic definitions. In Section III we describe local approximate steering methods based on the use of approximations. These local methods are used in Section IV for devising a globally convergent steering algorithm in the absence of singularities, validated through simulation in Section V.

II. NONHOLONOMIC CONTROL SYSTEMS

We recall some basic tools used in sub-Riemannian geometry following [1].

Let Ω be an open connected subset of $\mathbb{R}^n$, and $VF(\Omega)$ the set of C^∞ vector fields on Ω. Consider a nonholonomic control system

$$\dot{x} = \sum_{i=1}^{m} g_i(x) u_i, \qquad x \in \Omega, \tag{1}$$

where $g_1, \ldots, g_m$ belong to $VF(\Omega)$ and the input $u(t) = (u_1(t), \ldots, u_m(t))$ is an integrable vector function which takes values in $\mathbb{R}^m$. This system is characterized by the m-tuple $g = (g_1, \ldots, g_m) \in VF^m(\Omega)$.

Definition 1: Given the function $u(t)$, $t \in [0,T]$, the *length* of u is defined as

$$\ell(u) = \int_0^T \sqrt{u_1^2(t) + \ldots + u_m^2(t)}\, dt.$$

Given $\overline{x} \in \Omega$, let $x_u(t)$, $t \in [0,T]$ be a trajectory of (1) originating from $\overline{x}$ under an input function $u(t)$. We define its *length* as

$$\ell(x_u(t)) = \ell(u).$$

A point $x = x_u(t)$, for $t \in [0,T]$, is *accessible* from $\overline{x}$.

Definition 2: System (1) induces a *sub-Riemannian distance* d on Ω, defined as

$$d(x_1, x_2) = \inf_{x_u} \ell(x_u), \quad (2)$$

where the infimum is taken over all trajectories x_u joining x_1 to x_2.

Note that $d(x_1, x_2) < \infty$ if and only if x_1 and x_2 are accessible from each other. Chow's Theorem states that any two points in Ω are accessible from each other if the elements of the Lie Algebra $\mathcal{L}_g$ generated by the g_i's form an n-dimensional vector space at each point. As system (1) is driftless, Chow's condition implies controllability in any usual sense [8]. Throughout this paper, we assume that system (1) is controllable.

Take $\overline{x} \in \Omega$ and let $L^s(\overline{x})$ be the vector space generated by the values at $\overline{x}$ of the brackets of the elements of g of length $\leq s$, $s = 1, 2, \ldots$ (input vector fields are brackets of length 1). Controllability guarantees that there exists a smallest integer $r = r(\overline{x})$ such that $\dim L^r(\overline{x}) = n$. This integer is called the *degree of nonholonomy* at $\overline{x}$.

Definition 3: Let $n_s(x) = \dim L^s(x)$, $s = 1, \ldots, r$, the sequence $(n_1(x), \ldots, n_r(x))$ is the *growth vector* of g at x.

Point $\overline{x}$ is said to be *regular* if the growth vector remains constant in a neighborhood of $\overline{x}$; otherwise $\overline{x}$ is *singular*. Points at which the degree of nonholonomy changes are singular. Regular points form an open and dense set in Ω.

Consider a smooth real-valued function f. Call *first-order nonholonomic derivatives* of f the Lie derivatives $g_i f$ of f along g_i, $i = 1, \ldots, m$. Call $g_i(g_j f)$, $i, j = 1, \ldots, m$, the *second-order nonholonomic derivatives* of f, and so on.

Definition 4: A function f is *of order* $\geq s$ at $\overline{x}$ if its nonholonomic derivatives of order $\leq s-1$ vanish at $\overline{x}$. If f is of order $\geq s$ and not of order $\geq s+1$ at $\overline{x}$, it is *of order* s at $\overline{x}$.

Equivalently, if f is of order $\geq s$ at $\overline{x}$, then $f(x) = O(d^s(x, \overline{x}))$.

Definition 5: A vector field h is *of order* $\geq q$ at $\overline{x}$ if, for every s and every f of order s at $\overline{x}$, hf has order $\geq q+s$ at $\overline{x}$. If h is of order $\geq q$ but not $\geq q+1$, it is *of order* q at $\overline{x}$.

It is easy to show that every element of g has order ≥ -1, bracket $[g_i, g_j]$, $i, j = 1, \ldots, m$, has order ≥ -2, and so on.

Definition 6: Let the integer w_j, $j = 1, \ldots, n$, be defined by setting $w_j = s$ if $n_{s-1} < j \leq n_s$, with $n_s = n_s(\overline{x})$ and $n_0 = 0$. Local coordinates $z_1, \ldots, z_n$ centered at $\overline{x}$ form a system of *privileged coordinates* if the order of z_j at $\overline{x}$ equals w_j (called the *weight* of coordinate z_j), for $j = 1, \ldots, n$.

The order of functions and vector fields expressed in privileged coordinates can be computed in an algebraic way:

- The order of the monomial $z_1^{\alpha_1} \ldots z_n^{\alpha_n}$ is equal to its weighted degree $w(\alpha) = w_1\alpha_1 + \cdots + w_n\alpha_n$.
- The order of a function $f(z)$ at $z = 0$ (the image of $\overline{x}$) is the least weighted degree of the monomials actually appearing in the Taylor expansion of f at 0.
- The order of a vector field $h(z) = \sum_{j=1}^{n} h_j(z)\partial_{z_j}$ at $z = 0$ is the least weighted degree of the monomials actually appearing in the Taylor expansion of h at 0:

$$h(z) \sim \sum_{\alpha,j} a_{\alpha,j} z_1^{\alpha_1} \ldots z_n^{\alpha_n} \partial_{z_j},$$

considering the term $a_\alpha z_1^{\alpha_1} \ldots z_n^{\alpha_n} \partial_{z_j}$ as a monomial and assigning to ∂_{z_j} the weight $-w_j$.

Definition 7: Given the system $z_1, \ldots, z_n$ of privileged coordinates at $\overline{x}$, the function

$$\|z\|_{\overline{x}} = |z_1|^{1/w_1} + \cdots + |z_n|^{1/w_n},$$

where $w_1, \ldots, w_n$ are the coordinate weights at $\overline{x}$, is called *pseudonorm* at $\overline{x}$.

Denoting by $B(x, r)$ an open sub-Riemannian ball of radius r centered at x, we have the following:

Definition 8: A *continuously varying system of privileged coordinates on* Ω is a mapping Φ, with values in $\mathbb{R}^n$, defined and continuous on a neighborhood of the diagonal in $\Omega \times \Omega$, and such that the partial mapping $z = \Phi(\overline{x}, \cdot)$ is a system of privileged coordinates at $\overline{x}$. In this case, there exists a continuous function $\sigma : \Omega \to (0, +\infty)$ such that the coordinates $\Phi(\overline{x}, \cdot)$ are defined on $B(\overline{x}, \sigma(\overline{x}))$; we call σ an *injectivity radius* of Φ.

If $x \in \Omega$ is a regular point, then there exists a continuously varying system of privileged coordinates on a neighborhood of x (see for instance [1] and [11]).

Privileged coordinates provide an estimate of the sub-Riemannian distance d, according to the following result.

Theorem 1 (Ball-Box Theorem): Consider $g \in VF^m(\Omega)$, a point $\overline{x} \in \Omega$ and a system of privileged coordinates z at $\overline{x}$. There exist positive constants $C'(\overline{x})$ and $\varepsilon'(\overline{x})$ such that, for all x with $d(x, \overline{x}) < \varepsilon'(\overline{x})$,

$$\frac{1}{C'(\overline{x})} \|z(x)\|_{\overline{x}} \leq d(x, \overline{x}) \leq C'(\overline{x}) \|z(x)\|_{\overline{x}}. \quad (3)$$

If Ω contains only regular points and if Φ is a continuously varying system of privileged coordinates on Ω, then there exist continuous positive functions $C'(\cdot)$ and $\varepsilon'(\cdot)$ on Ω such that inequality (3) holds with $z = \Phi(\overline{x}, \cdot)$ at all $(x, \overline{x})$ satisfying $d(x, \overline{x}) < \varepsilon'(\overline{x})$.

III. Local approximate steering methods

In this section, we define local approximate steering methods based on the use of system approximations, and introduce notions and criteria of contraction.

A. First-order approximations

Let $g = (g_1, \ldots, g_m) \in VF^m(\Omega)$.

Definition 9: A m-tuple $\hat{g} = (\hat{g}_1, \ldots, \hat{g}_m)$ defined on a neighborhood of $\overline{x}$ is a *first-order approximation of* g *at* $\overline{x}$ if the vector fields $g_i - \hat{g}_i$, $i = 1, \ldots, m$, are of order ≥ 0 at $\overline{x}$. A *first-order approximation of* g *on* Ω is a mapping A that associates to each $\overline{x} \in \Omega$ a first-order approximation $\hat{g} = A(\overline{x})$ of g at $\overline{x}$ defined on a ball $B(\overline{x}, \rho(\overline{x}))$. The function $\rho : \Omega \to (0, +\infty)$ is called the *approximation radius* of A.

Since first-order approximations are always used in this paper, they are referred to simply as 'approximations'.

Definition 10: A *continuous approximation* of g on Ω is an approximation $A : \overline{x} \mapsto \hat{g}$ of g on Ω such that the mapping

$$(x, \overline{x}) \mapsto \hat{g}(x) \in \mathbb{R}^n,$$

is defined and continuous on a neighborhood of the diagonal in $\Omega \times \Omega$, and the approximation radius ρ of A is continuous.

An additional property of approximations, useful for control design, is nilpotency.

Definition 11: Let $s \in \mathbb{N}$. We say that an approximation $A : \overline{x} \mapsto \hat{g}$ on Ω is *nilpotent of step* s if, for all $\overline{x} \in \Omega$, the Lie algebra generated by $\hat{g}$ is nilpotent of step s.

An explicit procedure for constructing continuous nilpotent approximations is given in [1], extended to systems with singularities in [11].

First-order approximations are used in this paper to design approximate steering laws for the original system. Privileged coordinates allow to measure the error obtained when we replace g by a first-order approximation (see [1, Prop. 7.29]).

Lemma 1: Consider a point $\overline{x} \in \Omega$, a system of privileged coordinates z at $\overline{x}$, and a first-order approximation $\hat{g}$ of g at $\overline{x}$. Then, there exist positive constants $C''(\overline{x})$ and $\varepsilon''(\overline{x})$ such that, for all $x \in \Omega$ with $d(x, \overline{x}) < \varepsilon''(\overline{x})$ and all integrable control functions $u(\cdot)$ with $\ell(u) < \varepsilon''(\overline{x})$, we have

$$\|z(x_u(T)) - z(\hat{x}_u(T))\|_{\overline{x}} \leq C''(\overline{x}) \max\left(\|z(x)\|_{\overline{x}}, \ell(u)\right) \ell(u)^{1/r}, \quad (4)$$

where r is the degree of nonholonomy at $\overline{x}$ and x_u and $\hat{x}_u$ are the trajectories of $\dot{x} = \sum_{i=1}^m g_i(x)\, u_i$ and $\dot{x} = \sum_{i=1}^m \hat{g}_i(x)\, u_i$ respectively which are defined by the same initial condition $x_u(0) = \hat{x}_u(0) = x$ and the same control function $u(\cdot)$.

If Ω contains only regular points, Φ is a continuously varying system of privileged coordinates on Ω and A a continuous approximation on Ω, then there exist continuous positive functions $C''(\cdot)$ and $\varepsilon''(\cdot)$ such that inequality (4) holds, with $z = \Phi(\overline{x}, \cdot)$ and $\hat{g} = A(\overline{x})$, for all $(x, \overline{x})$ with $d(x, \overline{x}) < \varepsilon''(\overline{x})$ and all integrable control functions $u(\cdot)$ with $\ell(u) < \varepsilon''(\overline{x})$.

B. Approximate steering

We need to define precisely the notion of steering law for an approximation. Let $A : \overline{x} \mapsto \hat{g}$ be an approximation on Ω and ρ its approximation radius.

Definition 12: A *steering law* of A is a mapping which, to every pair $x, \overline{x} \in \Omega$ satisfying $d(x, \overline{x}) < \rho(\overline{x})$, associates an integrable control function $\hat{u}(t)$, $t \in [0, T]$ (henceforth called a *steering control*) such that the trajectory $\hat{x}_{\hat{u}}(\cdot)$ with $\hat{x}_{\hat{u}}(0) = x$ is defined on $[0, T]$ and satisfies $\hat{x}_{\hat{u}}(T) = \overline{x}$. In other terms, $\hat{u}(\cdot)$ steers $A(\overline{x})$ from x to $\overline{x}$.

For example, a systematic design of the steering law is possible when nilpotent approximations are used [3].

Given g, an approximation A of g, and a steering law for A, we define a local approximate steering method for g as:

Definition 13: Fix $\overline{x} \in \Omega$. For a point $x \in B(\overline{x}, \rho(\overline{x}))$, let $\hat{u}(\cdot)$ be the steering control of $A(\overline{x})$ between x and $\overline{x}$. We denote by $x_{\hat{u}}(\cdot)$ the trajectory of the control system (1) originating from $x_{\hat{u}}(0) = x$ under this control function $\hat{u}(\cdot)$. The *local approximate steering* (LAS) method associated to A and its steering law is the function defined by:

$$\mathrm{AppSteer}(x, \overline{x}) = x_{\hat{u}}(T).$$

Definition 14: A LAS method is *contractive* if, for any $\overline{x} \in \Omega$, there exist positive constants $\varepsilon(\overline{x})$ and $C(\overline{x})$ such that, for any x sufficiently close to $\overline{x}$, $d(x, \overline{x}) < \varepsilon(\overline{x})$ implies

$$d(\mathrm{AppSteer}(x, \overline{x}), \overline{x}) \leq C(\overline{x})\, d(x, \overline{x})^{1+1/p},$$

where p is a positive constant independent of $\overline{x}$. A LAS method is *uniformly contractive* if it is contractive and if $\varepsilon(\cdot)$ and $C(\cdot)$ can be chosen as continuous functions.

A useful result which is later needed is the following.

Lemma 2: Consider a contractive LAS method $\mathrm{AppSteer}(\cdot, \cdot)$ associated to A. Then, $\forall x, \overline{x} \in \Omega$, $\exists\, \beta(\overline{x}) > 0$ such that if $d(x, \overline{x}) < \beta(\overline{x})$ both the following inequalities are satisfied:

$$d(\mathrm{AppSteer}(x, \overline{x}), \overline{x}) \leq \frac{1}{2} d(x, \overline{x}) \quad (5)$$

$$\|z(\mathrm{AppSteer}(x, \overline{x}))\|_{\overline{x}} \leq \frac{1}{2} \|z(x)\|_{\overline{x}}. \quad (6)$$

If Ω contains only regular points and the LAS method is uniformly contractive, then the function $\beta(\cdot)$ is continuous.

Proof: Straightforward from Definition 14 and the Ball-Box Theorem. ■

C. Quasi-optimal steering methods

It is possible to show that a contractive LAS method is obtained whenever the steering law satisfies a special condition.

Definition 15: We say that a steering law of A is *quasi-optimal* if there exists a constant $K > 0$ such that, for any $x, \overline{x} \in \Omega$ with $d(x, \overline{x}) < \rho(\overline{x})$, the control $\hat{u}(\cdot)$ steering $A(\overline{x})$ from x to $\overline{x}$ satisfies:

$$\ell(\hat{u}) \leq K\, d(x, \overline{x}) = K\, d(\hat{x}_{\hat{u}}(0), \hat{x}_{\hat{u}}(1)).$$

Note that, due to the definition of the sub-Riemannian distance, quasi-optimal steering laws always exist.

Proposition 1: A LAS method based on a quasi-optimal steering law is contractive. Moreover, if Ω contains only regular points and if the approximation A is continuous on Ω, then the method is uniformly contractive.

Proof: Let A be an approximation of g on Ω, and AppSteer a local approximate steering method associated to A based on a quasi-optimal steering law.

Given $\overline{x} \in \Omega$, let z be a system of privileged coordinates at $\overline{x}$, and $\hat{g} = A(\overline{x})$. Reducing if needed the approximation radius $\rho(\overline{x})$ of A, we assume that $\rho(\overline{x})$ is an injectivity radius of z. Given another point $x \in B(\overline{x}, \rho(\overline{x}))$, let $\hat{u}(\cdot)$ be the steering control of $\hat{g}$ between x and $\overline{x}$, and set $x_{\hat{u}}(T) = \mathrm{AppSteer}(x, \overline{x})$. Using Lemma 1, the fact that $\hat{x}_{\hat{u}}(T) = \overline{x}$ by construction and that the image of $\overline{x}$ in the privileged coordinates at $\overline{x}$ is the origin, we obtain, that, if $d(x,\overline{x})$ and $\ell(u)$ are smaller than $\varepsilon''(\overline{x})$, then

$$\|z(x_{\hat{u}}(T))\|_{\overline{x}} \leq C''(\overline{x}) \max\left(\|z(x)\|_{\overline{x}}, \ell(\hat{u})\right)\, \ell(\hat{u})^{1/r}.$$

Now, since $\hat{u}(\cdot)$ is quasi-optimal, we have $\ell(\hat{u}) \leq Kd(x,\overline{x})$. Using then the Ball-Box Theorem, we obtain

$$\frac{1}{C'(\overline{x})}\, d(\overline{x}, x_{\hat{u}}(T)) \leq C''(\overline{x}) K^{1/r} \max(C'(\overline{x}), K)\, d(x,\overline{x})^{1+1/r},$$

provided that $d(x,\overline{x})$ is smaller than a constant $\varepsilon(\overline{x})$ depending on $C'(\overline{x})$, $C''(\overline{x})$, $\varepsilon'(\overline{x})$, $\varepsilon''(\overline{x})$, $\rho(\overline{x}) = \sigma(\overline{x})$ and K. We then obtain that the method is contractive.

Assume now that A is continuous and that Ω contains only regular points. There exists a continuously varying system Φ of privileged coordinates on Ω. Thus $C'(\cdot)$, $C''(\cdot)$, $\varepsilon'(\cdot)$, $\varepsilon''(\cdot)$, $\rho(\cdot)$, $\sigma(\cdot)$ – and so $\varepsilon(\cdot)$ – can be chosen as continuous functions, and the LAS method is uniformly contractive. ■

IV. The global approximate steering algorithm

Assume that the domain of definition Ω for system (1) contains only regular points. Assume further that there exist a continuously varying system of privileged coordinates and an associated continuous approximation A of g on Ω. This hypothesis will be removed at the end of the section.

In this section, we will devise an algorithm to steer system (1) from any $x_0 \in \Omega$ to the origin (assumed w.l.o.g. to be the goal) using a contractive LAS designed on the basis of A. To have an infinite injectivity radius, we use the algebraic privileged coordinates defined in [1].

First, note that a locally convergent approximate steering algorithm can be easily built using a contractive LAS method, as follows.

Take a starting point x_0 such that $d(x_0,\overline{x}) < \varepsilon(\overline{x})$ and $d(x_0,\overline{x}) < \beta(\overline{x})$, and let e be a given tolerance.

Local_Approximate_Steering$(x_0,\overline{x})$

1. $k := 0$;
2. $x^k := x_0$;
3. while $\|z(x^k)\|_{\overline{x}} > e$
4. $\quad x^{k+1} = \text{AppSteer}(x^k,\overline{x})$;
5. $\quad k := k+1$;

Based on this local algorithm, the construction of the global approximate steering method to be presented is inspired to the following idea. Consider a parameterized path[2] γ connecting x_0 to 0, and choose a finite sequence of intermediate goals $\{x_0^d = x_0, x_1^d, \ldots, x_n^d = 0\}$ on γ, such that $d(x_{i-1}^d, x_i^d) < \beta(x_i^d)/2$, $i = 0,\ldots,n$, with $\beta(x_i^d)$ as in Lemma 2 (this is always possible thanks to the regularity assumption, which guarantees that $\beta(\cdot)$ is continuous and, hence, has a positive lower bound on the compact set γ). It is possible to prove that the iterated application of a contractive steering method AppSteer(x_{i-1}, x_i^d) from the current state to the next subgoal (having set $x_i^d = 0$, $\forall i \geq n$) yields an approximate steering algorithm[3] which is globally convergent to 0.

However, the above algorithm requires the a priori knowledge of $\beta(\cdot)$, which in practice is not available. An algorithm which achieves global convergence to 0 without knowing $\beta(\cdot)$ is described in Fig. 1.

The function Subgoal is the following.

Subgoal$(x_{i,0}, \eta_i, x_{i,j-1}^d)$

1. $t_j := \max(0, 1 - \frac{j\eta_i}{\|z(x_{i,0})\|_0})$;
2. $x_{i,j}^d := \delta_{0,t_j}(x_{i,0})$

Here, δ_{0,t_j} denotes the dilation in privileged coordinates at 0, with parameter t. That is, $\delta_{0,t_j}(x_{i,0})$ is computed by first dilating $z(x_{i,0})$ and then mapping back to the original coordinates. The formula for generating t_j guarantees that $\|z(x_{i,j}^d) - z(x_{i,j-1}^d)\|_0 = \eta_i$ if $\|z(x_{i,j-1}^d)\|_0 > \eta_i$; otherwise, $t_j = 0$ and then $x_{i,j}^d = 0$.

The global convergence of the approximate steering algorithm is established in the following result. For the sake of simplicity we assume to work on a compact set $K \subset \Omega$.

Proposition 2: If the LAS method AppSteer$(\cdot,\cdot)$ used in Step 7 is uniformly contractive, then the algorithm Global_Approximate_Steering terminates in a finite number of steps for any choice of x_0 and of the tolerance e.

Proof: Set $\beta = \min_{x\in K} \beta(x)$. The LAS method being uniformly contractive, the function $\beta(\cdot)$ is continuous and so the constant β is positive.

Note first that the contraction property of the LAS method guarantees that if $d(x_{i,j-1}, x_{i,j}^d) < \beta$, for some i and for $j = 1,2,\ldots$, then condition (6) is verified and the conditional statement of Step 8 is not true (i.e., the system is approaching the origin). In this case, the error $\|z(x_{i,j})\|_0$ is reduced at each iteration and the algorithm stops when it becomes smaller than a given tolerance e.

Another preliminary remark is that, due to the continuity of the control distance and of the pseudonorm at zero in the regular case, there exists μ such that, for any $x, y \in K$,

$$\|z(x) - z(y)\|_0 < \mu \Rightarrow d(x,y) < \frac{\beta}{2}. \qquad (7)$$

In the following, we will prove by induction that if

$$\|z(x_{i,j-1}^d) - z(x_{i,j}^d)\|_0 = \eta_i < \mu,$$

for some i and for $j = 1,2,\ldots$, then

$$d(x_{i,j-1}, x_{i,j}^d) < (1/2 + \cdots + (1/2)^j)\beta < \beta.$$

For $j = 1$, it is $x_{i,0} = x_0$ and by construction

$$\|z(x_{i,0}) - z(x_{i,1}^d)\|_0 = \eta_i < \mu.$$

In view of (7) we have then $d(x_{i,0}, x_{i,1}^d) < \beta/2$.

[2]If no such path exists, Ω is not arc-connected and the steering problem has no solution in Ω.

[3]A similar idea is proposed in [3].

Global_Approximate_Steering$(x_0, 0)$

1. $i := 0;\ j := 0;$
2. $x_{i,0} := x_0;$
3. $\eta_i := \|z(x_0)\|_0;$ — *initial choice of the maximum step size;*
4. while $\|z(x_{i,j})\|_0 > e$ — *while the pseudonorm at 0 of the state is above a given tolerance e…;*
5. $\quad j := j + 1;$
6. $\quad x^d_{i,j} :=$ Subgoal$(x_{i,0}, \eta_i, x^d_{i,j-1});$ — *choose the subgoal $x^d_{i,j}$ at a distance η_i from $x^d_{i,j-1}$;*
7. $\quad x_{i,j} :=$ AppSteer$(x_{i,j-1}, x^d_{i,j});$ — *steer the system from $x_{i,j-1}$ using an approximate steering control with destination $x^d_{i,j}$;*
8. $\quad$ if $\|z(x_{i,j})\|_{x^d_{i,j}} > \frac{1}{2}\|z(x_{i,j-1})\|_{x^d_{i,j}}$ — *if the system is not approaching the subgoal…;*
9. $\quad\quad \eta_{i+1} := \min(\frac{\eta_i}{2}, \frac{\|z(x_{i,j-1})\|_0}{2});$ — *reduce the maximum step size;*
10. $\quad\quad x_{i+1,0} := x_{i,j};\ i := i + 1;\ j := 0;$

Fig. 1. The approximate steering algorithm

Assume now that for $j = n - 1 > 1$ we have:

$$d(x_{i,n-2}, x^d_{i,n-1}) < (1/2 + \cdots + (1/2)^{n-1})\beta. \tag{8}$$

For $j = n$ we can write

$$d(x_{i,n-1}, x^d_{i,n}) \le d(x_{i,n-1}, x^d_{i,n-1}) + d(x^d_{i,n-1}, x^d_{i,n}).$$

By construction, it is

$$\|z(x^d_{i,n-1}) - z(x^d_{i,n})\|_0 = \eta_i < \mu,$$

which implies $d(x^d_{i,n-1}, x^d_{i,n}) < \beta/2$. The induction hypothesis (8) implies that

$$d(x_{i,n-1}, x^d_{i,n-1}) \le \frac{1}{2} d(x_{i,n-2}, x^d_{i,n-1}).$$

Finally, we have

$$\begin{aligned} d(x_{i,n-1}, x^d_{i,n}) &\le \frac{1}{2} d(x_{i,n-2}, x^d_{i,n-1}) + d(x^d_{i,n-1}, x^d_{i,n}) \\ &\le (1/2 + \cdots + (1/2)^n)\beta. \end{aligned}$$

If, for some (i, j), $\eta_i \ge \mu$ the conditional statement of step 8 could be false. In this case, η_i is decreased as in step 9 and the algorithm is restarted from the current point. The updating law of η_i guarantees that there exists i such that $\eta_i < \mu$ for $j = 1, 2, \ldots$, i.e., there exists i such that the error $\|z(x_{i,j})\|_0$ is reduced at each iteration. ■

For the sake of simplicity, we have assumed through this section the existence on Ω of a continuously varying system of privileged coordinates and an associated continuous approximation of g. This hypothesis can be removed by noting that, since Ω contains only regular points, the existence is guaranteed on a neighbourhood of every point in Ω. As suggested in [6], the working space $K \subset \Omega$ can then be covered by a finite number of compact sets, the assumption being true on each one of these compact sets. The proof of Proposition 2 should be accordingly modified by assigning the appropriate minimum value of $\beta(x)$ over K.

V. SIMULATION RESULTS

The proposed algorithm has been tested on the plate-ball manipulation system, a well-known example of general nonholonomic system without singularities. The generalized coordinates are (u, v, ψ, x, y), where u and v are the coordinates of the contact point on the sphere, ψ is the orientation of the sphere, and x, y are the coordinates of the contact point on the plane. A feedback transformation allows to put the system in a triangular form, with the first three equations in chained form, as described in [6]. We refer the reader to this work for the original and the transformed model of the system as well as for the definition of the privileged coordinates and the computation of the nilpotent approximation. In the same paper it is also shown that the steering problem for this system can be splitted in two phases: in the first one, the u, v and ψ coordinates are driven to the origin while x and y drift; in the second phase, x and y are steered approximately to the origin while the first three coordinates perform a cyclic motion returning to the origin at the end of this steering phase.

For simplicity, we assume that the first three coordinates are already at the origin. This allows to compute the privileged coordinates and nilpotent approximations only at points with the first three coordinates equal to zero. It can be shown [6] that if the parameters of the steering law:

$$w_1 = a_1^{II} \cos \omega t + a_2^{II} \cos 4\omega t \tag{9}$$

$$w_2 = b_1^{II} \cos 2\omega t, \tag{10}$$

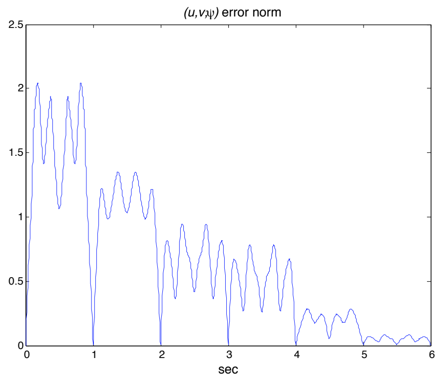

Fig. 2. Euclidean norm of the first three coordinates.

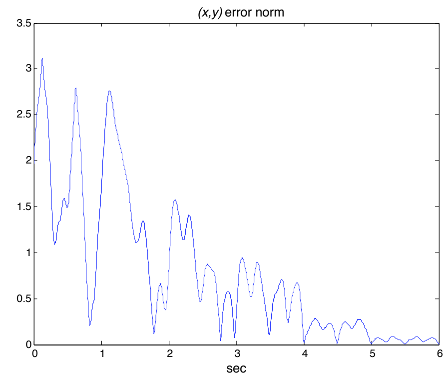

Fig. 3. Euclidean norm of x and y.

with $a_1^{II}, a_2^{II}, b_1^{II} \in \mathbb{R}$ and $\omega = 2\pi/T$, are chosen as:

$$a_1^{II} = \left(\frac{z_4^i}{k_1 b_1^{II}}\right)^{1/2}, \qquad a_2^{II} = \frac{z_5^i}{k_2 (b_1^{II})^2},$$

$$b_1^{II} = -\text{sign}(z_4^i) \cdot \left| \left(\begin{array}{c} z_4^i \\ z_5^i \end{array} \right) \right|^{1/3},$$

where $k_1 = -T^3/32\pi^2$ and $k_2 = T^3/128\pi^2$, then the control (9–10): *(i)* steers the system from z^i to $z = 0$, respectively the image of the current and goal point in privileged coordinates centered at the latter, *(ii)* drives u, v, ψ exactly back to the origin, *(iii)* is uniformly contractive.

In the proposed simulation, the initial point has been chosen as $(0, 0, 0, -1, -1.7)$ and the tolerance e as 10^{-5}. The algorithm converges (i.e., the pseudonorm at zero is below e) after six iterations. Figure 2 shows the norm of the first three coordinates. Note that, at the end of each iteration the norm is zero, confirming that the first three coordinates are driven exactly to zero, while its maximum value is reduced along the iterations. The euclidean norm of the (x, y) subvector is reported in fig. 3 and shows how the error is reduced both at the end and during each iteration.

The values of the pseudonorm of the state used at step 8 of the algorithm, are reported in Table I. Observe that the error is not reduced to half of the initial error at the end of iterations 1 and 2. The step size η_i is correspondingly updated to the value in the fourth column of the table.

TABLE I

PSEUDONORM OF THE STATE AT STEP 8 OF THE ALGORITHM

iteration	$\frac{1}{2}\|\|z(x_{i,j})\|\|_{x^d_{i,j-1}}$	$\|\|z(x_{i,j})\|\|_{x^d_{i,j}}$	η_i
1	0.45	0.6471	0.9
2	0.2250	0.2332	0.45
3	0.1125	0.0188	0.2250
4	0.0664	0.0062	0.2250
5	0.0031	0.0002	0.2250
6	0.0001	0.0000	0.2250

VI. CONCLUSIONS

In this paper we have presented a globally convergent steering algorithm for regular nonholonomic systems. In particular, we have defined local approximate steering methods based on the use of system approximations, and introduced notions and criteria of contraction. On this basis we have constructed a steering algorithm which is proven to be globally convergent. We have tested the algorithm by simulation on the plate-ball manipulation system.

We are currently extending the presented algorithm to systems with singularities relying on the tools devised in [11]. In particular, we use nonhomogeneous nilpotent approximations built on continuously varying systems of privileged coordinates therein developed to generalize the proposed steering algorithm to systems with singularities.

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