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Brief Paper

Robust stabilization via iterative state steering with an application to chained-form systems☆

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Abstract

An approach is presented for the robust stabilization of non-linear systems. The proposed strategy can be adopted whenever it is possible to compute a control law that steers the state in finite time from any initial condition to a point closer to the desired equilibrium. Under suitable assumptions, such control law can be applied in an iterative fashion, obtaining uniform asymptotic stability of the equilibrium point, with exponential rate of convergence. Small non-persistent perturbations are rejected, while persistent perturbations induce limited errors. In order to show the usefulness of the presented theoretical developments, the approach is applied to chained-form systems and, for illustration, simulations results are given for the robust stabilization of a unicycle.

Keywords: Stabilization; Robustness; Iterative methods; Non-smoothly stabilizable systems; Mobile robots

1. Introduction

This paper deals with the asymptotic stabilization of equilibrium points for controllable systems. For linear systems, controllability and smooth stabilizability imply each other, whereas this is not true for non-linear systems. In fact, there exist systems which are controllable but cannot be asymptotically stabilized at equilibrium points by a continuous time-invariant feedback, e.g., non-holonomic systems. This circumstance has motivated the search for time-varying (Pomet, 1992; Samson, 1995) and/or discontinuous (Sørdalen & Egeland, 1995) feedback controllers for particular classes of systems. Hence, there is a clear interest in control strategies other than the classical continuous state feedback, for they can achieve stabilization of systems that do not satisfy the necessary condition for smooth stabilizability due to Brockett (1983).

The stabilization strategy described here, first introduced in Lucibello and Oriolo (1996), is called *iterative state steering*; it can be summarized as follows. Suppose that we can compute a control law, not necessarily in feedback, which steers the system in finite time from any point to another closer to the desired equilibrium. By applying such control law in an iterative fashion, one obtains a time-varying feedback which, under suitable assumptions, produces exponential convergence to the desired equilibrium, guaranteeing at the same time certain robustness properties. In particular, small non-persistent perturbations are rejected, while ultimate boundedness is achieved in the presence of persistent perturbations.

The above idea is simple and, in fact, not completely new. Related approaches were proposed by Hermes (1980), resulting in a computationally intensive stabilization technique for low-dimensional systems, by Bennani and Rouchon (1995) for chained-form systems, and by Lucibello (1992) for steering a flexible spacecraft. Also, there are similarities with receding horizon control and predictive control techniques; in particular, the former have been applied to non-holonomic systems by Alamir and Bornard (1996).

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With respect to these works, the distinctive points of our contribution are the following:

- It formalizes an intuitive control approach that is applicable to non-linear systems for which an approximate steering control can be computed, identifying conditions under which asymptotic stability with exponential convergence is achieved. In particular, our approach can be applied to systems with drift, whereas the methods in Hermes (1980), Bennani and Rouchon (1995) and Alamir and Bornard (1996) deal only with driftless systems.
- It addresses explicitly the robustness problem, exhibiting a direct proof of disturbance rejection that gives guidelines for robust control design. None of the aforementioned techniques does this, with the notable exception being the one by Bennani and Rouchon (1995), which however heavily relies on the special structure of chained-form systems.
- It can provide a simple solution to difficult problems, such as the stabilization of an underactuated satellite (Lucibello & Oriolo, 1998), of a non-flat non-holonomic system (Vendittelli & Oriolo, 2000), or the robust stabilization of the unicycle (a canonical instance of the class of non-holonomic mechanical system) considered in this paper.

With reference to the last remark, we mention that robust control for non-holonomic mobile robots was also addressed by Canudas de Wit and Khennouf (1995) for the case of a unicycle with perturbations on the translational velocity, and by D'Andréa-Novel, Campion and Bastin (1995) for the trajectory tracking problem, which is however known to be much easier than point stabilization in non-holonomic systems.

The paper is organized as follows. Section 2 introduces the fundamental ideas on which iterative state steering is based, while the robustness properties of the scheme are analyzed in Section 3. In order to show the usefulness of the presented approach, we devise a stabilizing control for the (2,3) chained form in Section 4. Its effectiveness is illustrated by simulating the control of a unicycle whose radius is not exactly known.

2. Iterative state steering

The proposed stabilization strategy relies on the iterative application of a suitable control law: during each iteration, the state of the system is steered from the current point to another, closer to the desired equilibrium. In this section, we shall give a result that formalizes this simple idea.

Consider the control system

$$\dot{x}(t) = f(x(t), u(t), t), \quad x(t_0) = x_0 \in B_x, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control vector, and B_x is an open ball around the origin of the state space. Without loss of generality, we assume the origin to be the desired equilibrium; hence, it is $f(0,0,t) = 0$ for any $t \in \mathbb{R}$.

Consider now an infinite sequence of time instants $\{t_k\}$, for $k = 0, 1, 2, \ldots$, with $t_{k+1} = t_k + T_{k+1}$, and $0 < T_m \leq T_{k+1} \leq T_M < \infty$. For compactness, we let $x(t_k) = x_k$ hereafter. On each time interval $I_{k+1} = [t_k, t_{k+1})$, the following steering control is applied:

$$u(t) = u_{k+1}(t) = \alpha(x(t), x_k, t), \quad t \in I_{k+1}. \tag{2}$$

The steering control (2) may be specified either in open-loop (i.e., as a function of only the current initial condition x_k and time t) or in feedback. Let

$$\dot{x} = f(x(t), \alpha(x(t), x_k, t), t) = \tilde{f}(x(t), x_k, t), \quad t \in I_{k+1}, \; k = 0, 1, 2, \ldots \tag{3}$$

be the closed-loop system, i.e., the dynamics of system (1) under the repeated application of the steering control (2).

Throughout the paper, we suppose that the steering control $\alpha(\cdot)$ has been chosen so as to satisfy the following assumption (the symbol $|\cdot|$ denotes the euclidean norm):

Assumption 1. *The steering control function α is such that*:

(a) $\alpha(x, 0, t) = 0$, *for any* $(x, t) \in \mathbb{R}^n \times I_{k+1}$;
(b) $\tilde{f}$ *is locally Lipschitzian in x (with Lipschitz constant λ), continuous in x_k and piecewise-continuous*[1] *in t, for $t \in I_{k+1}$*;
(c) *the following condition (contraction) is satisfied*:

$$|x_{k+1}| \leq \beta |x_k|, \quad \beta < 1, \; \forall x_k \in B_x. \tag{4}$$

A first consequence of Assumption 1 (point *b*) is that the solution of Eq. (3) exists and is unique (Hale, 1980). Therefore, once a steering control (2) has been selected, the state $\varphi(t, x_0, t_0)$ at time $t \geq t_0$ is uniquely determined by t and the initial conditions (x_0, t_0).

Moreover, points *a* and *c* of Assumption 1 imply that the origin is an exponentially stable equilibrium point of the discrete-time system

$$x_{k+1} = \varphi(t_{k+1}, x_k, t_k) \tag{5}$$

since

$$|x_{k+1}| \leq \beta^{k+1} |x_0|, \quad k = 0, 1, 2, \ldots. \tag{6}$$

We define *geometric* the strong type of exponential convergence implied by Eq. (6).

As x_k approaches the origin, one would also like the state of the continuous-time system to converge exponentially to zero; moreover, stability in the sense of

[1] This means that $\tilde{f}$ is continuous in I_{k+1} except, possibly, for a finite set of points where it may have finite jump discontinuities (Khalil, 1992).

Lyapunov is desirable. Below, we give conditions under which this situation is achieved.

Theorem 1. *Under Assumption* 1, *for the controlled system* (3):

(1) *The origin is a uniformly asymptotically stable equilibrium point.*
(2) *If the additional condition holds*

$$|\tilde{f}(0, x_k, t)| \leq \bar{\mu}|x_k|^r, \quad t \in I_{k+1}, \ \bar{\mu} \geq 0, \ r > 0, \tag{7}$$

then the rate of convergence is exponential. In particular, if $r \geq 1$ *the origin is an exponentially stable equilibrium point.*

Proof.

(1) Assumption 1 implies that (point *a*) the origin is an equilibrium point also of the continuous-time controlled system, and that (points *a* and *b*)

$$|\tilde{f}(x, x_k, t) - \tilde{f}(0, x_k, t)| \leq \lambda|x|, \quad x, x_k \in B_x, \ t \in I_{k+1},$$

with $\lambda > 0$ and $|\tilde{f}(0, x_k, t)|$ continuous in x_k and such that $|\tilde{f}(0,0,t)| = 0$. Therefore, we have

$$|\tilde{f}(x, x_k, t)| \leq \lambda|x| + |\tilde{f}(0, x_k, t)|, \quad x, x_k \in B_x, \ t \in I_{k+1}. \tag{8}$$

Since $|\tilde{f}(x, x_k, t)|$ is piecewise-continuous in t in the closed interval I_{k+1}, $|\tilde{f}(0, x_k, t)|$ can be bounded as

$$|\tilde{f}(0, x_k, t)| \leq \mu(|x_k|), \quad x_k \in B_x, \ t \in I_{k+1}, \tag{9}$$

with $\mu(|x_k|)$ continuous in x_k and such that $\mu(0) = 0$.
On the other hand, since

$$x(t) = x_k + \int_{t_k}^{t} \tilde{f}(x(\tau), x_k, \tau)\, d\tau, \quad t \in I_{k+1},$$

we have

$$|x(t)| \leq |x_k| + \int_{t_k}^{t} |\tilde{f}(x(\tau), x_k, \tau)|\, d\tau \leq |x_k| + \int_{t_k}^{t} \lambda|x(\tau)|\, d\tau + \int_{t_k}^{t} \mu(|x_k|)\, d\tau,$$

where we have exploited Eqs. (8) and (9). By using the Gronwall inequality (Hale, 1980) and the fact that $t - t_k \leq T_m$ for $t \in I_{k+1}$, one gets

$$|x(t)| \leq (|x_k| + T_M \mu(|x_k|))e^{\lambda T_M}, \quad t \in I_{k+1}. \tag{10}$$

Since μ is continuous at zero and $\mu(0) = 0$, Eq. (10) shows that, as x_k converges to zero, $x(t)$ also converges to zero. In particular, asymptotic stability is achieved, as $|x(t)|$ can be arbitrarily bounded, for all t, by appropriately choosing x_0 (recall that $|x_k| < \beta^k|x_0|$).

(2) If the additional condition (7) holds, Eq. (10) gives

$$|x(t)| \leq (|x_k| + \bar{\mu}T_M|x_k|^r)e^{\lambda T_M}, \quad t \in I_{k+1}.$$

Hence,

$$|x_k| \leq 1, \ r < 1 \Rightarrow |x(t)| \leq l|x_k|^r, \quad t \in I_{k+1}, \tag{11}$$

$$|x_k| \leq 1, \ r \geq 1 \Rightarrow |x(t)| \leq l|x_k|, \quad t \in I_{k+1}, \tag{12}$$

where $l = (1 + \bar{\mu}T_M)e^{\lambda T_M}$ is a bounded positive quantity. Since $|x_k| \leq \beta^k|x_0|$, we obtain

$$|x_k| \leq 1, \ r < 1 \Rightarrow |x(t)| \leq l\beta^{kr}|x_0|^r, \quad t \in I_{k+1}, \tag{13}$$

$$|x_k| \leq 1, \ r \geq 1 \Rightarrow |x(t)| \leq l\beta^k|x_0|, \quad t \in I_{k+1}, \tag{14}$$

i.e., $x(t)$ converges exponentially to zero, with rate $r|\log\beta|/T_M$ if $r < 1$ or $|\log\beta|/T_M$ if $r \geq 1$. It is easy to verify that convergence is exponential also when $|x_k| > 1$, with convergence rate $|\log\beta|/T_M$ if $r < 1$ or $r|\log\beta|/T_M$ if $r \geq 1$ in that region.

If $r \geq 1$, the origin is locally exponentially stable in the Lyapunov sense (Hahn, 1967, p. 113), as the following estimate holds

$$|x(t)| \leq \frac{l}{\beta}|x_0|\, e^{-(|\log\beta|/T_M)\,t}$$

for $|x_0| \leq 1$. If $r = 1$, the origin is globally exponentially stable. □

Remark 1. With reference to Assumption 1, note that the continuity requirement on $\tilde{f}$ (and thus, on the steering control α) in x_k is essential for the proposed stabilization strategy. In particular, condition (7) is known as *Hölder-continuity* of order r at the origin.

Remark 2. In our formulation, the duration T_{k+1} of the $(k+1)$th iteration is not necessarily fixed, and may take values within an interval $[T_m, T_M]$. In fact, as the current initial condition x_k approaches the origin, a smaller time T will typically be sufficient for achieving the contraction condition (4). Note also that only the upper bound T_M has influence on the convergence rate estimate.

Remark 3. Eqs. (13) and (14) hold for the $(k+1)$th iteration, with $k = 1, 2, \ldots$, while Eqs. (11) and (12) hold also for the first iteration. In particular, if $\beta = 0$ in Eq. (4), the steering controller drives the system to zero in one iteration, and thus $x(t) = 0$, $\forall t \geq t_1$.

3. Robustness analysis

Suppose that the given control system (1) is perturbed as follows:

$$\dot{z}(t) = f(z(t), u(t), t) + \varepsilon g(z(t), u(t), t) \tag{15}$$

with $\varepsilon \in \mathbb{R}^+$, and $z \in \mathbb{R}^n$ the state of the perturbed system. During the $(k+1)$th iteration, the controlled system becomes

$$\dot{z} = \tilde{f}(z(t), z_k, t) + \varepsilon \tilde{g}(z(t), z_k, t), \quad t \in I_{k+1}. \tag{16}$$

The solution of Eq. (16) exists and is unique under Assumption 1 and the following

Assumption 2. *Function $\tilde{g}$ is locally Lipschitzian in z (with Lipschitz constant η), continuous in z_k and piecewise-continuous in t, for $t \in I_{k+1}$.*

It is customary (Hahn, 1967) to classify the perturbations as

persistent, $\tilde{g}(0, 0, t) \neq 0$,

non-persistent, $\tilde{g}(0, 0, t) = 0$.

For *dynamical* systems, converse Lyapunov theorems can be used in order to infer that exponential stability of an equilibrium is preserved under small non-persistent perturbations, while limited errors occur under persistent perturbations (Hahn, 1967, Theorems 56.2 and 56.4). However, even in the case of exponential stability, this kind of result does not apply here, because — as shown in the appendix — the nominal system (3) does not satisfy the definition of dynamical system as given, e.g., by Hahn (1967).

Nevertheless, a robust stability property can be established for the continuous-time system (3) under certain assumptions. To prove ultimate boundedness or complete rejection of the perturbation for the continuous-time system (16), we first analyze the stability of the associated discrete-time system. To this end, let us recall a preliminary result which is valid for geometrically stable discrete-time systems. The proof is immediate and therefore omitted.

Lemma 1. *Consider the system*

$$\xi_{k+1} = \phi(\xi_k, k), \quad \xi_k \in \mathbb{R}^n$$

for which

$$|\phi(\xi_k, k)| \leq \beta |\xi_k|, \quad \beta < 1$$

and the perturbed system

$$\zeta_{k+1} = \phi(\zeta_k, k) + \gamma(\zeta_k, k), \quad \zeta_k \in \mathbb{R}^n.$$

(1) *If the perturbation is bounded as*

$$|\gamma(\zeta_k, k)| \leq \varepsilon,$$

ζ is ultimately bounded (namely, confined to the ball $\{\zeta : |\zeta| \leq \varepsilon/(1-\beta)\}$).

(2) *More stringently, if the perturbation γ satisfies the following estimate:*

$$|\gamma(\zeta_k, k)| \leq \varepsilon |\zeta_k|^s, \quad s \geq 1, \tag{17}$$

the contraction (and hence, the geometric stability) is locally preserved for sufficiently small ε (namely, for $|\xi_k| \leq 1$ and $\varepsilon < 1 - \beta$). In the particular case $s = 1$, the stability is globally preserved for sufficiently small ε.

Remark 4. To appreciate the role of the condition $s \geq 1$ in Eq. (17), consider the system

$$\zeta_{k+1} = \beta \zeta_k + \varepsilon \sqrt{|\zeta_k|}, \quad \zeta \in \mathbb{R}, \ 0 \leq \beta < 1, \ \varepsilon \geq 0.$$

While $\zeta = 0$ is a geometrically stable equilibrium for $\varepsilon = 0$ (nominal system), it is easy to show that it is an unstable equilibrium for any $\varepsilon > 0$ (perturbed system). However, the state ζ is found to be ultimately bounded.

In view of Lemma 1, to prove ultimate boundedness or asymptotic stability of the perturbed continuous-time system (16), it is necessary to check whether the perturbation induced by $\varepsilon \tilde{g}(x(t), x_k, t)$ on the nominal discrete-time system (5) satisfies the hypothesis of the lemma. As shown in the following Theorem 2 (point 1), ultimate boundedness of the error is guaranteed for sufficiently small ε. As for asymptotic stability, if the system equation (16) can be explicitly integrated, testing condition (17) becomes simple and possibly straightforward. If this is not the case, one may still apply point 2 of Theorem 2, that gives sufficient conditions on $\tilde{g}(x(t), x_k, t)$ under which, condition (17) is satisfied. Both types of perturbations (i.e., integrable and non-integrable) will be illustrated by means of the case study analyzed in the next Section.

Theorem 2. *Under Assumptions* 1 *and* 2, *and for sufficiently small ε:*

(1) *The state of the perturbed system* (16) *is ultimately bounded.*

(2) *For $t \in I_{k+1}$, denote by $z(t)$ the state of the perturbed system* (16) *and by $w(t)$ be the state of the nominal system* (3), *both initialized at z_k, and let $\chi(t) = z(t) - w(t)$. If the additional condition*

$$|\tilde{g}(z, z_k, t)| \leq \bar{\eta} |\chi| + \bar{\psi} |z_k|^s,$$

$$t \in I_{k+1}, \ \bar{\eta} \geq 0, \ \bar{\psi} > 0, \ s \geq 1, \tag{18}$$

holds, the origin is locally a uniformly asymptotically stable equilibrium point of the perturbed system (16). *In particular, if $s = 1$ the asymptotic stability of the origin becomes global.*

(3) *If both conditions* (7) *and* (18) *are satisfied, the rate of local convergence of the perturbed system* (16) *to the origin is exponential. In particular, if $r = 1$ in Eq.* (7) *and $s = 1$ in Eq.* (18) *the origin is globally exponentially stable.*

Proof.

(1) For $t \in I_{k+1}$, the state of the perturbed system is

$$z(t) = z_k + \int_{t_k}^{t} (\tilde{f}(z(\tau), z_k, \tau) + \varepsilon \tilde{g}(z(\tau), z_k, \tau))\, d\tau$$

with $z(t) \in B_x$ for sufficiently small ε, while

$$w(t) = z_k + \int_{t_k}^{t} \tilde{f}(w(\tau), z_k, \tau)\, d\tau$$

is the state of the nominal system initialized at z_k. Defining

$$\chi(t) = z(t) - w(t) = \int_{t_k}^{t} (\tilde{f}(z, z_k, \tau) - \tilde{f}(w, z_k, \tau) + \varepsilon \tilde{g}(z, z_k, \tau))\, d\tau,$$

we get

$$|\chi(t)| \le \int_{t_k}^{t} \lambda |\chi(\tau)|\, d\tau + \varepsilon \int_{t_k}^{t} (\eta |z(\tau)| + |\tilde{g}(0, z_k, \tau)|)\, d\tau, \tag{19}$$

having used Assumption 1 (point *b*) and the fact that $|\tilde{g}(z, z_k, t)| \le \eta |z| + |\tilde{g}(0, z_k, \tau)|$ (a consequence of Assumption 2).

Since $|\tilde{g}(z, z_k, t)|$ is piecewise-continuous in t in the closed interval I_{k+1}, $|\tilde{g}(0, z_k, t)|$ can be bounded as

$$|\tilde{g}(0, z_k, t)| \le \psi(|z_k|), \quad z_k \in B_x,\ t \in I_{k+1} \tag{20}$$

with $\psi(|x_k|)$ continuous in x_k. Using $z(t) = \chi(t) + w(t)$ and Eq. (20) in Eq. (19) we obtain

$$|\chi(t)| \le (\lambda + \varepsilon\eta) \int_{t_k}^{t} |\chi(\tau)|\, d\tau + \varepsilon\eta \int_{t_k}^{t} |w(\tau)|\, d\tau + \varepsilon T_M \psi(|z_k|).$$

Now, using Eq. (10) we get

$$|\chi(t)| \le (\lambda + \varepsilon\eta) \int_{t_k}^{t} |\chi(\tau)|\, d\tau + \varepsilon T_M [\psi(|z_k|) + \eta(|z_k| + T_M \mu(|z_k|))\, e^{\lambda T_M}], \tag{21}$$

and finally applying the Gronwall inequality, we obtain

$$|\chi(t)| \le \varepsilon T_M [\psi(|z_k|) + \eta(|z_k| + T_M \mu(|z_k|))\, e^{\lambda T_M}]\, e^{(\lambda + \varepsilon\eta) T_M},$$

which shows that, for sufficiently small ε, the perturbation $\chi(t_{k+1})$ on the discrete-time system is arbitrarily bounded. Therefore, Lemma 1 (in particular, point 1) can be invoked to conclude that the state of the perturbed discrete-time system is ultimately bounded. Owing to the continuous dependence of the continuous-time system (16) on the initial condition z_k, its state is also ultimately bounded.

(2) If the additional condition (18) holds, in place of Eq. (21) one obtains

$$|\chi(t)| \le (\lambda + \varepsilon\bar{\eta}) \int_{t_k}^{t} |\chi(\tau)|\, d\tau + \varepsilon T_M \bar{\psi} |z_k|^s$$

and using the Gronwall inequality

$$|\chi(t)| \le \varepsilon T_M \bar{\psi} |z_k|^s\, e^{(\lambda + \varepsilon\bar{\eta}) T_M}, \quad s \ge 1.$$

This expression directly yields, for the perturbation $\chi(t_{k+1})$ on the discrete-time system, an estimate in form (17). Therefore, Lemma 1 indicates that for sufficiently small ε the geometric stability is locally preserved for the discrete-time system. Through Theorem 1 (point 1), the origin of the continuous-time system (16) is uniformly asymptotically stable in a local sense. If $s = 1$ in Eq. (18), the geometric stability is globally preserved, and thus the asymptotic stability becomes global.

(3) Similar to the above point, the proof of this fact is immediately obtained combining the thesis of Theorem 1 with the thesis of Lemma 1. □

Remark 5. Point 1 of Theorem 2 covers the case of persistent perturbations, whereas only non-persistent perturbations can satisfy Eq. (18) of point 2.

So far, we have discussed the features of the iterative state steering approach from a general point of view. In order to apply the proposed technique to a specific control system, one must be able to compute an appropriate steering control $\alpha(x, x_k, t)$. To be precise, α should satisfy Assumption 1 (essentially, continuity in x_k and contraction) in order to obtain asymptotic stability. If exponential convergence is desired, the hypothesis of point 2 of Theorem 1 must also be met (Hölder-continuity in x_k). Robustness with respect to perturbations can then be analyzed using Lemma 1 and/or Theorem 2.

In the next section, we consider a class of non-linear systems for which a suitable steering control can be computed with reasonable effort, i.e., systems that can be put in chained form. For illustration, we also work out a case study for an example of this class.

4. Application to chained-form systems

Consider the driftless control system

$$\begin{aligned} \dot{x}_1 &= u_1, \\ \dot{x}_2 &= u_2, \\ \dot{x}_3 &= x_2 u_1, \\ &\vdots \\ \dot{x}_n &= x_{n-1} u_1 \end{aligned} \tag{22}$$

with 2 inputs and n states. System (22) is called a $(2, n)$ *chained form* and is readily verified to be controllable via the Lie algebra rank condition.

Murray (1993) has established necessary and sufficient conditions for converting a generic two-input driftless system

$$\dot{q} = f_1(q)v_1 + f_2(q)v_2$$

into chained form by means of an input transformation $v = \alpha(q)u$ and a change of coordinates $x = \beta(q)$. In particular, controllable driftless systems with two inputs and $n \leq 4$ state variables can be always put in chained form.

By using the necessary conditions due to Brockett (1983), it can be shown that system (22) cannot be stabilized at a point by using continuous time-invariant feedback. Time-varying (Samson, 1995) and/or discontinuous (Sørdalen & Egeland, 1995) feedback controllers have been proposed, but their robustness has not been analyzed so far. The iterative state steering technique represents a natural approach for the design and the analysis of robust control laws, the essential element being a suitable steering control to be applied iteratively.

The design of open-loop control laws for chained systems is relatively simple. Among the available techniques, we cite sinusoidal steering (Murray, 1993) and piecewise-constant inputs (Monaco & Normand-Cyrot, 1992). Hereafter, we show how to design a piecewise-continuous open-loop control satisfying Assumption 1 as well as the hypothesis of Theorem 1.

4.1. The (2,3) chained form

Consider the (2,3) chained form

$$\begin{aligned} \dot{x}_1 &= u_1, \\ \dot{x}_2 &= u_2, \\ \dot{x}_3 &= x_2 u_1 \end{aligned} \tag{23}$$

with $x(t_0) = x_0 = (x_{10}, x_{20}, x_{30})$ and $t_0 = 0$. We wish to design an open-loop control law with steering interval $T = 1$, whose iterative application guarantees robust stabilization of the origin; in particular, an exact ($\beta = 0$) steering control can be computed in this case, although this is not required by the iterative steering paradigm.

Let

$$u_1 = \begin{cases} -c_1 - 2x_{10}, & t \in [0,1/2), \\ c_1, & t \in [1/2,1), \end{cases} \tag{24}$$

$$u_2 = \begin{cases} 0, & t \in [0,1/2), \\ c_2 + c_3(t - \frac{1}{2}), & t \in [1/2,1) \end{cases} \tag{25}$$

with c_1, c_2, c_3 constant values to be determined as a function of the initial conditions. In particular, it must be $u_1 = u_2 = 0$ for $x_0 = 0$. By forward integration, it is easily verified that $x_1(1) = 0$ always, while imposing $x_2(1) = 0$ and $x_3(1) = 0$ one obtains, respectively,

$$c_2 = -\left(2x_{20} + \frac{c_3}{4}\right) \tag{26}$$

and

$$c_3 = \frac{96}{c_1}\left(x_{30} - x_{10}x_{20} - \frac{x_{20}c_1}{4}\right). \tag{27}$$

At this point, letting

$$c_1(x_0) = |x_0|^{1/(1+\nu)}, \quad \nu > 0, \tag{28}$$

one obtains that c_1 is a continuous function at zero, with $c_1(0) = 0$, and guarantees the same properties for c_2 and c_3 as given by Eqs. (26) and (27), respectively. In particular, c_1 is Hölder-continuous of order $1/(1+\nu)$, while c_2 and c_3 are Hölder-continuous of order $\nu/(1+\nu)$.

Substituting the expressions of c_1, c_2 and c_3 (with x_k in place of x_0 during the $(k+1)$th iteration) in the steering control (24) and (25), and the latter in the system dynamics (23), it is easy to verify that the closed-loop dynamics within each iteration satisfies Assumption 1 and condition (7). To be precise, function $\mu(|x_k|)$ is found to be Hölder-continuous of order $1/(1+\nu)$ if $\nu \geq 1$, of order $\nu/(1+\nu)$ otherwise. Therefore, point 2 of Theorem 1 guarantees that the (2,3) chained form is uniformly asymptotically stable under the iterative application of the above steering control, with exponential convergence.[2] Moreover, Theorem 2 indicates that small non-persistent perturbations satisfying condition (18) are rejected, while persistent perturbations give rise to limited errors.

Using the same approach, it is very simple to work out extensions of the above iterative steering technique for the general case of $(2, n)$ chained-form systems.

In the remainder of this section, we apply the proposed method in order to obtain robust stabilization of a wheeled mobile cart. This system, which can be converted in (2,3) chained form, will provide an example of physically motivated chained form perturbations.

4.2. Case study: Unicycle

Many types of wheeled mobile robots with multiple wheels have a kinematic model equivalent to a *unicycle* (see Fig. 1A). The generalized coordinates are $q = (p_x, p_y, \theta)$, where (p_x, p_y) are the Cartesian

[2] The resulting feedback controller is time-varying (in particular, piecewise-continuous in t), consistently with the indication of Brockett's theorem on smooth stabilizability, and Hölder-continuous with respect to the state variables. Note also that a feedback law continuous in t could easily be obtained by choosing a different open-loop control law.

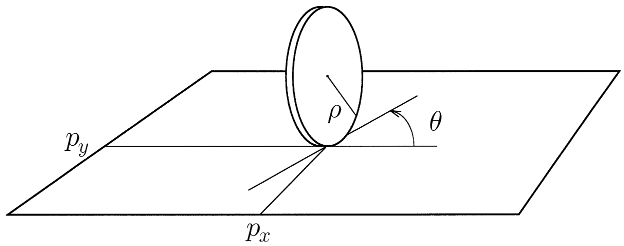

Fig. 1. Generalized coordinates for the unicycle.

coordinates of the contact point and θ is the orientation of the vehicle with respect to the horizontal axis.

The *pure rolling* non-holonomic constraint is expressed as

$$\dot{p}_x \sin\theta - \dot{p}_y \cos\theta = [\sin\theta \quad -\cos\theta \quad 0]\dot{q} = 0,$$

which can be solved for the generalized velocities as

$$\begin{bmatrix} \dot{p}_x \\ \dot{p}_y \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} \rho v_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} v_2, \tag{29}$$

with ρ being the wheel radius, v_1 the *driving* and v_2 the *steering* velocity (respectively, the wheel angular velocity about its horizontal and vertical axis).

System (29) may be put in (2,3) chained form via the change of coordinates

$$x_1 = \theta,$$

$$x_2 = p_x \cos\theta + p_y \sin\theta,$$

$$x_3 = p_x \sin\theta - p_y \cos\theta,$$

together with the input transformation

$$v = \begin{bmatrix} x_3/\rho & 1/\rho \\ 1 & 0 \end{bmatrix} u.$$

The above input and state transformation are globally defined.

To test the robustness of our stabilization strategy, we have introduced a perturbation on the wheel radius, whose true value is assumed to be $\rho + \Delta\rho$. As a consequence, (2,3) chained form is found to be perturbed[3] as

$$\dot{z}_1 = u_1,$$

$$\dot{z}_2 = u_2 + \frac{\Delta\rho}{\rho}(u_2 + z_3 u_1), \tag{30}$$

$$\dot{z}_3 = z_2 u_1.$$

[3] For the sake of clarity, we use here the same symbols of Section 3. Hence, x and z denote, respectively, the state of the nominal and of the perturbed system.

When the steering control of Section 4.1 is iteratively applied to the perturbed chained form, the closed-loop perturbation is represented by the following vector field:

$$\varepsilon\tilde{g} = \varepsilon\tilde{g}_1 + \varepsilon\tilde{g}_2 = \frac{\Delta\rho}{\rho}\begin{pmatrix} 0 \\ u_2 \\ 0 \end{pmatrix} + \frac{\Delta\rho}{\rho}\begin{pmatrix} 0 \\ z_3 u_1 \\ 0 \end{pmatrix}$$

with u_1, u_2 given by Eqs. (24) and (25). Note that $\tilde{g}$ satisfies Assumption 2.

It is convenient to analyze first the effect on the nominal system of the perturbation term $\varepsilon\tilde{g}_1$ alone. Since the integral of the second control input (25) between 0 and $T = 1$ is $-x_{20}$ by construction, we have

$$\int_0^1 \frac{\Delta\rho}{\rho} u_2(t)\,\mathrm{d}t = -\frac{\Delta\rho}{\rho} x_{20}.$$

Using this fact and integrating the third equation of the nominal system, it is easy to verify that the effect of perturbation $\varepsilon\tilde{g}_1$ on the discrete-time system satisfies condition (17) with $s = 1$. Hence, global asymptotic stability of the continuous-time system is preserved for sufficiently small $\Delta\rho$.

We now redefine the nominal system as the original unperturbed system (i.e., the (2,3) chained form) plus the first perturbation term $\varepsilon\tilde{g}_1$, and analyze the effect of the second perturbation term $\varepsilon\tilde{g}_2$ by applying Theorem 2 to the new nominal system. Note that the latter satisfies both Assumption 1 and condition (7).

For $t \in I_{k+1}$, denote by $z(t)$ the state of system (30) and by $w(t)$ the state of the new nominal system, both initialized at z_k, and let $\chi(t) = z(t) - w(t)$. By integrating the third equation of the nominal system, one readily finds the following estimate

$$|w_3(t)| \le \sigma|z_k|, \quad \sigma > 0,$$

and thus

$$|\tilde{g}(z, z_k, t)| = |z_3(t)u_1(t)|$$
$$= |\chi_3(t)u_1(t) + w_3(t)u_1(t)| \le \bar{\eta}|\chi(t)| + \bar{\psi}|z_k|,$$

where the expressions of $\bar{\eta}$ and $\bar{\psi}$ can be computed from Eqs. (24) and (25). The perturbation term $\varepsilon\tilde{g}_2$ is thus non-persistent and, in particular, satisfies condition (18).

Wrapping up, the iterative controller of Section 4.1 designed on the original unperturbed system (i.e., for $\Delta\rho = 0$) guarantees robust asymptotic stability, with exponential convergence rate, for sufficiently small $\Delta\rho$. Moreover, this property holds globally, since condition (18) is verified with $s = 1$.

The simulation results reported in Figs. 2A and 3A are in accordance with this prediction. The robot must perform a *parallel parking* maneuver, moving from (0,2,0) to (0,0,0). The control law is given by Eqs. (24) and (25), with $v = 1$ in Eq. (28). A 20% perturbation on the nominal

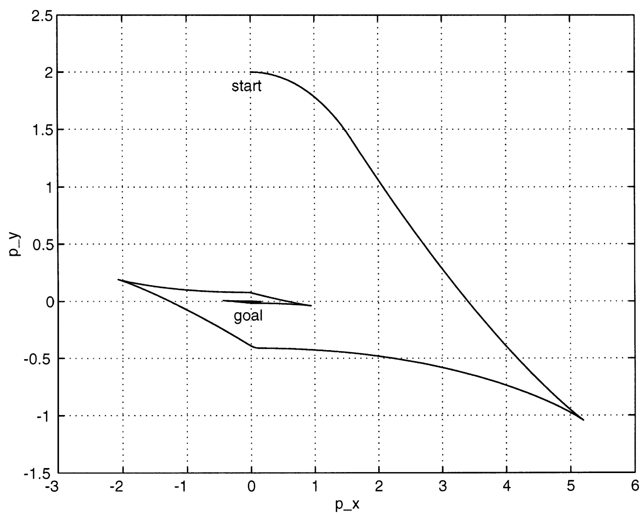

Fig. 2. Cartesian motion of the unicycle.

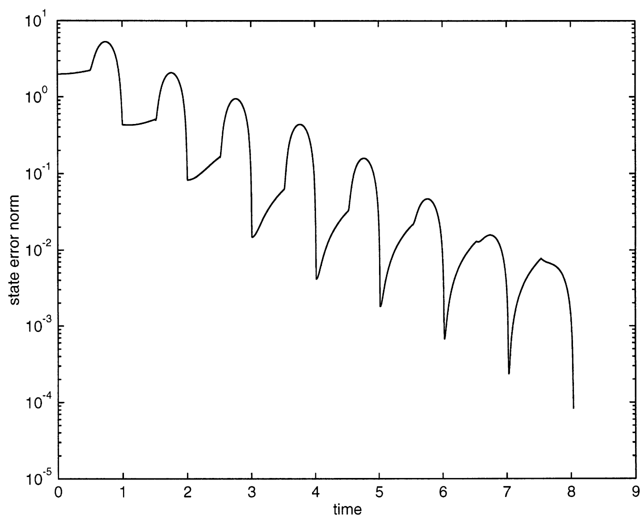

Fig. 3. Time evolution of the state error Euclidean norm.

wheel radius $\rho = 1$ was included in simulation. The cartesian motion of the unicycle is shown in Fig. 2a. Fig. 3A is a logarithmic plot of the error euclidean norm after eight iterations; note the exponential convergence rate.

We conclude this case study by pointing out that different trajectories can be obtained by choosing different steering controls in place of (24) and (25). For example, it is possible to compute a sinusoidal steering control that meets our requirements. The essential point here is that, as long as the conditions stated in Theorems 1 and 2 are satisfied, the iterated application of the control law will yield robust asymptotic stability, with exponential convergence rate. The possibility of 'shaping' the system trajectory while guaranteeing at the same time robust stabilization is an advantage of our method.

5. Conclusions

We have presented an approach for the robust stabilization of non-linear systems. The essential tool is a control law α that steers the system closer to the desired equilibrium point (assumed to be the origin) in a finite time. Whenever such a control is computable, its iterated application (i.e., from the state x_k attained at the end of the previous iteration) yields exponential convergence to the origin, provided that α is Hölder-continuous with respect to x_k. The resulting stabilization scheme can reject a class of small non-persistent perturbations, while small persistent perturbations induce limited errors.

For illustration, we have applied the proposed method to a particular class of controllable non-linear systems, i.e., chained forms. These are a family of driftless systems that cannot be stabilized by smooth time-invariant feedback, and represent a fairly canonical form for mechanical systems subject to non-holonomic constraints. In particular, we have presented simulation results for a wheeled mobile robot with the kinematics of a unicycle. We mention that iterative state steering can also provide stabilization of non-holonomic systems that do not admit a chained-form transformation, such as the general trailer system (Vendittelli & Oriolo, 2000).

It appears that several other difficult control problems can be successfully addressed by using this approach. In particular, the proposed strategy is proved to be effective in the control of underactuated mechanical systems, such as the underactuated 2R manipulator (De Luca, Mattone & Oriolo, 2000) and the satellite with a failed control (Lucibello & Oriolo, 1998).

Appendix

By means of an example, we show below that the solution $\varphi(t, x_0, t_0)$ of Eq. (3) may not satisfy the semigroup condition

$$\varphi(t, x_0, t_0) = \varphi(t, \varphi(\tau, x_0, t_0), \tau), \quad t \geq \tau. \tag{A.1}$$

Therefore, the controlled system (3) does not satisfy the definition of *dynamical system* as given (e.g., Hale, 1980; Hahn, 1967).

Consider the simple system

$$\dot{x}(t) = u(t), \quad x(t), u(t) \in \mathbb{R}, \quad x(t_0) = x_0,$$

and let

$$u(t) = -\frac{x_k}{2}, \quad t \in [t_k, t_{k+1}),$$

where x_k is the state at time t_k, being $t_{k+1} - t_k = 1$, $\forall k$. With this choice, the state is steered halfway to zero at each iteration.

Letting $x_0 = 1$, $t_0 = 0$, we have

$$\varphi(0.5, x_0, t_0) = \tfrac{3}{4}, \quad \varphi(1, x_0, t_0) = \tfrac{1}{2}.$$

On the other hand,

$$\varphi(1, \varphi(0.5, x_0, t_0), 0.5) = \varphi(1, \tfrac{3}{4}, 0.5) = \tfrac{9}{16}$$

and, thus,

$$\varphi(1, x_0, t_0) \neq \varphi(1, \varphi(0.5, x_0, t_0), 0.5),$$

showing that the solution does not satisfy the semigroup condition (A.1).

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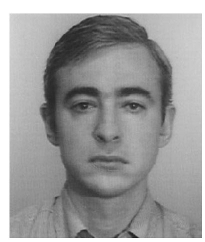

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