2001 IEEE/ASME International Conference on
Advanced Intelligent Mechatronics Proceedings
8—12 July 2001 • Como, Italy

Rest-to-Rest Motion of a Two-link Robot with a Flexible Forearm

Alessandro De Luca   Giandomenico Di Giovanni

Università degli Studi di Roma "La Sapienza"
Dipartimento di Informatica e Sistemistica
Via Eudossiana 18, 00184 Roma, Italy
deluca@dis.uniroma1.it
http://labrob.ing.uniroma1.it

Abstract— **We consider the problem of finding the torque commands that provide rest-to-rest motion in a given time for the FLEXARM, a two-link planar manipulator with a flexible forearm and nonlinear dynamics. The basic idea is to design a set of two outputs with respect to which the system has no zero dynamics. Planning smooth interpolating trajectories for these outputs imposes a unique rest-to-rest motion to the whole robot, with bounded link deformations. The nominal rest-to-rest torque is obtained by standard inverse dynamics computation. In the multi-input nonlinear case, this approach requires in general the use of a dynamic linearizing extension. Numerical results are presented and possible extensions discussed.**

I. Introduction

Steering a robot manipulator in prescribed time between two given configurations is a basic problem in robotics. In the presence of link flexibility, typically encountered in long reach and slender/lightweight robot arms [1], the problem is very critical since large and simultaneous motion of the links induce oscillations that persist beyond the nominal final completion time. When fast and precise positioning is the highest priority of a robotic task, the vanishing of link vibrations due to damping injected by a feedback controller [2] and/or inherent in the structure [3] may not be a satisfactory solution.

In order to cope with vibrational behavior of manipulators with flexible links, it is necessary to carefully model the distributed arm flexibility [4] and then to utilize the complete robot dynamics in the design of feedforward commands and feedback control laws [5]. This is particularly true when trying to obtain a rest-to-rest motion in given time. Unfortunately, a general solution technique to this problem is not yet available in the case of multi-link flexible manipulators.

For one-link flexible arms, characterized by linear dynamics, two common model-based techniques that generate the open-loop torque command for rest-to-rest maneuvers are input shaping [6] and inverse dynamics trajectory design [7]. Input shaping filters the step input reference command according to the characteristic frequencies of the system. Being based on the properties of the impulse response, this technique is intrinsically linear. Inverse dynamics trajectory design was pioneered by Bayo in the frequency domain [7] and revisited later in the time domain in [8]. It consists in the stable input-output inversion of an end-effector trajectory so as to achieve the desired reconfiguration with bounded link deformations. This procedure has been extended to multi-link flexible manipulators based on iterative algorithms [9], [10], [11] and, more in general, to nonlinear systems with non-minimum phase zero dynamics through the numerical solution of a two-point boundary value problem [12]. However, the cosidered problem of rest-to-rest motion in given time is only partially solved: since the computed nominal torques are non-causal in time, extending both before start and after completion of the end-effector trajectory, the definition of a motion interval at end of which the robot is at rest is related in accuracy to the finite window in time (or in the frequency domain) used in the implementation of the method.

In [13] and [14], a different technique has been proposed for a general linear model of a one-link flexible arm with an arbitrary number of deformation modes. In order to generate a rest-to-rest motion, a system output is designed having maximum relative degree, i.e., such that no zeros appear in the trasfer function from the input torque to the defined output. The problem is then solved by fitting to this output a smooth interpolating polynomial between the start and final rest configurations and using inverse dynamics.

In this paper, we present an extension of the result in [14] to a multi-link flexible manipulator with nonlinear dynamics. This generalization is feasible since the required computations rely only on time and state-space concepts. In particular, zero dynamics is well defined also for nonlinear systems [15]. Therefore, one should design auxiliary outputs for a flexible manipulator such that the system is invertible and has no associated zero dynamics. The solution of this problem relies in general on a dynamic feedback linearization scheme, which has been already used in robotics also for manipulators with elasticity concentrated at the joints [16].

The approach is detailed here with reference to the laboratory prototype FLEXARM, a two-link planar manipulator with a flexible forearm. Numerical results show the effectiveness of the method in the case of one flexible mode. Finally, we discuss possible extensions and feedback applications of this dynamic linearization approach.

II. Dynamic modeling of the FLEXARM

Consider the two-link manipulator FLEXARM in Fig. 1. Both the first rigid link and the flexible forearm move in the horizontal plane. By its construction, the second link can only bend in the plane of rigid motion and is stiff with respect to axial forces and torsion. The robot is driven by two direct-drive DC motors located at the joints. For a detailed description of the mechanical design, sensors and actuators, and interface electronics, see [17].

A nonlinear dynamic model of this two-link manipulator has been derived following a Lagrangian approach. Small deformations are assumed for the forearm, leading to a linear dynamics of the flexible part. The main nonlinearities in the model arise then from the rigid body interaction between the two links and from the interaction of rigid and flexible dynamics.

Let $\theta_1(t)$ be the angle of the first link of length ℓ_1 and inertia J_1 w.r.t. the first joint axis. The inertia of the first actuator is J_{01}. The actuator driving the second link has mass m_{02} and inertia J_{02}. To compute its deformation modes, the flexible forearm link is modeled as an Euler-Bernoulli beam of length ℓ_2, uniform density ρ, Young modulus E, and inertia of the cross section I. Thus, the second link has mass $m_2 = \rho\ell_2$ and equivalent rigid inertia w.r.t. the second joint axis $J_2 = m_2\ell_2^2/3$. Let $\theta_2(t)$ be the angle of a line pointing from the second joint axis to the instantaneous center of mass of the flexible forearm (pinned angle) w.r.t. the rotated frame associated to the first link. The transversal bending deflection $w(x,t)$ at a point $x \in [0, \ell_2]$ along the second link is described w.r.t. this line (pinned frame). The second link may carry a tip payload of mass m_p and inertia J_p.

The deformation eigenfunctions and eigenfrequencies of the flexible forearm are computed according to [18], [19], by including J_{02}, m_p, and J_p in the dynamic boundary conditions associated to the partial differential equation for $w(x,t)$. By separation in space and time, using a finite number n_e of deformation mode shapes $\phi_i(x)$ with associated deformation coordinates $\delta_i(t)$,

$$w(x,t) = \sum_{i=1}^{n_e} \phi_i(x)\delta_i(t),$$

the free evolution of the second link (when the first link is at rest) is characterized by the solutions, for $i = 1, \ldots, n_e$, to

$$\begin{aligned} EI\phi_i''''(x) - \rho\omega_i^2\phi_i(x) &= 0 \\ \ddot{\delta}_i(t) + \omega_i^2\delta_i(t) &= 0, \end{aligned}$$

being ω_i the angular eigenfrequencies of the flexible arm, with spatial boundary conditions

$$\begin{aligned} \phi_i(0) &= 0 \\ EI\phi_i''(0) + \omega_i^2 J_{02}\,\phi_i'(0) &= 0 \\ EI\phi_i''(\ell_2) - \omega_i^2 J_p\,\phi_i'(\ell_2) &= 0 \\ EI\phi_i'''(\ell_2) + \omega_i^2 m_p\,\phi_i(\ell_2) &= 0, \end{aligned}$$

where a prime denotes spatial derivative w.r.t. x. The general solutions are in the form

$$\begin{aligned} \phi_i(x) &= A_i\sin(\beta_i x) + B_i\cos(\beta_i x) \\ &\quad + C_i\sinh(\beta_i x) + D_i\cosh(\beta_i x), \end{aligned} \tag{1}$$

with $\beta_i^4 = \rho\omega_i^2/EI$ and $\beta_1, \ldots, \beta_{n_e}$ being the first n_e roots of the following characteristic equation

$$\begin{aligned} &(c\,sh - s\,ch) - \frac{2m_p}{\rho}\beta_i\,s\,sh - \frac{2J_p}{\rho}\beta_i^3\,c\,ch \\ &-\frac{J_{02}}{\rho}\beta_i^3(1 + c\,ch) - \frac{m_p}{\rho^2}\beta_i^4(J_{02} + J_p)(c\,sh - s\,ch) \\ &+\frac{J_{02}J_p}{\rho^2}\beta_i^6(c\,sh + s\,ch) - \frac{J_{02}J_p m_p}{\rho^3}\beta_i^7(1 - c\,ch) = 0, \end{aligned}$$

where $s = \sin(\beta_i\ell_2)$, $c = \cos(\beta_i\ell_2)$, $sh = \sinh(\beta_i\ell_2)$, and $ch = \cosh(\beta_i\ell_2)$. The coefficients $(A_i, \ldots, D_i)$ in eq. (1) are determined, up to a scaling factor which is chosen through normalization, from the imposed boundary conditions.

Starting from this analysis, the Lagrangian dynamics of the FLEXARM is derived in the standard way as

$$B(q)\ddot{q} + n(q,\dot{q}) + Kq = G\tau, \tag{2}$$

where $q = (\theta, \delta) = (\theta_1, \theta_2, \delta_1, \ldots, \delta_{n_e}) \in I\!R^{2+n_e}$, with symmetric inertia matrix $B > 0$, nonlinear Coriolis and centrifugal terms n, elasticity matrix $K > 0$, and input matrix G which transforms the actuating torques $\tau = (\tau_1, \tau_2)$ into generalized forces performing work on q.

In order to express the single dynamic terms in eq. (2), the following coefficients are defined [1]:

$$J_{1t} = J_{01} + J_1 + \left[m_{02} + m_2 + m_p\right]\ell_1^2$$

[1] Note that there was a wrong index in the definition of h_i in [17].

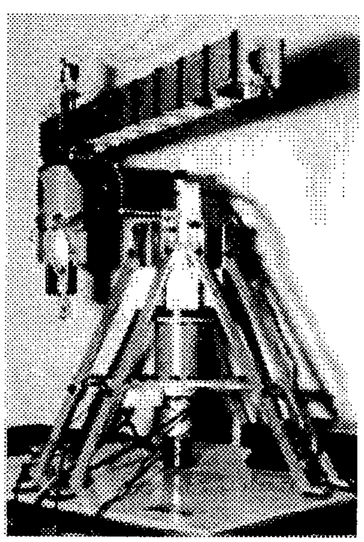

Fig. 1. The FLEXARM robot

$$
\begin{aligned}
J_{2t} &= J_{02} + J_2 + J_p + m_p \ell_2^2 \\
v_i &= \rho \int_0^{\ell_2} \phi_i(x)\, dx, \qquad i = 1, \ldots, n_e \\
h_i &= \Big[v_i + m_p \phi_i(\ell_2) \Big] \ell_1, \qquad i = 1, \ldots, n_e \\
h_{n_e+1} &= \Big[m_2 \frac{\ell_2}{2} + m_p \ell_2 \Big] \ell_1 .
\end{aligned}
$$

Since the eigenfunctions $\phi_i(x)$ automatically satisfy proper orthonormality conditions, relevant simplifications arise in the terms of the dynamic model. In particular, by neglecting in the kinetic energy of the system terms which are quadratic in the deformation variables δ_i, the inertia matrix becomes

$$
B(q) = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \ldots & b_{1,n_e+2} \\ b_{12} & J_{2t} & 0 & \ldots & 0 \\ b_{13} & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ b_{1,n_e+2} & 0 & \ldots & 0 & 1 \end{bmatrix},
$$

with elements

$$
\begin{aligned}
b_{11} &= J_{1t} + J_{2t} + 2h_{n_e+1}\cos\theta_2 - 2\sin\theta_2 \sum_{i=1}^{n_e} h_i \delta_i \\
b_{12} &= J_{2t} + h_{n_e+1}\cos\theta_2 - \sin\theta_2 \sum_{i=1}^{n_e} h_i \delta_i \\
b_{13} &= h_1 \cos\theta_2 \\
&\vdots \\
b_{1,n_e+2} &= h_{n_e} \cos\theta_2 .
\end{aligned}
$$

The definition $b_\delta = [\, b_{13} \;\; \ldots \;\; b_{1,n_e+2}]^T$ will also be used. The components of the Coriolis and centrifugal vector $n(q, \dot{q})$ are:

$$
\begin{aligned}
n_1 &= -\left(h_{n_e+1}\sin\theta_2 + \cos\theta_2 \sum_{i=1}^{n_e} h_i\delta_i \right) (2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) \\
&\quad - 2\sin\theta_2(\dot{\theta}_1 + \dot{\theta}_2) \sum_{i=1}^{n_e} h_i \dot{\delta}_i \\
n_2 &= \left(h_{n_e+1}\sin\theta_2 + \cos\theta_2 \sum_{i=1}^{n_e} h_i\delta_i \right) \dot{\theta}_1^2 \\
n_3 &= h_1 \sin\theta_2\, \dot{\theta}_1^2 \\
&\vdots \\
n_{n_e+2} &= h_{n_e} \sin\theta_2\, \dot{\theta}_1^2 .
\end{aligned}
$$

The elasticity matrix becomes

$$
K = \text{diag}\,\{0, 0, K_\delta\} = \text{diag}\,\{0, 0, \omega_1^2, \ldots, \omega_{n_e}^2\},
$$

while the input matrix takes the form

$$
G = \begin{bmatrix} I_{2\times 2} & \\ 0_{n_e\times 1} & \Phi'(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & \phi_1'(0) & \ldots & \phi_{n_e}'(0) \end{bmatrix}^T .
$$

A. Feedback transformation

Before proceeding, it is convenient to apply an invertible static state feedback in order to simplify the system equations. The dynamic model (2) can be rewriten in block form as

$$
\begin{bmatrix} B_{\theta\theta} & B_{\theta\delta} \\ B_{\theta\delta}^T & I \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{\delta} \end{bmatrix} + \begin{bmatrix} n_\theta \\ n_\delta \end{bmatrix} + \begin{bmatrix} 0 \\ K_\delta \delta \end{bmatrix} = \begin{bmatrix} \tau \\ \Phi'(0)\tau_2 \end{bmatrix},
$$

partitioned according to the dimensions of $\boldsymbol{\theta}$ and δ. Solving for $\ddot{\delta}$ from the second block of equations and substituting into the first yields

$$
\left(B_{\theta\theta} - B_{\theta\delta} B_{\theta\delta}^T \right) \ddot{\theta} + n_\theta - B_{\theta\delta}\,(n_\delta + K_\delta\delta) = \tau - B_{\theta\delta}\Phi'(0)\tau_2 .
$$

Here, the matrix multiplying $\ddot{\theta}$ is always nonsingular by the positive definiteness of the inertia matrix. One can define a global nonlinear feedback law for τ so that the equations of the joint variables θ are fully linearized and decoupled. Exploiting the internal structure of the block elements, it is straightforward to see that by choosing

$$
\begin{aligned}
\tau \;=\; & \begin{bmatrix} 1 & b_\delta^T \Phi'(0) \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} b_{11} - b_\delta^T b_\delta & b_{12} \\ b_{12} & J_{2t} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \right. \\
& \left. + \begin{bmatrix} n_1 - b_\delta^T\,(n_\delta + K_\delta \delta) \\ n_2 \end{bmatrix} \right), \qquad (3)
\end{aligned}
$$

where a_1 and a_2 are new acceleration inputs, we obtain an equivalent dynamic model of the FLEXARM in the form:

$$
\begin{aligned}
\ddot{\theta}_1 &= a_1 \\
\ddot{\theta}_2 &= a_2 \qquad\qquad (4) \\
\ddot{\delta} &= -b_\delta a_1 - (n_\delta + K_\delta\delta) + \Phi'(0)\,(b_{12}a_1 + J_{2t}a_2 + n_2) .
\end{aligned}
$$

III. Rest-to-rest motion of the FLEXARM

Consider a rest-to-rest motion task for the FLEXARM. The manipulator should be moved from an initial undeformed configuration $q_i = (\theta_i, 0)$ at $t_i = 0$ to a final undeformed configuration $q_f = (\theta_f, 0)$ at time $t_f = T$, with $\dot{q}(0) = \dot{q}(T) = 0$.

In order to solve this problem, extending the idea in [14], we look for a two-dimensional design output $y = (y_1, y_2)$ associated to which the system has no zero dynamics and is input-output invertible. This means that we should be able to differentiate w.r.t. time a specific number of times the output y until the available two-dimensional input finally appears in a nonsingular way. The addition of integrators on one of the two input channels may be needed at a given step so as to avoid subsequent differentiation of the relative input. This extension process builds up the state of a dynamic feedback compensator. If the total number of output derivatives performed before the input appears equals the number of states of the flexible robot plus the number of added compensator states, then the system has no zero dynamics and can be transformed into two independent chains of integrators from auxiliary inputs to the chosen design outputs. This approach is named dynamic feedback linearization [15] and is possibly required only for nonlinear systems with more than one input. We shall present its application to the FLEXARM using a single deflection mode and starting from eqs. (4).

A. Dynamic feedback linearization

When considering only $n_e = 1$ mode for the flexible forearm, eqs. (4) becomes

$$\begin{aligned}
\ddot{\theta}_1 &= a_1\\
\ddot{\theta}_2 &= a_2\\
\ddot{\delta}_1 &= -h_1\left(\cos\theta_2 a_1 + \sin\theta_2\dot{\theta}_1^2\right) - \omega_1^2\delta + \phi_1'(0)\Big[J_{2t}(a_1+a_2)\\
&\quad + h_2\left(\cos\theta_2 a_1 + \sin\theta_2\dot{\theta}_1^2\right) + h_1\delta_1\left(\cos\theta_2\dot{\theta}_1^2 - \sin\theta_2 a_1\right)\Big].
\end{aligned}$$

The expression of $\ddot{\delta}_1$ can be rewritten as

$$\ddot{\delta}_1 = -\omega_1^2\delta_1+\phi_1'(0)J_{2t}(a_1+a_2)+[\,\phi_1'(0)h_1\delta_1 \quad \gamma_1\,]\,R(\theta_2)\begin{bmatrix}\dot{\theta}_1^2\\ a_1\end{bmatrix},$$

having set

$$\gamma_1 = \phi_1'(0)J_{2t} - h_1, \qquad R(\theta_2) = \begin{bmatrix}\cos\theta_2 & -\sin\theta_2\\ \sin\theta_2 & \cos\theta_2\end{bmatrix}. \tag{5}$$

Following [14], we choose as candidate design output

$$y = \begin{bmatrix} y_1\\ y_2\end{bmatrix} = \begin{bmatrix}\theta_1\\ \theta_2 + c_1\delta_1\end{bmatrix}, \tag{6}$$

where c_1 is a coefficient yet to be defined. Differentiating eq. (6) once

$$\dot{y} = \begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2 + c_1\dot{\delta}_1\end{bmatrix}, \tag{7}$$

and twice, yields

$$\ddot{y} = \begin{bmatrix} a_1\\ a_2 + c_1\phi_1'(0)J_{2t}(a_1+a_2) - c_1\omega_1^2\delta_1\\ +[\,c_1\phi_1'(0)h_1\delta_1 \quad c_1\gamma_1\,]\,R(\theta_2)\begin{bmatrix}\dot{\theta}_1^2\\ a_1\end{bmatrix}\end{bmatrix}.$$

Both inputs a_1 and a_2 appear at this level, but the total number of output derivatives $(2+2=4)$ does not yet cover the state space of the flexible manipulator which has dimension $2(n_e+2)=6$. Therefore, in order to make the matrix weighting the inputs in $\ddot{y}$ singular, we can choose the free coefficient c_1 as

$$c_1 = -\frac{1}{\phi_1'(0)J_{2t}}, \tag{8}$$

so that a_2 disappears from the expression of $\ddot{y}_2$. The definition (8) is consistent with the one made in [14] for the linear case of a one-link flexible arm with $n_e = 1$. In order to proceed with output differentiation, we need then a dynamic extension on the first input channel (i.e., a_1). Since the robot model has a second-order dynamics, we can directly add two integrators with states denoted by ξ_1 and ξ_2

$$\begin{aligned} a_1 &= \xi_1, \quad & \dot{\xi}_1 &= \xi_2, \quad & \dot{\xi}_2 &= \alpha_1,\\ a_2 &= \alpha_2, \end{aligned} \tag{9}$$

where $\alpha = (\alpha_1, \alpha_2)$ is the new input. As a result of (8) and (9),

$$\ddot{y} = \begin{bmatrix}\xi_1\\ -\xi_1 - c_1\omega_1^2\delta_1 + [\,c_1\phi_1'(0)h_1\delta_1 \quad c_1\gamma_1\,]\,R(\theta_2)\begin{bmatrix}\dot{\theta}_1^2\\ \xi_1\end{bmatrix}\end{bmatrix}, \tag{10}$$

and the third derivative of the output is

$$y^{[3]} = \begin{bmatrix}\xi_2\\ -\xi_2 - c_1\omega_1^2\dot{\delta}_1 + [\,c_1\phi_1'(0)h_1\dot{\delta}_1 \quad 0\,]\,R(\theta_2)\begin{bmatrix}\dot{\theta}_1^2\\ \xi_1\end{bmatrix}\\ +[\,c_1\phi_1'(0)h_1\delta_1 \quad c_1\gamma_1\,]\,R(\theta_2)\begin{bmatrix}2\dot{\theta}_1\xi_1\\ \xi_2\end{bmatrix}\\ +\dot{\theta}_2\,[\,c_1\phi_1'(0)h_1\delta_1 \quad c_1\gamma_1\,]\,\frac{dR}{d\theta_2}\begin{bmatrix}\dot{\theta}_1^2\\ \xi_1\end{bmatrix}\end{bmatrix}. \tag{11}$$

The set of equations (6)–(7) and (10)–(11) define a complete state-space transformation from the original state $(\theta_1, \theta_2, \delta_1, \dot{\theta}_1, \dot{\theta}_2, \dot{\delta}_1)$ of the FLEXARM and the state (ξ_1, ξ_2) of the dynamic compensator to the coordinates $(y, \dot{y}, \ddot{y}, y^{[3]}) \in \mathbb{R}^8$.

By differentiating once more the output we obtain

$$y^{[4]} = A(\theta_2, \delta_1, \dot{\theta}_1, \xi_1)\alpha + f(\theta_2, \delta_1, \dot{\theta}_1, \dot{\theta}_2, \dot{\delta}_1, \xi_1, \xi_2), \tag{12}$$

where the expressions of the so-called decoupling matrix A and of vector f are given in the Appendix. The decoupling matrix is nonsingular provided that its element a_{22} does not vanish, i.e.,

$$\omega_1^2 + [\,(c_1\gamma_1 - \phi_1'(0)h_1) \quad -c_1\phi_1'(0)h_1\delta_1\,]\,R(\theta_2)\begin{bmatrix}\dot{\theta}_1^2\\ \xi_1\end{bmatrix} \neq 0. \tag{13}$$

Under this assumption, the control law defined by the feedback from the extended (robot+compensator) state

$$\alpha = A^{-1}(\theta_2, \delta_1, \dot{\theta}_1, \xi_1)\left(v - f(\theta_2, \delta_1, \dot{\theta}_1, \dot{\theta}_2, \dot{\delta}_1, \xi_1, \xi_2)\right) \tag{14}$$

transforms completely the extended dynamic system into a linear controllable one made by two independent chains of four input-output integrators from the auxiliary input $v = (v_1, v_2)$ to the output $y = (y_1, y_2)$, or

$$y^{[4]} = v. \tag{15}$$

The complete expression of the two-dimensional dynamic feedback linearizing compensator in terms of the original torque inputs to the FLEXARM is obtained by merging eqs. (3), (9), and (14).

B. Rest-to-rest trajectory generation

From the initial rest state of the FLEXARM at $t = 0$,

$$\begin{aligned} \theta_1(0) &= \theta_{1i}, \quad \theta_2(0) = \theta_{2i}, \quad \delta_1(0) = 0,\\ \dot{\theta}_1(0) &= \dot{\theta}_2(0) = \dot{\delta}_1(0) = 0, \end{aligned}$$

the desired rest state at $t = T$,

$$\begin{aligned}\theta_1(T) &= \theta_{1f}, \quad \theta_2(T) = \theta_{2f}, \quad \delta_1(T) = 0,\\ \dot\theta_1(T) &= \dot\theta_2(T) = \dot\delta_1(T) = 0,\end{aligned}$$

and from $\xi_1(0) = \xi_2(0) = \xi_1(T) = \xi_2(T) = 0$, by using eqs. (6)–(7) and (10)–(11) one can derive two sets of boundary conditions for the reference output trajectory $y_d(t) = (y_{1d}(t), y_{2d}(t))$ and its derivatives up to the third one. These values can be simply interpolated by a polynomial trajectory of (at least) 7-th degree (one polynomial for each output) defined for $t \in [0, T]$.

From eqs. (14)–(15), setting $v = y_d^{[4]}$, we have

$$\begin{aligned}\alpha_d = {} & A^{-1}(\theta_{2d}, \delta_{1d}, \dot\theta_{1d}, \xi_{1d}) \cdot \\ & \left(y_d^{[4]} - f(\theta_{2d}, \delta_{1d}, \dot\theta_{1d}, \dot\theta_{2d}, \dot\delta_{1d}, \xi_{1d}, \xi_{2d})\right)\end{aligned}$$

where desired values of the extended state are obtained by inverting the transformation given by eqs. (6)–(7) and (10)–(11), in which $y \equiv y_d(t)$ is used at each $t \in [0, T]$. This inversion can be performed almost in closed form. It can be shown that the only solution variable that needs special attention is θ_{2d} which is computed solving (at every t) the nonlinear equation

$$\begin{aligned}g(\theta_{2d}) = {} & \ddot y_{2d} + \omega_1^2 y_{2d} + \ddot y_{1d} - \omega_1^2 \theta_{2d} \\ & - \left[\ \phi_1'(0)h_1\,(y_{2d} - \theta_{2d}) \quad c_1\gamma_1\ \right] R(\theta_{2d}) \begin{bmatrix} \dot y_{1d}^2 \\ \ddot y_{1d} \end{bmatrix} = 0\end{aligned}$$

with an efficient Newton method. All remaining variables are then determined analytically.

The whole process is feasible provided that the regularity condition (13) holds along the reference output trajectory. Noting that $\dot\theta_{1d} = \dot y_{1d}$ and $\xi_{1d} = \ddot y_{1d}$, we can give the following lover bound on $a_{22,d} = a_{22}(\theta_{2d}, \delta_{1d}, \dot\theta_{1d}, \xi_{1d})$:

$$\begin{aligned}|a_{22,d}| &\geq \omega_1^2 - \left\| \begin{bmatrix} (c_1\gamma_1 - \phi_1'(0)h_1) \\ -c_1\phi_1'(0)h_1\delta_{1d} \end{bmatrix} \right\| \cdot \left\| R(\theta_{2d}) \begin{bmatrix} \dot\theta_{1d}^2 \\ \xi_{1d} \end{bmatrix} \right\| \\ &= \omega_1^2 - \left\| \begin{bmatrix} (c_1\gamma_1 - \phi_1'(0)h_1) \\ -c_1\phi_1'(0)h_1\delta_{1d} \end{bmatrix} \right\| \cdot \sqrt{\dot y_{1d}^4 + \ddot y_{1d}^2}.\end{aligned}$$

Therefore, the condition $|a_{22,d}| > 0$ can be enforced by accurate planning of the reference output trajectory so as to satisfy the positivity of the above right-hand side. Since the only available parameter left is usually the transfer time T, it is clear that by slowing down the motion transfer, the nominal deformation δ_{1d} will also be reduced and the regularity condition will be enforced.

After substitutions, the nominal rest-to-rest torques for the FLEXARM with $n_e = 1$ mode are given by

$$\begin{aligned}\tau_{1d} = {} & \left(b_{11,d} - b_{13,d}^2\right)\xi_{1d} + b_{12,d}\,\alpha_{2d} + n_{1,d} \\ & - b_{13,d}\left(n_{3,d} + \omega_1^2\delta_{1d}\right) \\ & + b_{13,d}\,\phi_1'(0)\left(b_{12,d}\,\xi_{1d} + J_{2t}\,\alpha_{2d} + n_{2,d}\right) \\ \tau_{2d} = {} & b_{12,d}\,\xi_{1,d} + J_{2t}\,\alpha_{2d} + n_{2,d},\end{aligned}$$

where the added subscript d means that all dynamic model quantities are evaluated along the nominal state trajectory. We finally note that by choosing 9th-order polynomial trajectories we can also give continuity to the torques at $t = 0$ and $t = T$.

IV. Numerical results

The FLEXARM is characterized by the following data:

$$\begin{aligned}J_{01} &= 16.2 \cdot 10^{-4}\ \mathrm{kg\ m^2} \\ \ell_1 &= 0.3\ \mathrm{m} \\ m_1 &\simeq J_1 \simeq 0 \\ m_{02} &= 3.118\ \mathrm{kg} \\ J_{02} &= 6.35 \cdot 10^{-4}\ \mathrm{kg\ m^2} \\ \ell_2 &= 0.7\ \mathrm{m} \\ m_2 &= 1.853\ \mathrm{kg} \\ J_2 &= 0.1483\ \mathrm{kg\ m^2} \\ EI &= 2.4507\ \mathrm{N\ m^2} \\ m_p &= J_p = 0,\end{aligned}$$

where the mass of the first link has been neglected in comparison to the mass of the second actuator carried at its end. The associated frequency of the first mode of the flexible forearm is $f_1 = 3.7631$ Hz.

We have considered the following rest-to-rest task:

$$\theta_{1i} = \theta_{2i} = 0, \qquad \theta_{1f} = \theta_{2f} = 90°, \qquad T = 2\ \mathrm{s}.$$

For each output component in eq. (6), an 11-th order polynomial, with zero symmetric boundary conditions on its derivatives up to the fifth one, has been selected as reference trajectory. This guarantees also boundary continuity of the first derivative of the rest-to-rest torques.

The results in Figs. 2–6 indicate a natural behavior, with bounded deformation in the assumed linearity domain and maximum torques within the actuators capabilities. In particular, Fig. 3 shows the evolution of two variables of the flexible forearm that are of interest for feedback control solutions: the clamped joint angle $\theta_{c2} = \theta_2 + \phi_1'(0)\delta_1$, which is the angular position that can be directly measured by an encoder mounted on the actuator at the link base, and the tip angle $y_{t2} = \theta_2 + (\phi_1(\ell_2)/\ell_2)\delta_1$, which is the angle between a line pointing at the forearm tip and the x-axis of the pinned frame. In the experimental setup, the angular deflection at the tip $y_{t2} - \theta_{c2}$ can be directly measured by an optical sensor (see [13]). In the first half of the motion the clamped angle leads over the second output reference trajectory and the tip lags behind, while the situation is reversed in the second half. The maximum transversal displacement at the forearm tip is about 12 cm.

V. Conclusions

A method has been presented for generating the nominal torques needed to perform a rest-to-rest motion task in given time with the FLEXARM, a two-link planar manipulator with a flexible forearm. Although the result is specific,

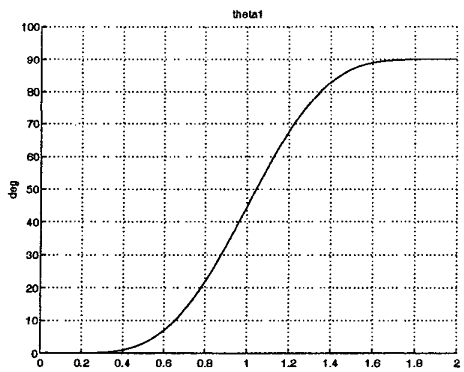

Fig. 2. Motion of first joint of FLEXARM

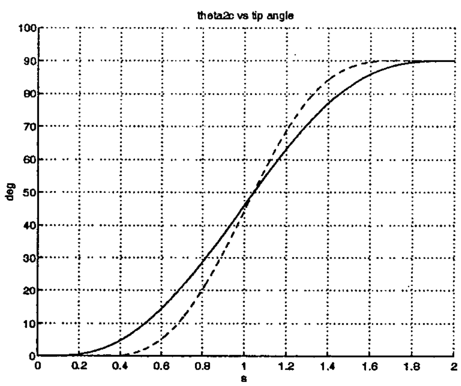

Fig. 3. Motion of clamped joint angle (—) and tip angle (- -) of the forearm

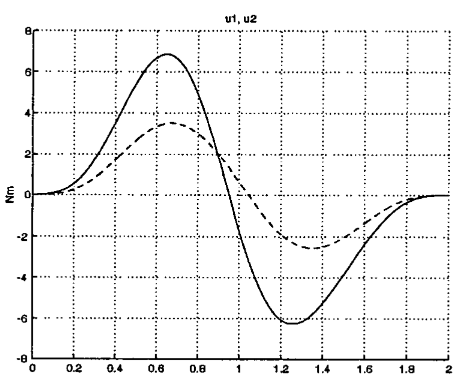

Fig. 4. Rest-to-rest torques for FLEXARM: τ_{1d} (—) and τ_{2d} (- -)

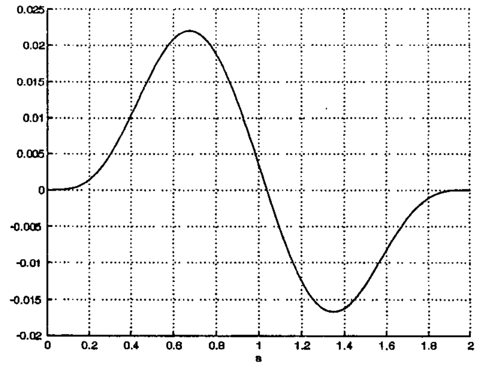

Fig. 5. Evolution of first flexible mode of the forearm

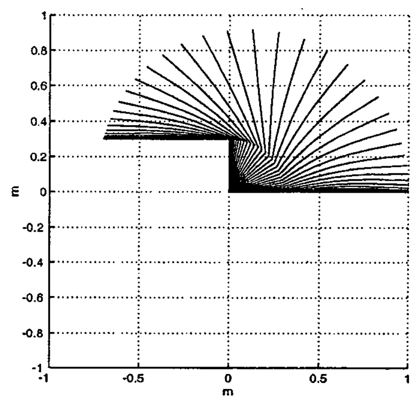

Fig. 6. Stroboscopic view of the FLEXARM motion

and limited to a single flexible mode, to the authors' knowledge this is the first available result in the literature solving the rest-to-rest motion problem for a multi-link flexible manipulator with nonlinear dynamics.

The method is based on the design of suitable outputs with no associated zero dynamics. When the input-output mapping is invertible, this is a sufficient condition for transforming the system into a linear controllable and input-output decoupled one by means of a nonlinear dynamic feedback compensator. This general result has been used here for motion planning of rest-to-rest maneuvers.

The computed nominal torques can also be incorporated as feedforward terms in a simple PD feedback controller, aimed at robustyfying the behavior in the presence of inaccurate information on the actual initial state and small disturbances. Only joint position and velocity measurements are needed, while the computed motion of the joints are used as references [13]. The resulting scheme, which follows the so-called nonlinear regulation paradigm for nonlinear systems, can be considered the counterpart for flexible manipulators of the widespread pre-computed torque plus joint PD feedback control of rigid robots.

The main current limitation of our approach is the consideration of only one deformation mode for the flexible forearm. We have applied this technique with satisfactory performance also to other simple nonlinear flexible structures finding the same obstruction. The use of a linear output function, which is the direct outgrow of the one designed for an arbitrary number of modes in a one-link flexible arm with linear dynamics [14], may indeed not be the right solution. More sophisticated choices should be explored in order to cover the case of multiple flexible links and/or multiple deformation modes.

Finally, experimental verification of the method is currently under way on the FLEXARM manipulator available at DIA, Università di Roma Tre, for rest-to-rest maneuvers of both the single flexible forearm alone (linear dynamics) and the full two-link flexible robot (nonlinear dynamics).

Acknowledgments

Work supported by MURST under MISTRAL project.

References

[1] W. J. Book, "Modeling, design, and control of flexible manipulator arms: A tutorial review," *29th IEEE Conf. on Decision and Control*, pp. 500–506, 1990.

[2] A. De Luca and B. Siciliano, "Regulation of flexible arms under gravity," *IEEE Trans. on Robotics and Automation*, vol. 9, no. 4, pp. 463–467, 1993.

[3] S. W. Kung and R. Singh, "Vibration analysis of beams with multiple constrained layer damping patches," *J. of Sound and Vibration*, vol. 212, no. 5; pp. 781–805, 1998.

[4] W. J. Book, "Recursive Lagrangian dynamics of flexible manipulator arms," *Int. J. of Robotics Research*, vol. 3, no. 3, pp. 87–101, 1984.

[5] A. De Luca and B. Siciliano, "Flexible links," in *Theory of Robot Control*, C. Canudas de Wit, B. Siciliano, G. Bastin (Eds.), pp. 219–261, Springer-Verlag, Berlin, 1996.

[6] N. C. Singer and W. P. Seering, "Preshaping command inputs to reduce system vibration," *ASME J. of Dynamic Systems, Measurements, and Control*, vol. 112, pp. 76–82, 1990.

[7] E. Bayo, "A finite-element approach to control the end-point motion of a single-link flexible robot," *J. of Robotic Systems*, vol. 4, no. 1, pp. 63–75, 1987.

[8] D.-S. Kwon and W. J. Book, "An inverse dynamic method yielding flexible manipulator state trajectories," *1990 American Control Conf.*, pp. 186–193, 1990.

[9] E. Bayo, M. A. Serna, P. Papadopoulus, and J. Stubbe, "Inverse dynamics and kinematics of multi-link elastic robots: An iterative frequency domain approach," *Int. J. of Robotics Research* vol. 8, no. 6, pp. 49–62, 1989.

[10] A. De Luca and R. De Simone, "Controllo di traiettoria per robot con bracci flessibili basato sull'inversione in frequenza della dinamica," *41° Conv. ANIPLA (Automation 1997)*, pp. 227–236, 1997.

[11] A. De Luca, S. Panzieri, and G. Ulivi, "Stable inversion control for flexible link manipulators," *1998 IEEE Int. Conf. on Robotics and Automation*, pp. 799–805, 1998.

[12] S. Devasia, D. Chen, and B. Paden, "Nonlinear inversion-based output tracking," *IEEE Trans. on Automatic Control*, vol. 41, no. 7, pp. 930–942, 1996.

[13] A. De Luca, "Feedforward/feedback laws for the control of flexible robots," *2000 IEEE Int. Conf. on Robotics and Automation*, pp. 233–240, 2000.

[14] A. De Luca and G. Di Giovanni, "Rest-to-rest motion of a one-link flexible arm," *2001 IEEE/ASME Int. Conf. on Advanced Intelligent Mechatronics*, July 2001.

[15] A. Isidori, *Nonlinear Control Systems*, 3rd Edition, Springer-Verlag, 1995.

[16] A. De Luca and P. Lucibello, "A general algorithm for dynamic feedback linearization of robots with elastic joints," *1998 IEEE Int. Conf. on Robotics and Automation*, pp. 504–510, 1998.

[17] A. De Luca, L. Lanari, P. Lucibello, S. Panzieri, and G. Ulivi, "Control experiments on a two-link robot with a flexible forearm," *29th IEEE Conf. on Decision and Control*, pp. 520–527, 1990.

[18] E. Barbieri and Ü. Özgüner, "Unconstrained and constrained mode expansions for a flexible slewing link," *ASME J. of Dynamic Systems, Measurement, and Control*, vol. 110, pp. 416–421, 1988.

[19] F. Bellezza, L. Lanari, and G. Ulivi, "Exact modeling of the slewing flexible link," *1990 IEEE Int. Conf. on Robotics and Automation*, pp. 734–739, 1990.

Appendix

The expression of the decoupling matrix A and of vector f in eq. (12) are given, after simplifications, by

$$A = \begin{bmatrix} 1 & 0 \\ a_{12} & a_{22} \end{bmatrix},$$

with

$$\begin{aligned} a_{12} &= -1 + [\, c_1\phi'_{p1}(0)h_1\delta_1 \quad c_1\gamma_1 \,]\, R(\theta_2) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ a_{22} &= \omega_1^2 + [\, (c_1\gamma_1 - \phi'_1(0)h_1) \quad -c_1\phi'_1(0)h_1\delta_1 \,]\, R(\theta_2) \begin{bmatrix} \dot\theta_1^2 \\ \xi_1 \end{bmatrix} \end{aligned}$$

and by

$$\begin{aligned} f &= c_1\omega_1^4\delta_1 + \omega_1^2\xi_1 + [\, \Gamma_1 \quad \Gamma_2 \,]\, R(\theta_2) \begin{bmatrix} \dot\theta_1^2 \\ \xi_1 \end{bmatrix} \\ &\quad + 2\, [\, (c_1\phi'_1(0)h_1\dot\delta_1 + c_1\gamma_1\dot\theta_2) \quad -\dot\theta_2 c_1\phi'_1(0)h_1\delta_1 \,] \\ &\qquad \cdot R(\theta_2) \begin{bmatrix} 2\dot\theta_1\xi_1 \\ \xi_2 \end{bmatrix} \\ &\quad + 2\, [\, c_1\phi'_1(0)h_1\delta_1 \quad c_1\gamma_1 \,]\, R(\theta_2) \begin{bmatrix} (\xi_1^2 + \dot\theta_1\xi_2) \\ 0 \end{bmatrix}, \end{aligned}$$

with

$$\begin{aligned} \Gamma_1 &= -c_1\phi'_1(0)h_1\delta_1 \left(2\omega_1^2 + \dot\theta_2^2\right) \\ &\quad + \phi'_1(0)h_1 \left(-\xi_1 + [\, c_1\phi'_1(0)h_1\delta_1 \quad c_1\gamma_1 \,]\, R(\theta_2) \begin{bmatrix} \dot\theta_1^2 \\ \xi_1 \end{bmatrix}\right) \\ \Gamma_2 &= -c_1\gamma_1 \left(\omega_1^2 + \dot\theta_2^2\right) - 2\dot\theta_1^2 c_1\phi'_1(0)h_1\dot\delta_1. \end{aligned}$$

In the above expressions, c_1 is given by eq. (8) and γ_1 and $R(\theta_2)$ are defined in eq. (5).